Information Choice Technologies: Technical Appendix

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A Derivations and Proofs

A.1 Proof of proposition 1: Multiple equilibria with fixed-cost information

Proof. **Proof of claim 1:** Suppose all agents acquire full information. In this case $\Pi(b_i) = 0$ and each agent has to pay c. Deviating from the equilibrium means that the agent receives an uninformative signal, i.e. $b_i \to \infty$, but does not have to pay c. From the definition we can easily see that as $b_i \to \infty$, $\tau_v \to 0$. In the full information equilibrium, the agent knows that $\bar{a} = s$ and thus $\mathbb{V}\text{ar}\,[\bar{a}|\mathcal{I}_i] = \mathbb{C}\text{ov}\,[\bar{a}, s|\mathcal{I}_i] = \mathbb{V}\text{ar}\,[s|\mathcal{I}_i]$. Put differently, the agent only needs to forecast s, as he knows the action of the other players. Thus, after deviation, the agent's payoff becomes

$$\begin{aligned}\Pi_i &= -\mathbb{V}\text{ar}[s|\mathcal{I}_i] \\ &= -\tau_s^{-1}\end{aligned}$$

This deviation is strictly profitable if and only if

$$\tau_s^{-1} < c$$

From there it follows that an equilibrium with full revelation is sustainable if and only if $c \leq \tau_s^{-1}$, which is what we wanted to show.

Proof of claim 2: Suppose we are in an equilibrium, in which no agent has acquired information. In this case, all agents know that no other agent has acquired information so that for each i we have $a_i = \bar{a} = \mathbb{E}[s] = \mu$. Thus, $\mathbb{V}\text{ar}[s|\mathcal{I}_i] = \tau_s^{-1}$ and $\mathbb{V}\text{ar}[\bar{a}|\mathcal{I}_i] = \mathbb{C}\text{ov}[s, \bar{a}|\mathcal{I}_i] = 0$. Hence, in equilibrium, the payoff for each agent π_i is given by

$$\pi_i = -(1-r)^2\tau^{-1}$$

If the agent were to deviate, he would fully learn s and incur a cost of c. He still knows that $\bar{a} = \mu$. Thus, he would play $a_i = (1-r)s + r\mu$. We see from there, as well as from the fact that $\mathbb{V}\text{ar}[s|\mathcal{I}_i] = \mathbb{V}\text{ar}[\bar{a}|\mathcal{I}_i] = \mathbb{C}\text{ov}[s, \bar{a}|\mathcal{I}_i] = 0$ that $\Pi = 0$. thus, the agent's payoff after deviation will be $-c$. From there it follows that a deviation is strictly profitable if and only if

$$(1-r)^2\tau^{-1} > c$$

Hence, an equilibrium in which no agent acquires any information is sustainable as long as
$c \geq (1-r)^2\tau_s^{-1}$, which is what we wanted to show.

Proof of claim 3: Suppose that we are in an equilibrium in which a fraction $\alpha \in (0, 1)$ of the agents is informed and a fraction $(1-\alpha)$ is not. Denote by $\mathcal{I}^I$ an informed agents' information set and by $\mathcal{I}^U$ the uninformed agents information set. The informed agents know the precise value of s, the uninformed agents have to rely on the prior alone. We conjecture that in equilibrium both, informed and uninformed agents play linear strategies of their signal and the prior:

$$\begin{aligned}a_i^I = \bar{a}^I &= \gamma_0^I\mu + (1-\gamma_0^I)s \\ a_i^U = \bar{a}^U &= \gamma_0^U\mu\end{aligned}$$

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Where we have used the fact that there is no idiosyncratic noise in the signals for the informed agents, since s is fully revealed. Also note that $\gamma_0^I, \gamma_0^U \in [0,1]$. Let us now find γ_0^I and γ_0^U in such an equilibrium. The average action is given by:

$$\bar{a} = (\alpha\gamma_0^I + (1-\alpha)\gamma_0^U)\mu + \alpha(1-\gamma_0^I)s$$

From there it follows that

$$\begin{aligned}\mathbb{E}\left[\bar{a}|\mathcal{I}^U\right] &= (\alpha + (1-\alpha)\gamma_0^U)\mu \\ \mathbb{E}\left[\bar{a}|\mathcal{I}^I\right] &= \bar{a} = (\alpha\gamma_0^I + (1-\alpha)\gamma_0^U)\mu + \alpha(1-\gamma_0^I)s\end{aligned}$$

From there, using the expression for the optimal action of agents we get:

$$\begin{aligned}\gamma_0^U\mu &= r(\alpha + (1-\alpha)\gamma_0^U)\mu + (1-r)\mu \\ \Rightarrow \quad \gamma_0^U &= r(\alpha + (1-\alpha)\gamma_0^U) + (1-r) \\ &= \frac{\alpha r + (1-r)}{1 - r(1-\alpha)} = 1\end{aligned}$$

With this, we can then solve for γ_0^I and $(1-\gamma_0^I)$:

$$\begin{aligned}\gamma_0^I\mu + (1-\gamma_0^I)s &= r(\alpha\gamma_0^I + (1-\alpha))\mu + (r\alpha(1-\gamma_0^I) + (1-r)s \\ \Rightarrow \quad \gamma_0^I &= r(\alpha\gamma_0^I + (1-\alpha)) \\ &= \frac{(1-\alpha)r}{1-\alpha r} \\ \Rightarrow \quad (1-\gamma_0^I) &= \frac{1-r}{1-\alpha r}\end{aligned}$$

Let us now turn to the sufficient statistics in such an equilibrium. Trivially, we have
$\mathbb{V}\text{ar}\left[s|\mathcal{I}^I\right] = \mathbb{C}\text{ov}\left[\bar{a}, s|\mathcal{I}^I\right] = \mathbb{V}\text{ar}\left[\bar{a}|\mathcal{I}^I\right] = 0$. For the uninformed agents, we get:

$$\begin{aligned}\mathbb{V}\text{ar}\left[s|\mathcal{I}^U\right] &= \tau_s^{-1} \\ \mathbb{C}\text{ov}\left[\bar{a}, s|\mathcal{I}^U\right] &= \alpha(1-\gamma_0^I)\tau_s^{-1} \\ \mathbb{V}\text{ar}\left[\bar{a}|\mathcal{I}^I\right] &= \alpha^2(1-\gamma_0^I)^2\tau_s^{-1}\end{aligned}$$

Denoting by Π^U and Π^I the payoff of uninformed and informed agents respectively, we know that a necessary condition for a mixed equilibrium is $\Pi^U = \Pi^I$ Since we have

$$\begin{aligned}\Pi^I &= r^2\mathbb{V}\text{ar}\left[s|\mathcal{I}^I\right] + 2(1-r)r\mathbb{C}\text{ov}\left[\bar{a}, s|\mathcal{I}^I\right] + (1-r)^2\mathbb{V}\text{ar}\left[\bar{a}|\mathcal{I}^I\right] - c = -c \\ \Pi^U &= r^2\mathbb{V}\text{ar}\left[s|\mathcal{I}^U\right] + 2(1-r)r\mathbb{C}\text{ov}\left[\bar{a}, s|\mathcal{I}^U\right] + (1-r)^2\mathbb{V}\text{ar}\left[\bar{a}|\mathcal{I}^U\right] \\ &= (r^2\alpha^2(1-\gamma_0^I)^2 + 2(1-r)\alpha(1-\gamma_0^I) + (1-r)^2)\tau_s^{-1} \\ &= [r\alpha(1-\gamma_0^I) + (1-r)]^2\tau_s^{-1}\end{aligned}$$

The necessary condition for a mixed equilibrium then becomes:

$$\begin{aligned}[r\alpha(1-\gamma_0^I) + (1-r)]^2 &= \tau_s c \\ r\alpha\frac{(1-r)}{1-\alpha r} + (1-r) &= \sqrt{\tau_s c} \\ 1-r &= \sqrt{\tau_s c}(1-\alpha r) \\ \alpha &= \frac{\sqrt{\tau_s c} - (1-r)}{r\sqrt{\tau_s c}}\end{aligned}$$

Since we require $\alpha > 0$, we get the first condition:

$$\sqrt{\tau_s c} - (1-r) > 0 \Rightarrow c > (1-r)^2\tau_s^{-1}$$

From $\alpha < 1$ we then get the second condition:

$$\begin{array}{rcl} & \sqrt{\tau_s c} - (1-r) & < r\sqrt{\tau_s c} \\ \Rightarrow & r\sqrt{\tau_s c} & < 1 \\ \Rightarrow & c & < \tau_s^{-1} \end{array}$$

□

A.2 Proof of proposition 2: Uniqueness with signal clarity choice

See Myatt and Wallace (2011).

A.3 Proof of Proposition 3: Uniqueness with sender noise choice

Proposition 3. *When $c(\psi^i)$ is a convex function and $\forall j$, $\frac{\partial c(\psi^i)}{\partial \psi^i_j}|_{\psi^i_j=0} = 0$ and $\forall j$, $\lim_{\psi^i_j \to \infty} \frac{\partial c(\psi^i)}{\partial \psi^i_j} = \infty$, then there is a unique symmetric equilibrium in the choice of the inverse signal clarity ψ^i and actions.*

Proof. The proof has four steps. First, we will show that our specified loss function is equal to the loss function in Myatt and Wallace (MW) plus a term that only depends on the average action a and the state s, and thus it does not influence the choice of a_i. We will then work with MW's objective function which is more tractable. Second, as an intermediate step to establish the convexity of MW loss function, we will show that at equilibrium a_i is not sensitive to $\bar{a}$. Third, the uniqueness of the choice a_i follows directly from the minimization of a convex function. Finally, we argue that uniqueness of individual choices in a symmetric equilibrium together with the absence of interaction with the average action at the optimum implies uniqueness of such equilibrium.

Step 1: Comparison between MW and our loss function.

Ignoring the expectations operator and the cost component, the loss function in Myatt and Wallace (2011) is given by:

$$L_{MW}(a_i, \bar{a}, s) \equiv (1-r)(a_i - s)^2 + r(a_i - \bar{a})^2$$

Our loss function is given by:

$$\begin{aligned} L(a_i, \bar{a}, s) &= [(1-r)(a_i - s) + r(a_i - \bar{a})]^2 \\ &= (1-r)^2(a_i - s)^2 + r^2(a_i - \bar{a})^2 + r(1-r)2(a_i - \bar{a})(a_i - s) \\ &= (1-r)^2(a_i - s)^2 + r^2(a_i - \bar{a})^2 + r(1-r)[(a_i - \bar{a})^2 + (a_i - s)^2 - (\bar{a} - s)^2] \\ &= [(1-r)^2 + r(1-r)](a_i - s)^2 + [r^2 + r(1-r)](a_i - \bar{a})^2 - r(1-r)(\bar{a} - s)^2 \\ &= (1-r)(a_i - s)^2 + r(a_i - \bar{a})^2 - r(1-r)(\bar{a} - s)^2 \\ &= L_{MW}(a_i, \bar{a}, s) - r(1-r)(\bar{a} - s)^2 \end{aligned}$$

Since the last term will not appear in the first order condition with respect to a_i, it will not affect its choice. Since there are no choice variables in the $(a-s)^2$ term, and we have expected utility, the information choice cannot affect the ex-ante expectation of this term. Therefore, adding this term to our utility function will also not affect the information choice. From now on, we will work with MW objective function.

Step 2: Reformulating loss functions in terms of signal variances.

Consider n signals where signal $j = 1, ..., n$ for agent i is given by

$$z^i_j = s + d^i_j u_j + b^i_j v^i_j$$

and $u_j \sim \mathcal{N}(0,1)$ is public noise, $v^i_j \sim \mathcal{N}(0,1)$ is private noise and $v \perp u \perp s$. The agents private noise component is determined by b^i_j which is fixed. Without loss of generality, we will assume that the fixed components are the same across agents: $\forall i\ b^i_j = b_j$.

Instead of letting agents choose the loading of the public noise d^i_j, we let them choose the inverse of the loading $\psi^i_j \equiv {d^i_j}^{-1}$. Each agent chooses his action a_i as well as his information strategy $\psi^i \equiv \{{d^i_j}^{-1}\}^n_{j=1}$, facing an information cost function $c(\psi^i)$ that is increasing and convex in ψ^i. Furthermore, we assume that it is never optimal to drive the public noise component to zero by assuming

$$\forall j, \quad \lim_{\psi_j^i \to \infty} \frac{\partial c(\psi^i)}{\partial \psi_j^i} = \infty$$

From Hellwig and Veldkamp (2009), we know that each agent chooses a linear strategy that assigns weights $\gamma_j^i \geq 0$ to each signal such that $\sum_{j=1}^n \gamma_j^i \leq 1$. Under this conjecture, their strategy is given by:

$$a_i = \mu + \sum_{j=1}^n \gamma_j^i \left(z_j^i - \mu \right) = \mu + \sum_{j=1}^n \gamma_j^i (s - \mu + d_j^i u_j + b_j v_j^i)$$

Furthermore, in any symmetric equilibrium, $\forall i \; d_j^i = d_j$ and $\gamma_j^i = \gamma_j$ for each signal j. Since idiosyncratic noise v_j^i washes out in the aggregate, the average action is then given by:

$$\bar{a} = \mu + \sum_{j=1}^n \gamma_j (s - \mu + d_j u_j)$$

Now we will express the objective as a function of the vectors $\gamma^i \equiv \{\gamma_j^i\}$ and ψ^i, assuming all other players opt for information choice ψ and weights γ:

$$L(\gamma_i, \psi^i; \gamma, \psi) = (1-r)\mathbb{E}[(a_i - s)^2] + r\mathbb{E}[(a_i - \bar{a})^2] + c(\psi^i)$$

For the first term, we have that:

$$\begin{aligned}
\mathbb{E}[(a_i - s)^2] &= \mathbb{E}\Big[\left\{ \left(1 - \sum_{j=1}^n \gamma_j^i \right) (\mu - s) + \sum_{j=1}^n \gamma_j^i ({\psi_j^i}^{-1} u_j + b_j v_j^i) \right\}^2 \Big] \\
&= \left(1 - \sum_{j=1}^n \gamma_j^i \right)^2 \mathbb{E}[(s-\mu)^2] + \sum_{j=1}^n {\gamma_j^i}^2 \left[{\psi_j^i}^{-2} \mathbb{E}[u_j^2] + b_j^2 \mathbb{E}[{v_j^i}^2] + 2{\psi_j^i}^{-1} b_j \mathbb{E}[u_j v_j^i] \right] \\
&= \left(1 - \sum_{j=1}^n \gamma_j^i \right)^2 \tau^{-1} + \sum_{j=1}^n {\gamma_j^i}^2 \left[{\psi_j^i}^{-2} + b_j^2 \right]
\end{aligned}$$

where we have used the independence of v and u.

For the second term, and using again the independence assumption:

$$\begin{aligned}
\mathbb{E}[(a_i - \bar{a})^2] &= \mathbb{E}\Big[\Big\{ \sum_{j=1}^n \left(\gamma_j^i - \gamma_j \right) (s - \mu) + \sum_{j=1}^n \gamma_j^i ({\psi_j^i}^{-1} u_j + b_j v_j^i) - \sum_{j=1}^n \gamma_j \psi_j^{-1} u_j \Big\}^2 \Big] \\
&= \mathbb{E}\Big[\Big\{ \sum_{j=1}^n \left(\gamma_j^i - \gamma_j \right) (s - \mu) + \sum_{j=1}^n \gamma_j^i b_j v_j^i + \sum_{j=1}^n (\gamma_j^i {\psi_j^i}^{-1} - \gamma_j \psi_j^{-1}) u_j \Big\}^2 \Big] \\
&= \left(\sum_{j=1}^n \gamma_j^i - \sum_{j=1}^n \gamma_j \right)^2 \tau^{-1} + \sum_{j=1}^n \left(\gamma_j^i b_j \right)^2 + \sum_{j=1}^n \left(\gamma_j^i {\psi_j^i}^{-1} - \gamma_j \psi_j^{-1} \right)^2
\end{aligned}$$

Substituting back in the loss function, we have the following expression:

$$\begin{aligned}
L(\gamma_i, \psi^i; \gamma, \psi) &= (1-r) \left(1 - \sum_{j=1}^n \gamma_j^i \right)^2 \tau^{-1} + (1-r) \sum_{j=1}^n {\gamma_j^i}^2 \left[{\psi_j^i}^{-2} + b_j^2 \right] \\
&\quad + r \left(\sum_{j=1}^n \gamma_j^i - \sum_{j=1}^n \gamma_j \right)^2 \tau^{-1} + r \sum_{j=1}^n \left(\gamma_j^i b_j \right)^2 + r \sum_{j=1}^n \left(\gamma_j^i {\psi_j^i}^{-1} - \gamma_j \psi_j^{-1} \right)^2 + c(\psi^i) \\
&= (1-r) \left(1 - \sum_{j=1}^n \gamma_j^i \right)^2 \tau^{-1} + \sum_{j=1}^n {\gamma_j^i}^2 \left[(1-r){\psi_j^i}^{-2} + b_j^2 \right] + c(\psi^i) \\
&\quad + r \left(\sum_{j=1}^n \gamma_j^i - \sum_{j=1}^n \gamma_j \right)^2 \tau^{-1} + r \sum_{j=1}^n (\gamma_j^i {\psi_j^i}^{-1} - \gamma_j \psi_j^{-1})^2
\end{aligned}$$

Define

$$\hat{L}(\gamma_i, \psi^i) = (1-r)\left(1 - \sum_{j=1}^{n} \gamma_j^i\right)^2 \tau^{-1} + \sum_{j=1}^{n} \gamma_j^{i\,2}\left[(1-r)\psi_j^{i\,2} + b_j^2\right]$$

$$\Delta(\gamma_i, \psi^i; \gamma, \psi) = r\left(\sum_{j=1}^{n} \gamma_j^i - \sum_{j=1}^{n} \gamma_j\right)^2 \tau^{-1} + r\sum_{j=1}^{n}(\gamma_j^i \psi_j^{i\,-1} - \gamma_j \psi_j^{-1})^2$$

Therefore, player i's optimal strategy minimizes $L(\gamma_i, \psi^i; \gamma, \psi)$, and $\left(\hat{\gamma}, \hat{\psi}\right)$ constitutes a symmetric Nash equilibrium, if and only if $\left(\hat{\gamma}, \hat{\psi}\right) \in \arg\min_{\left(\gamma_i, \psi^i\right)} L(\gamma_i, \psi^i; \hat{\gamma}, \hat{\psi})$.

Step 3: Transformation into a planning problem.

Next, we show that $\left(\hat{\gamma}, \hat{\psi}\right)$ is a symmetric Nash equilibrium, if and only if

$$\left(\hat{\gamma}, \hat{\psi}\right) \in \arg\min_{\left(\gamma_i, \psi^i\right)} \left(\hat{L}(\gamma_i, \psi^i) + c(\psi^i)\right)$$

That is, the symmetric equilibrium strategies are given by the solution to this associated planning problem.

1. if $r \in [0, 1]$, it is immediate that any $\left(\hat{\gamma}, \hat{\psi}\right) \in \arg\min_{\left(\gamma_i, \psi^i\right)} \left(\hat{L}(\gamma_i, \psi^i) + c(\psi^i)\right)$ forms a symmetric Nash equilibrium (to see his notice that $\Delta(\gamma, \psi; \gamma, \psi) = 0$ for any (γ, ψ), and therefore $\left(\hat{\gamma}, \hat{\psi}\right) \in \arg\min_{\left(\gamma_i, \psi^i\right)} \Delta(\gamma_i, \psi^i; \hat{\gamma}, \hat{\psi})$, but then it is immediate that $\left(\hat{\gamma}, \hat{\psi}\right) \in \arg\min_{\left(\gamma_i, \psi^i\right)} L(\gamma_i, \psi^i; \hat{\gamma}, \hat{\psi})$.

Converse:

Suppose now that $\left(\hat{\gamma}, \hat{\psi}\right)$ is a symmetric Nash equilibrium.

Since $\left(\hat{\gamma}, \hat{\psi}\right)$ is symmetric, we have $\Delta(\hat{\gamma}, \hat{\psi}; \hat{\gamma}, \hat{\psi}) = 0$. Furthermore, since $r \in [0, 1]$, $\left(\hat{\gamma}, \hat{\psi}\right) \in \arg\min_{\left(\gamma^i, \psi^i\right)} \Delta(\gamma_i, \psi_i; \hat{\gamma}, \hat{\psi})$. Since $\left(\hat{\gamma}, \hat{\psi}\right)$ is Nash, we also have
$\left(\hat{\gamma}, \hat{\psi}\right) \in \arg\min_{(\gamma_i, \psi_i)} L(\gamma_i, \psi_i, \hat{\gamma}, \hat{\psi})$.

There are two possibilities: either (i) $\left(\hat{\gamma}, \hat{\psi}\right) \in \arg\min_{\left(\gamma_i, \psi^i\right)} \left(\hat{L}(\gamma_i, \psi^i) + c(\psi^i)\right)$ or (ii) $\left(\hat{\gamma}, \hat{\psi}\right) \notin \arg\min_{\left(\gamma_i, \psi^i\right)} \Big(\hat{L}(\gamma_i, \psi^i) + c(\psi$
and there is no profitable deviation.

Consider (ii) first. Because of our assumptions on the cost function we know that we must be on the interior of the support of $L(\gamma^i, \psi^i, \gamma, \psi)$. Since we are at a minimum, that implies that the first-order effects of deviation are zero for $\Delta(\hat{\gamma}, \hat{\psi}; \hat{\gamma}, \hat{\psi})$. But since $\left(\hat{\gamma}, \hat{\psi}\right) \notin \arg\min_{\left(\gamma_i, \psi^i\right)} \left(\hat{L}(\gamma_i, \psi^i) + c(\psi^i)\right)$ and $\left(\hat{L}(\gamma_i, \psi^i) + c(\psi^i)\right)$ is continuous and convex at $\left(\hat{\gamma}, \hat{\psi}\right)$, there must be a profitable deviation. So this cannot be an equilibrium. Then, we are left with (i) as the only possibility and indeed $\left(\hat{\gamma}, \hat{\psi}\right) \in \arg\min_{\left(\gamma_i, \psi^i\right)} \left(\hat{L}(\gamma_i, \psi^i) + c(\psi^i)\right)$

2. $r \leq 0$. If $\left(\hat{\gamma}, \hat{\psi}\right)$ is a symmetric Nash equilibrium, then $\left(\hat{\gamma}, \hat{\psi}\right) \in \arg\min_{\left(\gamma_i, \psi^i\right)} \left(\hat{L}(\gamma_i, \psi^i) + c(\psi^i)\right)$. To see this, notice that $\hat{L}\left(\hat{\gamma}, \hat{\psi}\right) + c(\hat{\psi}) \leq \hat{L}(\gamma_i, \psi^i) + c(\psi^i) + r\Delta(\gamma_i, \psi^i; \hat{\gamma}, \hat{\psi})$ and $r\Delta(\gamma, \psi; \hat{\gamma}, \hat{\psi}) < 0$ for any $(\gamma_i, \psi^i) \neq \left(\hat{\gamma}, \hat{\psi}\right)$.

Converse:

Let us first show that a Nash equilibirum always exists. We will do so by using Kakutani's fixed point theorem. Consider first the problem of the agent choosing the best response weights for any given information choice and symmetric strategy of the other players $\gamma_i^*(\psi^i, \gamma, \psi)$:

$$\gamma_i^*(\psi^i, \gamma, \psi) = \arg\min_{\gamma_i} L(\gamma_i, \psi^i, \gamma, \psi)$$

Since the loss function is strictly convex in the weights, this will have a unique solution. Using this unicity result, we can find the best-response function of a player choosing his information ${\psi^i}^*(\gamma(\psi), \psi)$, which now only depends on aggregate information choice:

$$ {\psi^i}^*(\psi) = \arg\min_{\psi^i} L\left(\gamma_i^*(\psi^i, \gamma(\psi), \psi), \psi^i, \gamma(\psi), \psi\right) $$

Notice that in any symmetric Nash equilibrium, γ must be a best response to the information choice ψ. For that reason we, restrict our attention to symmetric strategies that satisfy $\gamma = \gamma^*(\psi, \tilde{\gamma}, \tilde{\psi})$ for some symmetric strategy $(\tilde{\gamma}, \tilde{\psi})$, which allows us to write the loss function as a function of information choices alone[1]. Because of our assumptions on the cost function we know that we can find some upper bound M such that for the primitives of the game no player i would ever choose a $\psi_j^i > M$ for any of the signals j. Because of this we can without loss of generality restrict the choice set of each player to $\psi_i \in [0, M]^l$. We then get the correspondence:

$$ BR : [0, M]^l \longrightarrow 2^{[0,M]^l} $$

which maps any symmetric strategy ψ into a best response $\psi^*(\psi)$.

First note that $[0, M]^l$ is a convex, compact subset of R^l. The best response correspondence is non-empty (since the agent is minimizing over a compact set), compact-valued and upper-hemicontinous (by applying Berge's theorem of the maximum) and convex-valued (since it is the set of maximizers). Combining all these observations, we can apply Kakutani's fixed point theorem and we know that the correspondence BR has at least one fixed point. But such a fixed point corresponds to a symmetric Nash equilibrium with each agent playing the fixed point. So we have shown that the game has at least one Nash equilibrium.

Now, proceed by contradiction. Suppose we have $\left(\hat{\gamma}, \hat{\psi}\right) \in \arg\min_{(\gamma_i,\psi^i)} \left(\hat{L}(\gamma_i, \psi^i) + c(\psi^i)\right)$ and $\left(\hat{\gamma}, \hat{\psi}\right)$ is not a Nash equilibrium. We know that a Nash Equilibrium always exists and we know that the strategy profile of such an equilibrium minimizes $\left(\hat{L}(\gamma_i, \psi^i) + c(\psi^i)\right)$. So we can pick (γ^*, ψ^*) that is Nash and hence $(\gamma^*, \psi^*) \in \arg\min_{(\gamma_i,\psi^i)} \left(\hat{L}(\gamma_i, \psi^i) + c(\psi^i)\right)$. But since $\left(\hat{L}(\gamma_i, \psi^i) + c(\psi^i)\right)$ is strictly convex it has a unique minimum. This implies $(\gamma^*, \psi^*) = \left(\hat{\gamma}, \hat{\psi}\right)$, a contradiction to our assumption that $\left(\hat{\gamma}, \hat{\psi}\right)$ is not a Nash equilibrium. So it must be that $\left(\hat{\gamma}, \hat{\psi}\right)$ is Nash.

Step 4: Uniqueness of symmetric equilibrium

The solution to this planning problem is unique whenever its objective is strictly convex. Since $\hat{L}(\gamma_i, \psi^i)$ is strictly convex, convexity and hence uniqueness is guaranteed whenever $c(\psi^i)$ is also convex. □

A.4 Adding correlation between a given agent's signals

This section shows how to compute the conditional state and average action variances and their covariance when the set of signals an agent observes are correlated with each other.

For each information choice vector χ, we construct a corresponding $m \times n$ matrix X of zeros and ones. The number of rows m is the number of ones in χ. If the j^{th} entry of χ is 1, then there is a row of X that is all zeros, except for a one in the j^{th} position. Agent i's information set is thus summarized by the vector of signals he observes: $X\mathbf{z}^i$.

We begin by computing posterior beliefs, conditional on an information choice and signal realizations. Agents are trying to forecast both the true state and the common signal noise which determines the average action of other agents. Therefore, we define a $k+1 \times 1$ vector of both variables $\omega = \left[\begin{array}{cc} s & \mathbf{u}' \end{array}\right]'$. This is the relevant state variable. It is normally distributed with mean 0 and $Var(\omega) = I_{k+1}$. The covariance of the observed signals and this state is $Cov(\omega, X\mathbf{z}^i) = [\mathbf{1}_m, XD]$. The distribution of posterior beliefs comes from standard formulas for the conditional distribution of bivariate normals. Conditional on observing $\mathcal{I}_i$, ω is normally distributed with posterior mean and variance-covariance matrix

$$ E\left(\omega|\mathcal{I}_i\right) = Cov(\omega, X\mathbf{z}^i)'\, Var(X\mathbf{z}^i)^{-1}\, X\mathbf{z}^i \tag{1} $$

$$ \Sigma\left(\chi\right) := Var(\omega|\mathcal{I}_i) = Var(\omega) - Cov(\omega, X\mathbf{z}^i)'\, Var(X\mathbf{z}^i)^{-1}\, Cov(\omega, X\mathbf{z}^i) \tag{2} $$

[1]Where we start with some arbitrary symmetric strategies that satisfy these conditions

where $Var(X\mathbf{z}^i) = X\Gamma\Gamma'X'$ and $\Gamma = \begin{bmatrix} \mathbf{1}_n & D & B \end{bmatrix}$.[2]

A.5 Updating with correlated signal errors

In this section we consider the case where signal errors are correlated across agents (as opposed to across a given agent's signals). We show how to do Bayesian updating on the state and the average action.

State variance Let us first consider the distribution of s, u_1, z_1, z_2:

$$\begin{pmatrix} s \\ u_1 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & d_1 & b_1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s \\ u_1 \\ v_1^i \\ d_2u_2 + b_2v_2^i \end{pmatrix} \sim \mathcal{N}(\mathbf{0}, \Sigma)$$

Since we know that

$$\begin{pmatrix} s \\ u_1 \\ v_1^i \\ d_2u_2 + b_2v_2^i \end{pmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{pmatrix} \tau_s^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d_2^2 + b_2^2 \end{pmatrix}\right)$$

We can easily find an expression for Σ:

$$\Sigma = \begin{pmatrix} \tau_s^{-1} & 0 & \tau_s^{-1} & \tau_s^{-1} \\ 0 & d_1^2 & d_1^2 & 0 \\ \tau_s^{-1} & d_1^2 & \tau_s^{-1} + d_1^2 + b_1^2 & \tau_s^{-1} \\ \tau_s^{-1} & 0 & \tau_s^{-1} & \tau_s^{-1} + d_2^2 + b_2^2 \end{pmatrix}$$

From standard Bayesian updating of normal variables we can now find an expression for $\mathbb{V}\mathrm{ar}\left(\begin{matrix} s \\ u_1 \end{matrix} |\mathcal{I}_i\right)$:

$$\begin{aligned} \mathbb{V}\mathrm{ar}\left(\begin{matrix} s \\ u_1 \end{matrix} |\mathcal{I}_i\right) &= \mathbb{V}\mathrm{ar}\begin{pmatrix} s \\ u_1 \end{pmatrix} - \mathbb{C}\mathrm{ov}(\begin{matrix} s & u_1 \end{matrix}, \begin{matrix} z_1 & z_2 \end{matrix})'\mathbb{V}\mathrm{ar}\begin{pmatrix} s \\ u_1 \end{pmatrix}^{-1} \mathbb{C}\mathrm{ov}(\begin{matrix} s & u_1 \end{matrix}, \begin{matrix} z_1 & z_2 \end{matrix}) \\ &= \begin{pmatrix} \tau_s^{-1} & 0 \\ 0 & d_1^2 \end{pmatrix} - \begin{pmatrix} \tau_s^{-1} & \tau_s^{-1} \\ d_1^2 & 0 \end{pmatrix} \begin{pmatrix} \tau_s^{-1} + d_1^2 + b_1^2 & \tau_s^{-1} \\ \tau_s^{-1} & \tau_s^{-1} + d_2^2 + b_2^2 \end{pmatrix} \begin{pmatrix} \tau_s^{-1} & d_1^2 \\ \tau_s^{-1} & 0 \end{pmatrix} \end{aligned}$$

Rearranging yields

$$\mathbb{V}\mathrm{ar}\left(\begin{matrix} s \\ u_1 \end{matrix} |\mathcal{I}_i\right) = \frac{1}{\tau_s + \tau_1 + \tau_2} \begin{pmatrix} 1 & -\tau_1 d_1^2 \\ -\tau_1 d_1^2 & d_1^2[\tau_1 + (1 - \tau_1 d_1^2)(\tau_s + \tau_2)] \end{pmatrix}. \quad (3)$$

Average action variance Using the formula for $\bar{a}$ from the previous step and integrating out the mean-zero private signal noise, we find that $\bar{a} = \gamma_0\mu + \sum_{j=1}^n \gamma_j(s + a_j u_j)$. To determine the matrix of variances and covariances $Var[s + D\mathbf{u}|\mathcal{I}_i]$, we use the formula for the conditional variance of bivariate normals,

$$\begin{aligned} Var[s + D\mathbf{u}|z^i] &= Var(s + D\mathbf{u}) - Cov(s + D\mathbf{u}, z^i)'Var(z^i)^{-1}Cov(s + D\mathbf{u}, z^i) & (4) \\ &= \tau_s^{-1} + D'D - (\tau_s^{-1} + D'D)'(\tau_s^{-1} + D'D + B'B)^{-1}(\tau_s^{-1} + D'D) & (5) \\ &= B'B - B'BHB'B, & (6) \end{aligned}$$

where $H \equiv (I_n\tau_s^{-1} + D'D + B'B)^{-1}$ is the precision matrix of the signals $\mathbf{z}$. Then letting γ be the $(n \times 1)$ vector of weights $\{\gamma_1, \ldots, \gamma_n\}$, we can write

$$Var[\bar{a}|\mathcal{I}_i] = \gamma'(B'B - B'BHB'B)\gamma$$

In scalar notation, this is

$$Var[\bar{a}|\mathcal{I}_i] = \sum_j \gamma_j^2 b_j^2 - \sum_j \sum_k \gamma_j\gamma_k b_j^2 b_k^2 H_{(j,k)}.$$

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[2] The relevant variance and covariance matrices are obtained by writing $\left[\omega' \;\; (X\mathbf{z}^i)'\right]'$ as

$$\begin{bmatrix} \omega \\ X\mathbf{z}^i \end{bmatrix} = \begin{bmatrix} I_{k+1} & 0 \\ & X\Gamma \end{bmatrix} \begin{bmatrix} \omega \\ \mathbf{v}^i \end{bmatrix}$$

and using the fact that if a vector x is distributed according to $x \sim \mathcal{N}(\mu, I)$, then $Cx \sim \mathcal{N}(C\mu, CC')$.

Expected utility Using $a_i = (1-\gamma)\mu + \gamma z_1$, $\bar{a} = (1-\gamma)\mu + \gamma(s + d_1 u_1)$, and (**??**), we can write the agent's payoff function as

$$u^i(b_1, b_2, \gamma) = -\left[\begin{array}{cc} (1-r+r\gamma) & r\gamma \end{array}\right] \mathbb{V}\mathrm{ar}\left(\begin{array}{c} s \\ u_1 \end{array} |z_1, z_2\right) \left[\begin{array}{c} (1-r+r\gamma) \\ r\gamma \end{array}\right]$$

We are now interested in the marginal value of observing the second signal with some precision, i.e. $\frac{\partial u^i(b_1,b_2,a_i)}{\partial b_2}|_{b_2=0}$. Substituting in the variance expression (3) yields utility.