The New Market for Volatility Trading

Jin E. Zhang*
Associate Professor
School of Economics and Finance
The University of Hong Kong
Pokfulam Road, Hong Kong, P. R. China

Jinghong Shu
Associate Professor
School of International Trade and Economics
University of International Business and Economics
Huixindongjie, Beijing, P. R. China

Menachem Brenner
Professor of Finance
Stern School of Business
New York University
New York, NY 10012, U.S.A.

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* Corresponding author, School of Economics and Finance, The University of Hong Kong, Pokfulam Road, Hong Kong, P. R. China. Tel: (852) 2859-1033, Fax: (852) 2548-1152, e-mail: jinzhang@hku.hk

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Abstract

This paper analyses the new market for trading volatility; VIX futures. We first use market data to establish the relationship between VIX futures prices and the index itself. We observe that VIX futures and VIX are highly correlated; the term structure of average VIX futures prices is upward sloping while the term structure of VIX futures volatility is downward sloping. To establish a theoretical relationship between VIX futures and VIX, we model the instantaneous variance using a simple square root mean-reverting process with a stochastic long-term mean level. Using daily calibrated long-term mean and VIX, the model gives good predictions of VIX futures prices under normal market situation. These parameter estimates could be used to price VIX options.

1. Introduction

Stochastic volatility was ignored for many years by academics and practitioners. Changes in volatility were usually assumed to be deterministic (e.g. Merton (1973)). The importance of stochastic volatility and its potential effect on asset prices and hedging/investment decisions has been recognized after the crash of ’87. The industry and academia have started to examine it in the late 80s, empirically as well as theoretically. The need to hedge potential volatility changes which would require a reference index has been first presented by Brenner and Galai (1989). It was also suggested that such an index should be the reference index for volatility derivatives that will be used to cope with stochastic volatility and its effect on portfolio returns. The dramatic volatility changes during the recent financial crisis, starting in September 2008, serve as a reminder of the need of volatility derivatives. Indeed, open interest and volume of exchange traded futures and options and over the counter derivatives, like variance swaps, have increased substantially during this period. Given the important role that these derivatives are playing in the securities market, present and future, this paper is trying to contribute to our knowledge regarding the market for volatility futures.

In 1993 the Chicago Board Options Exchange (CBOE) has introduced a volatility index based on the prices of index options. This was an implied volatility index based on option prices of the S&P100 and it was traced back to 1986. Until about 1995 the index was not a good predictor of realized volatility. Since then its forecasting ability has improved markedly (see Corrado and Miller (2005)), though it is biased upwards. Although many market participants considered the index to be a good predictor of short term volatility, daily or even intraday, only recently has the CBOE introduced volatility products based on the index. Our study focuses on the first exchange traded product, VIX futures, which was introduced in March 2004. Another market that has been, for some years now, trading volatility over-the-counter is the variance swaps market. This market has been thoroughly studied by Carr and Wu (2009) and by Egloff, Leippold and Wu (2009).

The current VIX is based on a different methodology than the previous VIX, renamed VXO, and uses the S&P500 European style options rather than the S&P100 American style options. Despite these two major differences the correlation between the levels of the two indices is about 98% (see Carr and Wu (2006)).

Carr and Madan (1998), and Demeterfi et al (1999) developed the original idea of replicating the realized variance by a portfolio of European options. In September 2003, the CBOE used their theory to design a new methodology to compute VIX, see Appendix for details.

On March 26, 2004, the newly created CBOE Futures Exchange (CFE) started to trade an exchange listed volatility product; VIX futures, a futures contract written on the VIX index. It is cash settled with the VIX. Since VIX is not a traded asset, one cannot replicate a VIX futures contract using the VIX and a risk free asset. Thus a cost-of-carry relationship between VIX futures and VIX cannot be established.

Though volatility futures did not exist back in the 1990s, Grünbichler and Longstaff (1996) have written the first theoretical paper on the valuation of futures and options on instantaneous volatility. They derive a closed form solution for the futures price assuming volatility follows the dynamics laid out in Heston (1993) and others. Naturally, their model does not deal with the existing futures contract and its specifications.

A recent paper by Zhang and Zhu (2006) is the first attempt to study the price of VIX futures. They developed a simple theoretical model for VIX futures prices and tested the model using the actual futures price on one particular day. Other related works include Dotsis, Psychoyios and Skiadopoulos (2007), which studies the continuous-time models of the volatility indices. Zhu and Zhang (2007) use the time-dependent long-term mean level in the volatility model. Lin (2007) incorporates jumps in both index return and volatility processes. Zhang and Huang (2009) study the CBOE S&P500 three-month variance futures market[1]. However they did not study empirically the behaviour of the VIX futures market and how it could be used to model futures prices.

Our objective is two fold; First, to use market data to analyze empirically the relationship between VIX futures prices and VIX, the term structure of VIX futures prices and the volatility of VIX futures prices. Second, to develop an efficient pricing model for VIX products and to find parameter estimates that best describe the empirical relationships and could be used in pricing VIX futures and options.

2. Data

In this paper, we use the daily VIX index and VIX futures data provided by the CBOE. The VIX index data, including open, high, low and close levels, are available from January 2, 1990 to the present. The VIX futures data, including open, high, low, close and settle prices, trading volume and open interest, are available from March 26, 2004 to the present.

Between March 26, 2004 and March 8, 2006, four futures contracts were listed for each day: two near term and two additional months on the February quarterly cycle. For example, on the first day of the listing, March 26, 2004, four contracts May 04, Jun 04, Aug 04 and Nov 04 were traded which stand for the four futures expiration months followed by the year respectively. On March 9, 2006, six futures contracts were listed. The number of contracts listed on each day increased to seven on April 24, 2006, to nine on October 23, 2006 and to

[1] Sepp (2008a,b) and Lin and Chang (2009) study VIX option pricing in an affine jump-diffusion framework. Lu and Zhu (2009) study the variance term structure using VIX futures market.

ten on April 22, 2008. Currently there are ten contracts traded on each day with maturity dates in each consecutive month.

The underlying value of the VIX futures contract used to be VIX times 10 under the symbol "VXB". The contract size was $100 times VXB. For example, with a VIX value of 17.33 on March 26, 2004, the VXB would be 173.3 and the contract size would be $17,330. On March 26, 2007, the underlying value was changed to be VIX and the futures price became one-tenth of the original value. But the contract size was changed to be $1,000 times VIX, so that the notional value of one futures contract remained unchanged.

Our empirical study covers the period of almost five years from March 26, 2004 to February 13, 2009, within which there were 63 contract months traded all together and 53 of them were matured. Table 1 provides a summary statistics of all of matured contracts. The average open interest for each contract was 4,404, which corresponded to a market value of 78 million dollars[2]. The average daily trading volume for each contract was 344, which corresponded to 6.1 million dollars. The shortest contract lasted 35 days, while the longest 524 days. The average futures price[3] for each contract changed from 18.56 for contracts that matured in May 2004 to 32.23 for contracts that matured in January 2009, while the VIX level ranged from 17.33 on March 26, 2004 to 42.93 on February 12, 2009. In general, the market expected future volatility decreased in the first three years, reached the historical lowest level of 9.89 on January 24, 2007. It increased dramatically in October 2008, reached the historical highest level of 80.86 on November 20, 2008, and quickly fell to current level of around 43.

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[2] Using the average VIX futures prices 176.29, we compute the market value as $17.63 \times 1000 \times 4{,}404 = 77{,}642{,}520$.

[3] The VIX futures price between March 26, 2004 and March 23, 2007 has been scaled down to be one-tenth of the original price in order to be consistent with the price after March 26, 2007.

2.1. VIX futures price pattern

In order to obtain some intuitions on what the data of VIX futures price is really like, we show in Figure 1a the time series of VIX and four VIX futures contracts, May 04, Jun 04, Aug 04 and Nov 04, listed on March 26, 2004. As we can see, the price of each contract started with a value relatively higher than its underlying variable VIX, and moved gradually downward and almost converged to VIX on maturity date. The downward trend the VIX futures price process indicates that the long-term mean level of volatility is higher than instantaneous volatility. Figure 1b shows the time series of VIX and five VIX futures, Sep 08, Oct 08, Nov 08, Dec 08 and Jan 09 during the period of recent global financial crisis. VIX level became extremely high and the term structure of VIX futures became strongly downward sloping.

2.2. Settlement procedure for VIX futures

VIX futures contracts settle on the Wednesday that is thirty days prior to the third Friday of the calendar month immediately following the month in which the applicable VIX futures contract expires. This means, for example, that the April 2008 VIX futures contract (J8) settled on Wednesday, April 16, 2008, which is thirty days prior to the settlement date of the corresponding May 2008 options on the Standard & Poor's 500 Stock Index (SPX) on Friday, May 16, 2008. If the third Friday of the month subsequent to expiration of the applicable VIX futures contract is a CBOE holiday, the final settlement date for the contract shall be thirty days prior to the CBOE business day immediately proceeding that Friday.

The Final Settlement Value for volatility index futures traded on CFE is equal to the Special Opening Quotation (SOQ) of the volatility index calculated from the sequence of opening prices on CBOE of the constituent options used to calculate the volatility index on the settlement date (Constituent Options). The opening price for any Constituent Options series in which there is no trade on CBOE will be the average of that option's bid price and ask price as determined at the opening of trading[4]. Because actual prices are used to compute the Final Settlement Value of VIX futures while mid-market options quotes are used to compute indicative volatility index values, there is an inherent risk of a significant disparity between the Final Settlement Value of an expiring VIX futures contract and the opening indicative volatility index value on the final settlement date. In Table 2, we compare the final settlement values of 53 matured VIX futures contracts and the previous closing VIX index level, opening and closing levels on maturity date. The average final settlement value is 1-2% lower than the average VIX index levels.

3. Empirical evidence

3.1. The relation between VIX futures and VIX

Because the underlying variable of VIX futures, i.e. VIX, is not a traded asset, we are not able to obtain a simple cost-of-carry relationship, arbitrage free, between the futures price, F_t^T, and its underlying, VIX_t. That is,

$$F_t^T \neq VIX_t e^{r(T-t)},$$

where r is the interest rate, and T is the maturity. Thus, we have gone to the data to see what we can learn about the relationship between VIX futures prices and VIX. We use this relationship to estimate the parameters, in a stochastic volatility model, that could be used to price volatility derivatives.

[4] The details of volatility index futures settlement is described in the CFE Information Circular IC07-21 available at http://cfe.cboe.com/.

There are four futures contracts available on a typical day. For example, on March 26, 2004, we had four VIX futures with maturities in May, June, August and November 2008, which corresponds to times to maturity of 54, 82, 145 and 236 days respectively. We construct 30-, 60-, 90- and 120-day futures prices by a linear interpolation technique. For example, the 30-day futures price is computed by using the market data of VIX and May futures on March 26, 2004. The 60-day futures price is computed by using the market data of May and June futures. The 90- and 120-day futures price is computed with June and August futures. We calculate these fixed time-to-maturity futures prices on each day and obtain four time series of 30-, 60-, 90- and 120-day futures prices. Table 3 provides summary statistics of their levels and returns. The return is computed as the logarithm of the price relative on two consecutive ends of day prices. Both the level and return of VIX futures price and returns are clearly non-normally distributed with positive skewness and excess kurtosis, while the S&P 500 index is negatively skewed. This shows that the volatility process is more likely to have positive jumps and the S&P 500 index process is more likely to have negative jumps. Figure 2 shows the time series of VIX and VIX futures for four fixed time-to-maturities. Table 4 presents the correlation matrix between the levels and returns of the S&P 500 index, VIX and VIX futures. All of the four futures series are negatively correlated with the S&P 500 index. VIX and VIX futures with four different maturities are very highly correlated, though there is no risk-free arbitrage relationship between them. Figure 2 also shows that the trading volume of VIX futures has been gradually increasing. The trading of VIX futures was not affected much by the global financial crisis.

We now explore the relationship between VIX futures and the underlying, VIX, using the data from March 26, 2004 to February 13, 2009. We examine the relationship using the following two equations

$$F_t^T = \alpha + \beta VIX_t + \varepsilon_t , \quad (1)$$

$$F_t^T = \alpha + \beta_1 VIX_t + \beta_2 VIX_t^2 + \varepsilon_t , \quad (2)$$

with fixed time-to-maturity, *T-t*. The regression results are reported in Table 5. As has already been observed, in Table 5, there is a strong correlation between the futures with different maturities and VIX, but the relationship is not linear. The slope coefficient, β_2, which relates the square of VIX to the futures contracts, is negative and highly significant for all of the three VIX futures. Though the magnitude of β_2 is rather small, its introduction in the regression affects strongly the VIX coefficient to be around 1. This indicates that the fixed time-to-maturity VIX future price is a nonlinear function of VIX. The R-square is about 0.95 for the 30-day VIX futures but drops to 0.9 for the 60-day VIX futures, and further decreases to about 0.88 for 90-day VIX futures and 0.87 for 120-day VIX futures, which reflects the fact that the prices of longer maturity futures are more uncertain than shorter ones. This is due to the lack of a no-arbitrage pricing relationship between the futures and the underlying index.

3.2. The term structure of VIX futures price

Over the period of March 26, 2004 to February 13, 2009, the average VIX was 18.99. The average VIX futures prices were 19.15, 19.36, 19.45 and 19.50 for 30-, 60-, 90- and 120-day maturities respectively. The term structure of the average VIX futures price is slightly upward sloping. As illustrated in Table 6, we find that the term structure of VIX futures price is upward sloping in 832 days, which is 67.6% of the total number of trading days, 1,231 days in the sample.

The upward sloping average VIX futures term structure indicates that the average short-term volatility is relatively low compared with the long-term mean level and that the volatility is increasing to the long-term higher level.

During the global financial crisis in October and November 2008, the market was very volatile. The short-term volatility, such as VIX was high and the term structure of VIX futures was downward sloping. In January and February 2009, VIX level fell to the current level of 43, the term structure of VIX futures became slightly humped.

3.3. The volatility of VIX and VIX futures

With the time series of VIX and fixed time-to-maturity VIX futures price, we compute the standard deviation of daily log price (index) relatives to obtain estimates of the volatility of these five series. During the five year period of our study we estimated the volatility of VIX to be 105.4%, while the volatilities of VIX futures prices are 50.6%, 38.6%, 33.8% and 30.6% for 30, 60, 90 and 120 days to maturity respectively. The longer the maturity, the lower is the volatility of volatility. The term structure of VIX futures volatility is downward sloping.

The phenomenon of downward sloping VIX futures volatility is consistent with the mean-reverting feature of the volatility. Since the long-term volatility approaches a fixed level, long-tenor VIX futures would be less volatile than short-tenor ones.

The empirical investigation provides us with some observations which will help us in our second objective of modelling the price of VIX futures.

4. A Theoretical Model of VIX Futures price

We now use a simple theoretical model to price the futures contracts using parameter estimates obtained from market data. We then test the extent to which model prices can explain market prices.

4.1. VIX futures price

In the risk-neutral measure, the dynamics of the S&P 500 index is assumed to be

$$dS_t = rS_t dt + \sqrt{V_t} S_t dB_{1t}^Q, \tag{3}$$

$$dV_t = \kappa(\theta_t - V_t)dt + \sigma_V \sqrt{V_t} dB_{2t}^Q, \tag{4}$$

$$d\theta_t = \sigma_\theta dB_{3t}^Q, \tag{5}$$

where r is the risk-free rate, V_t is the instantaneous variance of the index, θ_t, being the long-term mean level of the variance, is assumed to be a normal process, κ is the mean-reverting speed of the variance, σ_V measures the volatility of variance, dB_{1t}^Q, dB_{2t}^Q and dB_{3t}^Q are increments of three Brownian motions that describe the random noises in the index return, variance and long-term mean level. dB_{1t}^Q and dB_{2t}^Q are assumed to be correlated with a constant coefficient, ρ. dB_{3t}^Q is assumed to be independent of dB_{1t}^Q and dB_{2t}^Q. σ_θ, measuring the volatility of long-term mean level, is assumed to be very small.

The first three conditional (central) moments of the future variance, V_s, $0 < t < s$, can be evaluated as follows[5]

$$E_t^Q(V_s) = \theta_t + (V_t - \theta_t)e^{-\kappa(s-t)},$$

[5] The formulas for the first two moments were presented in Cox, Ingersoll and Ross (1985). The formula for the third central moments is not available in the literature.

$$E_t^Q\left[\left(V_s - E_t^Q(V_s)\right)^2\right] = \sigma_V^2 V_t e^{-\kappa(s-t)}\frac{1-e^{-\kappa(s-t)}}{\kappa} + \sigma_V^2\theta_t\frac{\left(1-e^{-\kappa(s-t)}\right)^2}{2\kappa},$$

$$E_t^Q\left[\left(V_s - E_t^Q(V_s)\right)^3\right] = \frac{3}{2}\sigma_V^4 V_t e^{-\kappa(s-t)}\frac{\left(1-e^{-\kappa(s-t)}\right)^2}{\kappa^2} + \frac{1}{2}\sigma_V^4\theta_t\frac{\left(1-e^{-\kappa(s-t)}\right)^3}{\kappa^2},$$

where E_t^Q stands for the conditional expectation in the risk-neutral measure.

The VIX index squared, at current time *t*, is defined as the variance swap rate over the next 30 calendar days. It is equal to the risk-neutral expectation of the future variance over the period of 30 days from *t* to $t+\tau_0$ with $\tau_0 = 30/365$,

$$\left(\frac{VIX_t}{100}\right)^2 = E_t^Q\left[\frac{1}{\tau_0}\int_t^{t+\tau_0} V_s ds\right] = \frac{1}{\tau_0}\int_t^{t+\tau_0} E_t^Q(V_s)ds$$

$$= \frac{1}{\tau_0}\int_t^{t+\tau_0}\left[\theta_t + \left(V_t - \theta_t\right)e^{-\kappa(s-t)}\right]ds = (1-B)\theta_t + BV_t, \qquad (6)$$

where $B = \frac{1-e^{-\kappa\tau_0}}{\kappa\tau_0}$ is a number between 0 and 1. Hence $\left(\frac{VIX_t}{100}\right)^2$ is the weighted average between long-term mean level θ_t and instantaneous variance V_t with B as the weight. Notice that the correlation, ρ, does not enter into the VIX formula, hence the VIX values do not capture the skewness of stock return.

The price of VIX futures with maturity *T* is then determined by

$$F_t^T = E_t^Q(VIX_T) = 100\times E_t^Q\left[\sqrt{(1-B)\theta_T + BV_T}\right]$$

$$= 100\times E_t^Q\left[\sqrt{(1-B)\theta_t + BV_T}\right] + O(\sigma_\theta^2), \qquad (7)$$

where $O(\sigma_\theta^2)$ stands for the terms with the order of σ_θ^2. In this paper, we will ignore the convexity adjustment from θ_t by assuming that σ_θ is very small.

We now study on the convexity adjustment from V_t. Expanding $\sqrt{(1-B)\theta_t + BV_T}$ with the Taylor expansion near the point of $E_t^Q(V_T)$ gives

$$
\begin{aligned}
\left[(1-B)\theta_t + BV_T\right]^{1/2} &= \left[(1-B)\theta_t + BE_t^Q(V_T)\right]^{1/2} \\
&+\frac{1}{2}\left[(1-B)\theta_t + BE_t^Q(V_T)\right]^{-1/2} B\left[V_T - E_t^Q(V_T)\right] \\
&-\frac{1}{8}\left[(1-B)\theta_t + BE_t^Q(V_T)\right]^{-3/2} B^2\left[V_T - E_t^Q(V_T)\right]^2 \\
&+\frac{1}{16}\left[(1-B)\theta_t + BE_t^Q(V_T)\right]^{-5/2} B^3\left[V_T - E_t^Q(V_T)\right]^3 + O\left(\left[V_T - E_t^Q(V_T)\right]^4\right),
\end{aligned}
$$

Taking expectation in the risk-neutral measure gives an approximate formula for the VIX futures price

$$
\begin{aligned}
\frac{F_t^T}{100} &= \left[\theta_t(1-Be^{-\kappa(T-t)}) + V_t Be^{-\kappa(T-t)}\right]^{1/2} \\
&-\frac{\sigma_V^2}{8}\left[\theta_t(1-Be^{-\kappa(T-t)}) + V_t Be^{-\kappa(T-t)}\right]^{-3/2} B^2\left[V_t e^{-\kappa(T-t)}\frac{1-e^{-\kappa(T-t)}}{\kappa} + \theta_t\frac{\left(1-e^{-\kappa(T-t)}\right)^2}{2\kappa}\right] \\
&+\frac{\sigma_V^4}{16}\left[\theta_t(1-Be^{-\kappa(T-t)}) + V_t Be^{-\kappa(T-t)}\right]^{-5/2} B^3\left[\frac{3}{2}V_t e^{-\kappa(T-t)}\frac{\left(1-e^{-\kappa(T-t)}\right)^2}{\kappa^2} + \frac{1}{2}\theta_t\frac{\left(1-e^{-\kappa(T-t)}\right)^3}{\kappa^2}\right],
\end{aligned}
\qquad (8)
$$

where terms with order $O(\sigma_\theta^2)$ and $O(\sigma_V^6)$ have been ignored.

4.2. Calibrating the VIX futures price model

Using the market prices of all traded S&P 500 three-month variance futures between May 18, 2004 and November 28, 2008, Zhang and Huang (2009) determined the two parameters of the variance process in Heston (1993) model as follows

$$\kappa = 2.4208, \qquad \theta = 0.03774.$$

Taking these two parameters as given, and using the market prices of all traded VIX futures during the same period, we determine the third parameter, σ_V, by solving following minimization problem:

$$\min_{\sigma_V}\sum_{i=1}^{I}\sum_{j=1}^{N_i}\left(F_{t_i\ mdl}^{T_j}(\sigma_V;VIX_{t_i},T_j-t_i,\kappa,\theta)-F_{t_i\ mkt}^{T_j}\right)^2 .$$

In this equation, index *i* stands for *i*th day, index *j* stands for *j*th contract on a particular day, $I=1{,}143$ is the total number of trading days in the sample between May 18, 2004 and November 28, 2008, N_i is the total number of contracts traded on *i*th day that ranges from 4 to 10. With the help of a computing software, such as Mathematica, after a few seconds of computation, we obtain a unique solution: $\sigma_V = 0.1425$.

The fixed long-term mean level, $\theta = 0.03774$, was determined unconditionally for the whole sample period. We now determine the process of θ_t by solving following minimization problem with $\kappa = 2.4208$ and $\sigma_V = 0.1425$:

$$\min_{\theta_t}\sum_{j=1}^{N_t}\left(F_{t\ mdl}^{T_j}(\theta_t;VIX_t,T_j-t,\kappa,\sigma_V)-F_{t\ mkt}^{T_j}\right)^2$$

on each day. The calibrated long-term mean level is presented in Figure 3. As observed from the figure, the value of θ_t stayed at a stable level of around 0.03 for the long period of two years from 2005 to 2007. It moved to a higher level of around 0.06 at the beginning of 2008. It became chaotic in October and November 2008 due to the sharp increase of short-term volatility during the global financial crisis.

To demonstrate how good our model fits to the market data, we perform a comparison in both cross-sectional and time series dimensions. Figure 4 shows the term structure of VIX futures price for a few sample days. Figure 5 depicts the model-fitted VIX future prices and the fixed time-to-maturity VIX futures prices constructed from the market data. Table 7 compares the market prices and the model-fitted prices. We may conclude that our model gives a reasonable fit under normal market situation. RMSE and MAE are relatively small, given the average VIX futures level around 18, the percentage of pricing error is less than 15%. For the four time series, almost 97% of the market prices fall into 95% interval of the model-fitted prices. The t-statistics fail to reject the null hypothesis that the model-fitted prices and market prices have the same mean. But our model slightly overprices VIX futures during the abnormal market in October and November 2008.

4.3. Model prediction of VIX futures price

We now examine the predicting power of our VIX futures pricing model. With two parameters $\kappa = 2.4208$, $\sigma_V = 0.1425$ calibrated from the market prices of three-month variance and VIX futures between May 18, 2004 and November 28, 2008, and θ_t calibrated from the market prices of all traded VIX futures on day *t*, we can compute the model price of VIX futures on the next day, *t+1*, given the VIX level on day *t+1*.

Figure 6 shows the calibrated process of θ_t between November 28, 2008 and February 13, 2009. Figure 7 shows the performance of the fitting exercises on three particular days. It seems to us that the model has some difficulties in fitting sharply-decreasing term structure, e.g., that on December 1, 2008. As a result, the model predicted prices are not impressively close to the market prices in December 2008 as shown in Figure 8. But the model predicted prices are getting much closer to the market price in February 2009 as the term structure becomes relatively flattened. Table 8 compares the model-predicted prices and fixed time-to-maturity VIX future prices constructed from the market data. We find our model has some ability in predicting market prices. The maximum relative RMSE and MAE is less than 5% comparing to the mean VIX futures price. Our model can predict the direction of changes of fixed time-to-maturity VIX futures prices correctly in 79% of times for VIXF30 and VIXF60, and 75% of times for VIXF90 and VIXF120 respectively. Almost 90% of the constructed fixed time-to-maturity VIX futures prices fall into 95% confidence interval of model-predicted prices, although out-of-sample performance is slightly worse than in-the-sample fit. The null hypothesis that the model-predicted prices and market data constructed prices have the same mean fails to be rejected even at 10% significance level.

5. Conclusion

With the enormous increase in derivatives trading and the focus on volatility came the realization that stochastic volatility is an important risk factor affecting pricing and hedging. A new asset class, volatility instruments, is emerging and markets that trade these instruments are created. The first exchange traded instrument is, VIX futures. It has been trading on the CBOE Futures Exchange since March 26 2004.

In this paper, we first study the behaviour of VIX futures prices using the market data from March 26, 2004 to February 13, 2009. We observe three stylized facts:

1. The index, VIX, and the four fixed time-to-maturity VIX futures prices are negatively correlated with the S&P 500 index. VIX and VIX futures with three different maturities are very highly correlated.
2. The term structure of average VIX futures prices is upward sloping.
3. The volatility term structure of VIX futures is downward sloping.

The first observation, which has been coined the ”leverage effect”, has been noted back in the 70s by Fischer Black with regard to volatility computed from stock prices and has several alternative explanations, none of which is fully satisfactory. The second observation is that the long term mean level of volatility is expected to be higher than the short term volatility which is explained by the historically low implied volatilities in 2006-07. This expectation has changed recently following the high volatilities during the recent financial crisis. The third observation indicates that the volatility of volatility is getting lower as we go out further in time. This is consistent with the observations that smaller time intervals contain more noise which shows up in the volatility estimates.

In the second part of the paper we use a simple model of mean-reverting variance process with stochastic long-term mean level to establish the theoretical relationship between VIX futures prices and its underlying spot index. Using the mean-reverting speed, κ, and volatility of variance, σ_V, calibrated with historical data and long-term mean level, θ_t, calibrated with the market data at *t*, we can price VIX futures at time *t+1* conditional on VIX at time *t+1*. An empirical study shows that our model provides prices that are close to the market prices. Our model captures the dynamics of VIX futures price reasonably well.

To sum, our main two contributions are: First, we provide a detailed empirical analysis of the VIX futures market since its inception. Second, we explain fairly well VIX futures prices using a simple stochastic volatility model calibrated to the data.

Appendix

VIX is computed from the option quotes of all available calls and puts on the S&P500 (SPX) with a non-zero bid price (see the CBOE white paper[6]) using following formula

$$\sigma^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T}\left(\frac{F}{K_0} - 1\right)^2, \qquad (1)$$

where the volatility σ times 100 gives the value of the VIX index level. T is the 30 day volatility estimate. In practice options with 30-day maturity might not exist. Thus, the variances of the two near-term options, with at least 8 days left to expiration, are combined to obtain the 30-day variance. F is the implied forward index level derived from the nearest to the money index option prices by using put-call parity. K_i is the strike price of ith out-of-money options, ΔK_i is the interval between two strikes, K_0 is the first strike that is below the forward index level. R is the risk-free rate to expiration. $Q(K_i)$ is the midpoint of the bid-ask spread of each option with strike K_i.

We now briefly review the theory behind equation (1). If we assume that the strike price is distributed continuously from 0 to $+\infty$ and neglect the discretizing error, equation (1) becomes

$$\sigma^2 = \frac{2}{T}\left[e^{RT}\int_0^{K_0} \frac{1}{K^2} p(K)dK + e^{RT}\int_{K_0}^{+\infty} \frac{1}{K^2} c(K)dK\right] + \frac{2}{T}\left[\ln\frac{F}{K_0} - \left(\frac{F}{K_0} - 1\right)\right]. \qquad (2)$$

By construction, K_0 is very close to F, hence $\frac{F}{K_0} - 1$ is very small but always positive. With a Taylor series expansion we obtain

[6] The CBOE white paper, first drafted in 2003, was revised in 2009. The revised version can be retrieved from http://www.cboe.com/micro/vix/vixwhite.pdf

$$\ln\frac{F}{K_0}=\ln\left[1+\left(\frac{F}{K_0}-1\right)\right]=\left(\frac{F}{K_0}-1\right)-\frac{1}{2}\left(\frac{F}{K_0}-1\right)^2+O\left(\frac{F}{K_0}-1\right)^3.$$

By omitting the third order terms, $O\left(\frac{F}{K_0}-1\right)^3$, the last term of equation (2) becomes that of equation (1). Carr and Madan (1998) and Demeterfi et al (1999) show that due to the following mathematical identity,

$$\ln\frac{S_T}{K_0}=\left(\frac{S_T}{K_0}-1\right)-\int_0^{K_0}\frac{1}{K^2}\max(K-S_T,0)dK-\int_{K_0}^{+\infty}\frac{1}{K^2}\max(S_T-K,0)dK,$$

the risk-neutral expectation of the log of the terminal stock price over strike K_0 is

$$E_0^Q\left[\ln\frac{S_T}{K_0}\right]=\left(\frac{F}{K_0}-1\right)-e^{RT}\int_0^{K_0}\frac{1}{K^2}p(K)dK-e^{RT}\int_{K_0}^{+\infty}\frac{1}{K^2}c(K)dK.$$

Hence equation (2) can be written as

$$\begin{aligned}\sigma^2&=\frac{2}{T}\left[\ln\frac{F}{K_0}-E_0^Q\left(\ln\frac{S_T}{K_0}\right)\right]=\frac{2}{T}\left[\ln\frac{F}{S_0}-E_0^Q\left(\ln\frac{S_T}{S_0}\right)\right]\\&=\frac{2}{T}E_0^Q\left[\int_0^T\frac{dS_t}{S_t}-d(\ln S_t)\right]=\frac{1}{T}E_0^Q\int_0^T\sigma_t^2dt,\end{aligned}$$

where the last equal sign is due to Ito's Lemma $d(\ln S_t)=\frac{dS_t}{S_t}-\frac{1}{2}\sigma_t^2dt$, under the assumption that the SPX index follows a diffusion process, $dS_t=\mu S_tdt+\sigma_tS_tdB_t$ with a general stochastic volatility process, σ_t. So VIX^2 represents the 30-day S&P 500 variance swap rate[7].

[7] In practice, the variance swap rate is quoted as volatility instead of variance. It should be noted that the realized variance can be replicated by a portfolio of all out-of-money calls and puts but the VIX index itself cannot be replicated by a portfolio of options because the computation of the VIX involves a square root operation against the price of a portfolio of options and the square root function is nonlinear.

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Table 1: Summary Statistics for the matured VIX Futures Contracts from March 26, 2004 to February 13, 2009

Contract Code	Maturity month	No. of observations	Period covered		VIX Futures prices		Open interest	Volume
			Start	End	Mean	Std	Mean	Mean
K4	May 04	38	03/26/04	05/19/04	18.56	0.92	1166	148
M4	Jun 04	56	03/26/04	06/16/04	18.68	1.43	1256	170
N4	July 04	35	05/24/04	07/14/04	16.82	1.32	1981	192
Q4	Aug 04	100	03/26/04	08/18/04	19.32	1.23	1697	135
U4	Sep 04	42	07/19/04	09/15/04	17.86	2.28	1533	158
V4	Oct0 4	37	08/23/04	10/13/04	15.68	1.41	1259	128
X4	Nov 04	164	03/26/04	11/17/04	19.01	2.56	2794	145
F5	Jan 05	62	10/21/04	01/19/05	14.61	1.16	1210	75
G5	Feb 05	168	06/21/04	02/16/05	17.22	2.92	3092	125
H5	Mar 05	37	01/24/05	03/16/05	12.74	0.79	1181	154
K5	May 05	168	09/20/04	05/18/05	15.65	1.72	2629	157
M5	June 05	61	03/21/05	06/15/05	14.29	1.24	1178	138
Q5	Aug 05	186	11/22/04	08/17/05	14.94	1.54	4427	173
V5	Oct 05	86	06/20/05	10/19/05	14.30	0.54	826	66
X5	Nov 05	188	02/22/05	11/16/05	15.06	0.83	3303	129
Z5	Dec 05	42	10/24/05	12/21/05	12.66	1.26	950	86
F6	Jan 06	39	11/21/05	01/18/06	12.52	0.66	313	53
G6	Feb 06	186	05/23/05	02/15/06	15.07	1.16	3416	146
H6	Mar 06	42	01/23/06	03/22/06	12.57	0.76	693	65
J6	Apr 06	41	02/21/06	04/19/06	12.51	0.40	664	62
K6	May 06	186	08/22/05	05/17/06	14.92	1.57	5657	283
M6	Jun 06	42	04/24/06	06/21/06	14.97	2.28	1950	285
N6	Jul 06	41	05/22/06	07/19/06	15.83	1.29	1325	178
Q6	Aug 06	163	12/22/05	08/16/06	15.14	1.13	10552	470
U6	Sep 06	42	07/24/06	09/20/06	14.08	1.29	3788	412
V6	Oct 06	42	08/21/06	10/18/06	13.50	1.23	8416	701
X6	Nov 06	176	03/09/06	11/15/06	14.85	1.56	15015	608
Z6	Dec 06	64	09/21/06	12/20/06	12.77	1.65	7988	541
F7	Jan 07	58	10/23/06	01/17/07	12.57	0.69	2218	190
G7	Feb 07	236	03/09/06	02/14/07	15.01	1.69	6534	364
H7	Mar 07	102	10/23/06	03/21/07	13.57	1.08	6311	399
J7	Apr 07	121	10/23/06	04/18/07	13.93	0.84	2533	285
K7	May 07	289	03/23/06	05/16/07	15.36	1.25	6046	262
M7	Jun 07	121	11/20/06	06/20/07	14.19	0.60	4248	338

N7	Jul 07	124	01/22/07	07/18/07	14.53	6.59	3413	383
Q7	Aug 07	295	06/21/06	08/22/07	16.16	2.46	6233	562
U7	Sep 07	124	03/26/07	09/19/07	17.71	4.01	4376	506
V7	Oct 07	125	04/23/07	10/17/07	18.16	3.24	6966	608
X7	Nov 07	252	11/21/06	11/21/07	17.53	3.29	12921	845
Z7	Dec 07	149	05/21/07	12/19/07	20.13	3.32	8133	798
F8	Jan 08	124	07/23/07	01/16/08	22.28	2.20	6860	523
G8	Feb 08	252	02/20/07	02/19/08	19.85	4.28	5064	383
H8	Mar 08	123	09/24/07	03/19/08	24.11	2.44	6176	726
J8	Apr 08	122	10/22/07	04/16/08	24.82	1.62	6268	738
K8	May 08	524	04/24/06	05/21/08	19.09	3.56	3712	230
M8	Jun 08	251	06/21/07	06/18/08	22.35	2.65	6142	404
N8	Jul 08	141	12/24/07	07/16/08	24.26	1.51	5009	556
Q8	Aug 08	251	08/23/07	08/20/08	23.23	1.76	5421	421
U8	Sep 08	125	03/24/08	09/17/08	23.57	1.51	6237	698
V8	Oct 08	130	04/21/08	10/22/08	26.95	9.29	5405	812
X8	Nov 08	251	11/23/07	11/19/08	27.49	9.78	7393	513
Z8	Dec 08	231	01/22/08	12/17/08	29.11	1.12	7577	498
F9	Jan 09	190	04/22/08	01/21/09	32.23	1.20	1982	225
	Average				17.63		4404	344

Note: The futures contract code is the expiration month code followed by a digit representing the expiration year. The expiration month codes follow the convention for all commodities futures, which is defined as follows: January-F, February-G, March-H, April-J, May-K, June-M, July-N, August-Q, September-U, October-V, November-X and December-Z. The VIX futures price data between March 26, 2004 and March 23, 2007 has been scaled down to be one-tenth of the original price in order to be consistent with the price after March 26, 2007.

Table 2: The settlement values for VIX futures contracts and VIX

Contract Code	Maturity date	Settlement value	VIX close on previous day	VIX open on maturity day	VIX close on maturity day
K4	05/19/04	18.355	19.33	18.48	18.93
M4	06/16/04	13.997	15.05	14.83	14.79
N4	07/14/04	13.134	14.46	14.9	13.76
Q4	08/18/04	17.489	17.02	17.55	16.23
U4	09/15/04	13.725	13.56	13.88	14.64
V4	10/13/04	13.157	15.05	13.92	15.42
X4	11/17/04	12.844	13.21	13.2	13.21
F5	01/19/05	12.772	12.47	12.47	13.18
G5	02/16/05	11.293	11.27	11.4	11.1
H5	03/16/05	13.626	13.15	13.3	13.49
K5	05/18/05	13.864	14.57	14.11	13.63
M5	06/15/05	11.012	11.79	11.22	11.46
Q5	08/17/05	12.819	13.52	13.35	13.3
V5	10/19/05	15.172	15.33	15.63	13.5
X5	11/16/05	12.306	12.23	12.22	12.26
Z5	12/21/05	10.175	11.19	10.71	10.81
F6	01/18/06	12.615	11.91	12.62	12.25
G6	02/15/06	12.043	12.25	12.43	12.31
H6	03/22/06	11.145	11.62	11.71	11.21
J6	04/19/06	11.941	11.4	11.52	11.32
K6	05/17/06	14.025	13.35	13.83	16.26
M6	06/21/06	17.285	16.69	16.67	15.52
N6	07/19/06	17.005	17.74	17.62	17.55
Q6	08/16/06	12.285	13.42	12.69	12.41
U6	09/20/06	11.289	11.98	11.75	11.39
V6	10/18/06	11.434	11.73	11.44	11.34
X6	11/15/06	10.268	10.5	10.47	10.31
Z6	12/20/06	10.053	10.3	10.3	10.26
F7	01/17/07	10.706	10.74	10.9	10.59
G7	02/14/07	9.954	10.34	10.19	10.23
H7	03/21/07	12.983	13.27	13.27	12.19
J7	04/18/07	12.03	12.14	12.48	12.42
K7	05/16/07	13.63	14.01	14.02	13.5
M7	06/20/07	13.01	12.85	12.77	14.67
N7	07/18/07	16.87	15.63	16.38	16
Q7	08/22/07	25.05	25.25	24.33	22.89
U7	09/19/07	20.29	20.35	19.96	20.03

V7	10/17/07	18.33	20.02	18.76	18.54
X7	11/21/07	26.7	24.88	26.3	26.84
Z7	12/19/07	22.08	22.64	22.62	21.68
F8	01/16/08	24.18	23.34	23.9	24.38
G8	02/19/08	25.51	25.02	25.39	25.59
H8	03/19/08	25.67	25.79	25.78	29.84
J8	04/16/08	21.78	22.78	22.03	20.53
K8	05/21/08	17.16	17.58	17.64	18.59
M8	06/18/08	21.54	21.13	21.67	22.24
N8	07/16/08	28.4	28.54	28.19	25.1
Q8	08/20/08	20.83	21.28	21.3	20.42
U8	09/17/08	31.54	30.3	31.96	36.22
V8	10/22/08	63.04	53.11	63.12	69.65
X8	11/19/08	67.22	67.64	68.46	74.26
Z8	12/17/08	51.29	52.37	52	49.84
F9	01/21/09	49.88	56.65	51.52	46.42
	Average	19.18	19.32	19.42	19.52

Note: The futures contract code is the expiration month code followed by a digit representing the expiration year. The expiration month codes follow the convention for all commodities futures, which is defined as follows: January-F, February-G, March-H, April-J, May-K, June-M, July-N, August-Q, September-U, October-V, November-X and December-Z. The VIX futures price data between March 26, 2004 and March 23, 2007 has been scaled down to be one-tenth of the original price in order to be consistent with the price after March 26, 2007.

Table 3: Summary statistics of levels and returns of the S&P 500 index, VIX and four fixed time-to-maturity VIX futures based on the market data from March 26, 2004 and February 13, 2009. The fixed time-to-maturity VIX futures prices are constructed by using the market data of available contracts with a linear interpolation technique. The return (daily continuously compounded) is defined as the logarithm of the ratio between the price on next day and the price on current day.

Panel A: Summary statistics of levels

	SPX	VIX	VIXF30	VIXF60	VIXF90	VIXF120
Mean	1265.22	18.99	19.15	19.36	19.45	19.50
Std Dev	162.78	11.66	9.62	8.51	7.68	7.04
Median	1266.74	14.74	15.23	15.56	15.88	16.12
Skewness	-0.54	2.67	2.35	2.30	2.18	2.05
Kurtosis	0.32	7.52	5.53	5.56	4.95	4.27
Minimum	752.44	9.89	11.26	12.23	12.52	12.85
Maximum	1565.15	80.86	65.68	60.14	54.67	50.33

Panel B: Summary statistics of returns

	SPX	VIX	VIXF30	VIXF60	VIXF90	VIX120
Mean	-0.0002	0.0007	0.0007	0.0006	0.0005	0.0005
Std Dev	0.0140	0.0664	0.0318	0.0243	0.0213	0.0193
Median	0.0007	-0.0046	-0.0012	-0.0004	-0.0006	-0.0008
Skewness	-0.3895	0.6134	0.5091	0.3847	0.3643	0.3776
Kurtosis	13.9505	4.7917	4.6030	3.3449	3.7196	4.2508
Minimum	-0.0947	-0.2999	-0.1800	-0.1268	-0.1166	-0.1078
Maximum	0.1096	0.4960	0.2282	0.1260	0.1079	0.1021

Note: SPX stands for S&P 500 index. VIXF30, VIXF60, VIXF90 and VIXF120 stand for the prices of 30-, 60-, 90-, and 120-day-to-maturity VIX futures respectively. The VIX futures price data between March 26, 2004 and March 23, 2007 has been scaled down to be one-tenth of the original price in order to be consistent with the price after March 26, 2007.

Table 4: The correlation matrix between levels and returns of S&P 500 index, VIX and four fixed time-to-maturity VIX futures computed based on the market data from March 26, 2004 to February 13, 2009. The fixed time-to-maturity VIX futures prices are constructed by using the market data of available contracts with a linear interpolation technique. The return (daily continuously compounded) is defined as the logarithm of the ratio between the price on next day and the price on current day.

Panel A: The correlation matrix between levels

	SPX	VIX	VIXF30	VIXF60	VIXF90	VIX120
SPX	1					
VIX	-0.5147	1				
VIXF30	-0.5206	0.9744	1			
VIXF60	-0.5209	0.9457	0.9932	1		
VIXF90	-0.5221	0.9390	0.9898	0.9988	1	
VIX120	-0.5217	0.9322	0.9856	0.9962	0.9989	1

Panel B: The correlation matrix between returns

	SPX	VIX	VIXF30	VIXF60	VIXF90	VIX120
SPX	1					
VIX	-0.7415	1				
VIXF30	-0.7716	0.8673	1			
VIXF60	-0.7656	0.8211	0.9472	1		
VIXF90	-0.7628	0.7992	0.9073	0.9635	1	
VIX120	-0.7631	0.7787	0.8816	0.9262	0.9728	1

Note: SPX stands for S&P 500 index. VIXF30, VIXF60, VIXF90 and VIXF120 stand for the prices of 30-, 60-, 90-, and 120-day-to-maturity VIX futures respectively. The VIX futures price data between March 26, 2004 and March 23, 2007 has been scaled down to be one-tenth of the original price in order to be consistent with the price after March 26, 2007.

Table 5: The OLS regression estimates of fixed time-to-maturity VIX futures and VIX prices. The estimates are obtained by running following two regressions:

$$F_t^T = \alpha + \beta VIX_t + \varepsilon_t,$$
$$F_t^T = \alpha + \beta_1 VIX_t + \beta_2 VIX_t^2 + \varepsilon_t$$

with fixed time-to-maturity, *T*-*t*. The dependent variable is the fixed time-to-maturity VIX futures price, while the independent variable is the corresponding VIX and the squared VIX. The sample period is from March 26, 2004 to February 13, 2009, a total of 1,231 trading days. The Newey and West standard errors are reported in parentheses. The default lags are 12 days, beyond which the sample autocorrelation is insignificant.

Dependent Variable	Independent Variables			
	Constant	VIX	VIX^2	R^2
VIXF30	3.8849	0.8038		0.9495
	(0.7444)*	(0.047)*		
	0.04508	1.1291	-0.0047	0.96
	(0.9578)	(0.0857)*	(0.00136)*	
VIXF60	6.2445	0.6904		0.894
	(1.005)*	(0.0632)*		
	1.3385	1.1061	-0.006	0.9164
	(1.229)	(0.1108)*	(0.0019)*	
VIXF90	7.697	0.6189		0.8817
	(0.9057)*	(0.0567)*		
	2.771	1.036	-0.0061	0.9089
	(1.098)*	(0.098)*	(0.0016)*	
VIXF120	8.7983	0.5632		0.8689
	(0.8747)*	(0.0519)*		
	3.64	1.003	-0.0063	0.9044
	(0.9796)*	(0.0858)*	(0.00137)*	

*significant at 5% level

Table 6: The shape of the term structure of VIX futures price based on the market data from March 26, 2004 to February 13, 2009.

	Observations (days)	Percentage
Upward sloping	832	67.6%
Humped	227	18.4%
Downward sloping	172	14.0%

Table 7: In-the-sample performance of the model-fitted VIX futures price. This Table reports statistical efficiency of the model-fitted VIX futures price. The mean squared error (MSE), root mean squared error (RMSE) and the mean absolute error (MAE) are reported. The *p*-value is for the null hypothesis that the model-fitted futures prices and the constructed market prices with constant time-to-maturity have equal mean. The percentage of violation reports the percentage of the observations of constructed VIX market prices that fall outside the 95% confidence interval of model-predicted price. The sample period is from May 18, 2004 to November 28, 2008.

	VIXF30	VIXF60	VIXF90	VIXF120
MSE	3.937	5.194	3.179	1.656
RMSE	1.984	2.279	1.783	1.287
MAE	0.812	0.867	0.663	0.465
p-value	0.455	0.402	0.389	0.315
Violation of 95% confidence interval	3.33%	2.97%	2.97%	2.97%

Note: VIXF30, VIXF60, VIXF90 and VIXF120 stand for the prices of 30-, 60-, 90-, and 120-day-to-maturity VIX futures respectively. The VIX futures price data between March 26, 2004 and March 23, 2007 has been scaled down to be one-tenth of the original price in order to be consistent with the price after March 26, 2007.

Table 8: Out-of-the-sample performance of the model-predicted VIX futures price. This Table reports 1-day ahead forecast ability of the model-predicted VIX futures price during the recent financial crisis. The sample period is from November 28, 2008 to February 13, 2009. The mean squared error (MSE), root mean squared error (RMSE) and the mean absolute error (MAE) are reported. MCP is the mean correct prediction of the direction, which is the percentage of times while the model-predicted future price changes have the same sign as the realized future price changes. The *p*-value is for the null hypothesis that the model-predicted futures prices and the constructed market prices with constant time-to-maturity have equal mean. The percentage of violation reports the percentage of the observations of constructed VIX market prices that fall outside the 95% confidence interval of model-predicted price.

	VIXF30	VIXF60	VIXF90	VIXF120
MSE	6.922	5.502	1.997	1.612
RMSE	2.631	2.346	1.413	1.270
MAE	2.132	1.902	1.102	0.909
MCP	78.85%	78.85%	75%	75%
p-value	0.1445	0.1506	0.7759	0.5765
Violation of 95% confidence interval	3.85%	5.77%	0%	11.54%

Note: VIXF30, VIXF60, VIXF90 and VIXF120 stand for the prices of 30-, 60-, 90-, and 120-day-to-maturity VIX futures respectively. The VIX futures price data between March 26, 2004 and March 23, 2007 has been scaled down to be one-tenth of the original price in order to be consistent with the price after March 26, 2007.

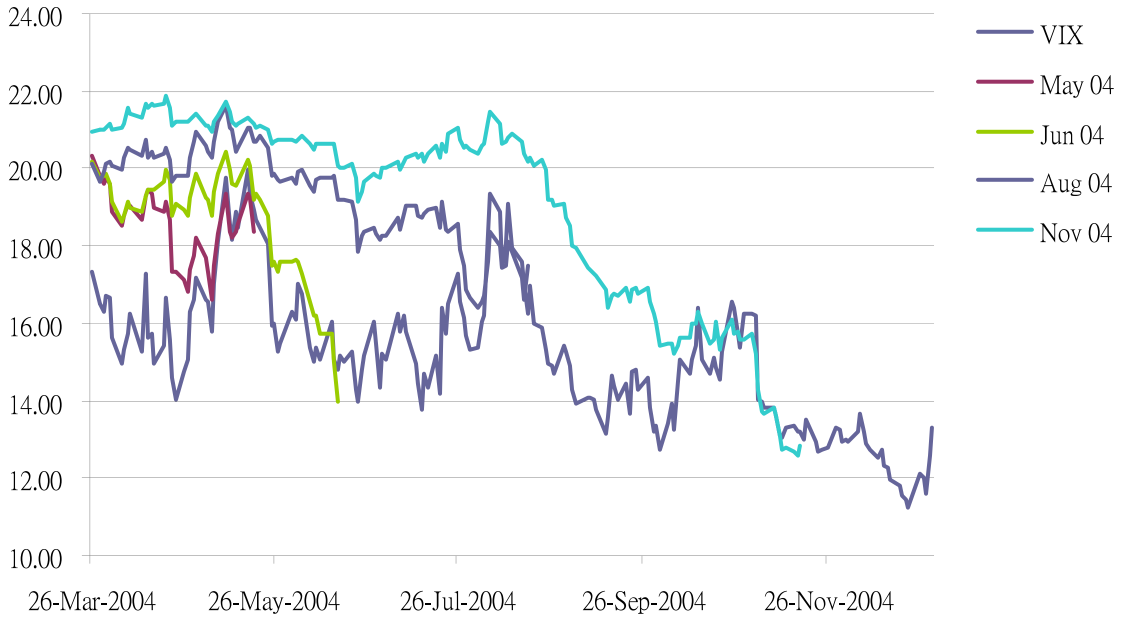


Figure 1a. The price pattern of VIX and four VIX futures contracts: May 04, Jun 04, Aug 04 and Nov 04 between March 26, 2004 and December 31, 2004.

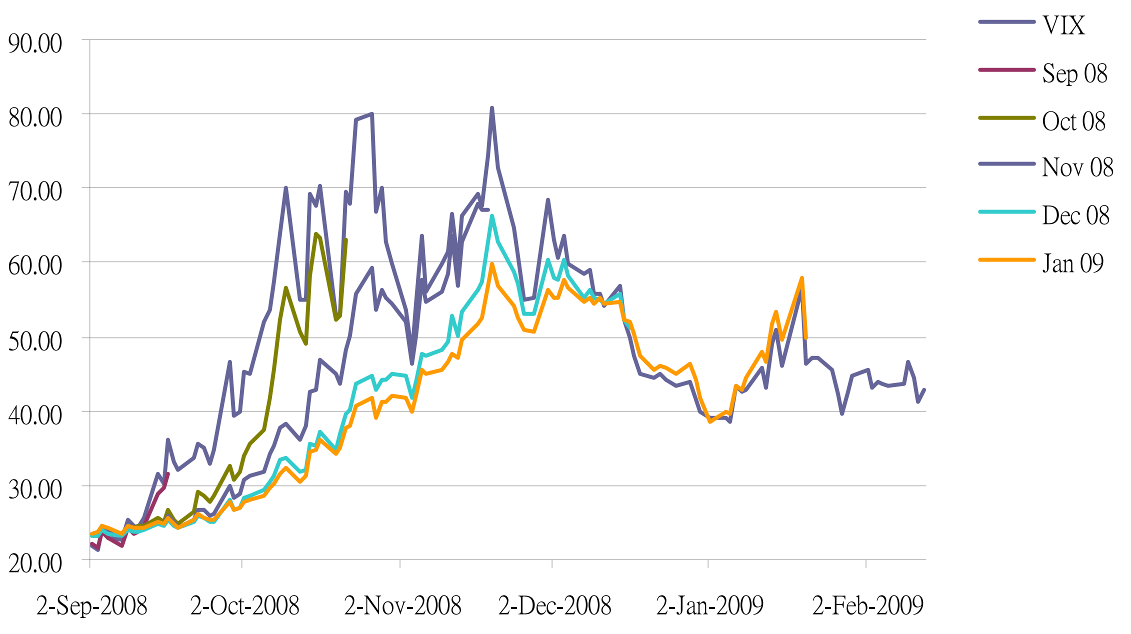


Figure 1b. The price pattern of VIX and five VIX futures contracts: Sep 08, Oct 08, Nov 08, Dec 08 and Jan 09 during the period of gloabal financial crisis between September 2, 2008 and February 13, 2009.

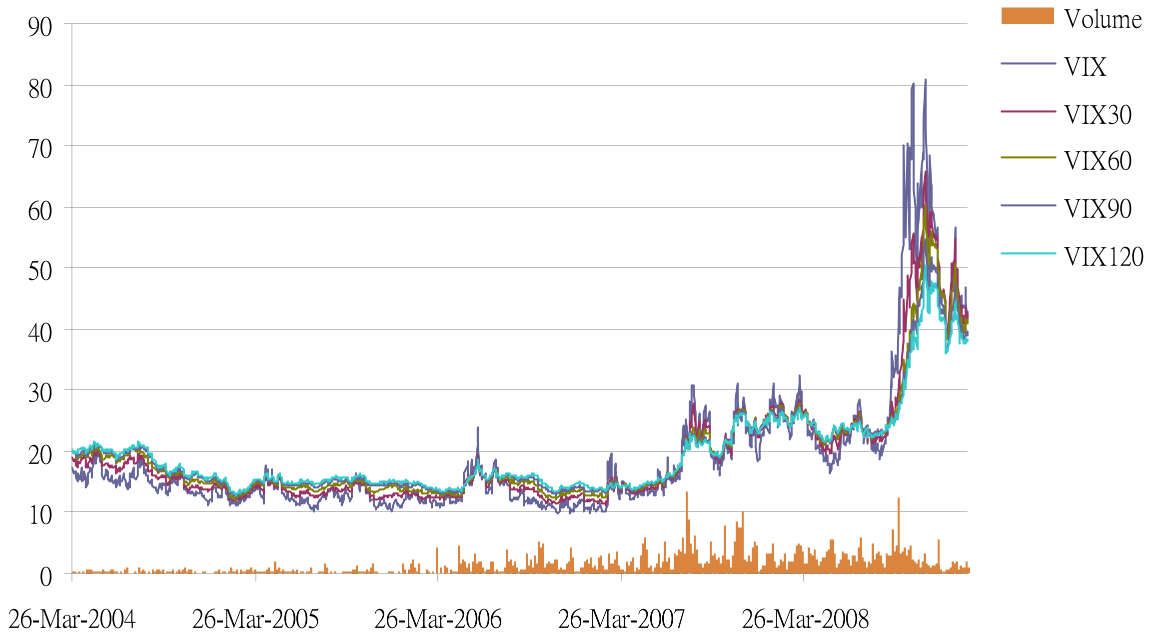
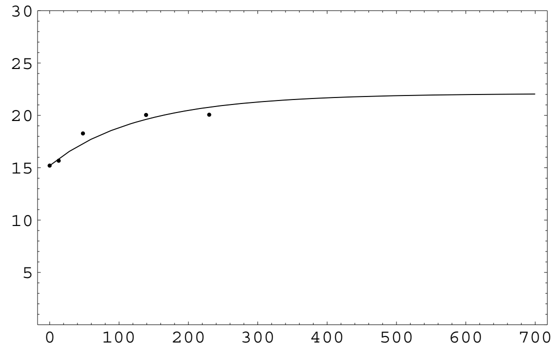
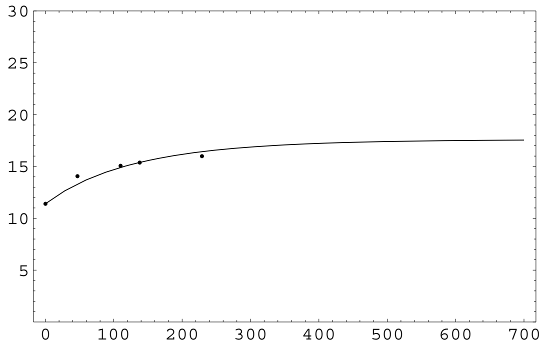


Figure 2. VIX and VIX futures prices with four fixed time-to-maturities between March 26, 2004 and February 13, 2009. The VIX time series is from the CBOE. The fixed maturity VIX futures prices are constructed by using the market data of available contracts with a linear interpolation technique. The bar chart shows the trading volume (normalized by 2,000 contracts) of futures of all maturities on each day.

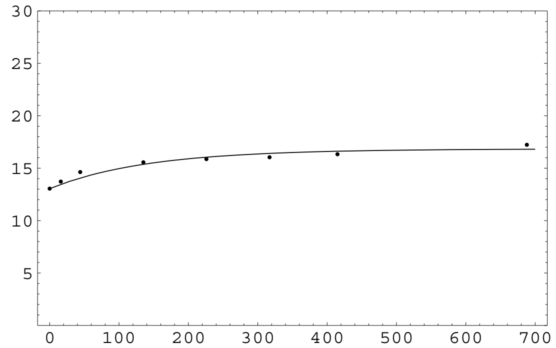
Figure 3. The process of long term mean level of variance, θ_t, calibrated daily from the market prices of VIX futures between May 18, 2004 and November 28, 2008 with $\kappa = 2.4208$ and $\sigma_V = 0.1425$.



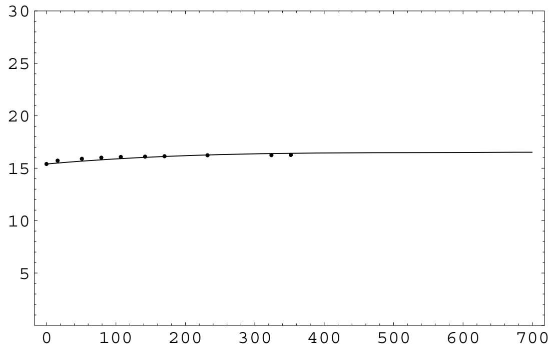
July 1, 2004, $VIX_t = 15.20$, $\theta_t = 0.04961$



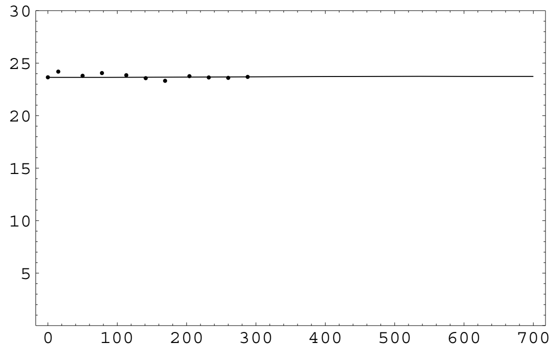
July 1, 2005, $VIX_t = 11.40$, $\theta_t = 0.03171$



July 3, 2006, $VIX_t = 13.05$, $\theta_t = 0.02911$



July 2, 2007, $VIX_t = 15.40$, $\theta_t = 0.02807$



July 1, 2008, $VIX_t = 23.65$, $\theta_t = 0.05716$

Figure 4. The term structures of VIX futures price on the first trading day in July in the five years from 2004-08. The solid line is the model-fitted price with $\kappa = 2.4208$, $\sigma_V = 0.1425$ and daily fitted θ_t. The dots are the market price.

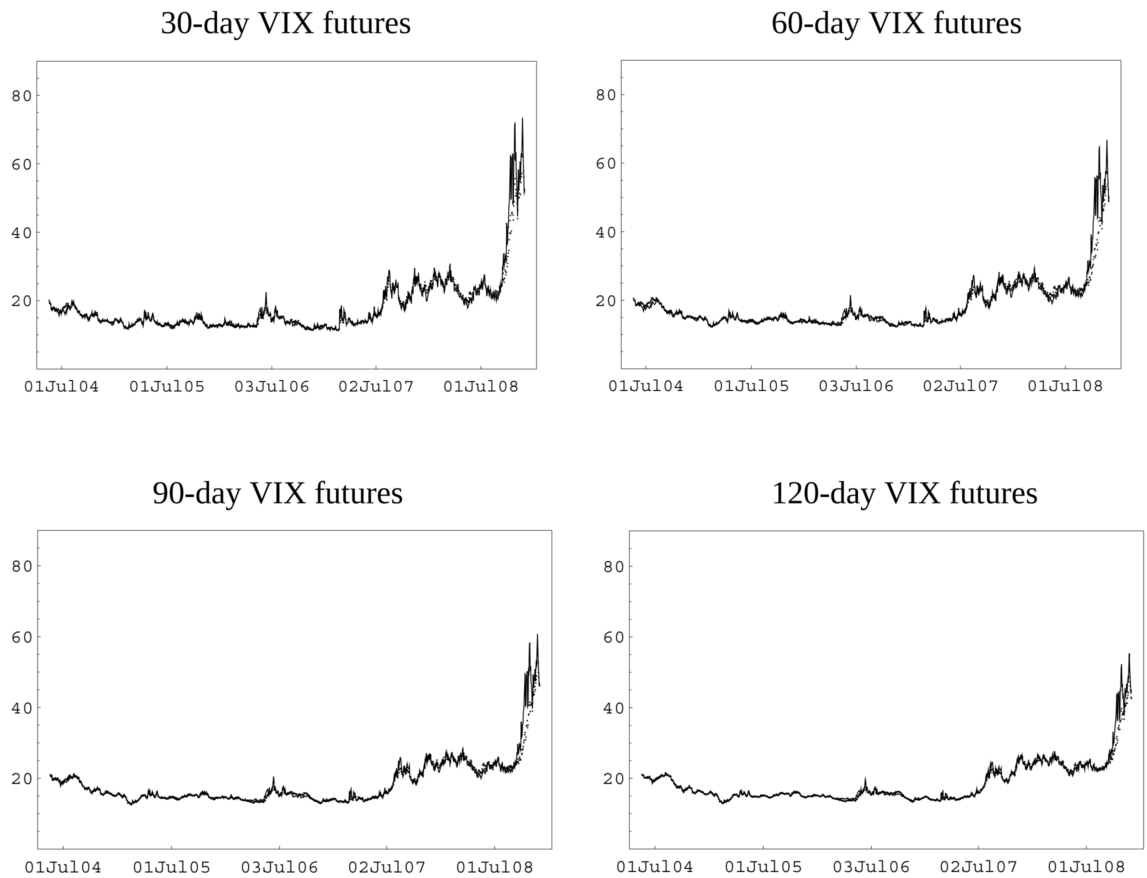


Figure 5. The time series of model-fitted prices and constructed prices of 30-, 60-, 90- and 120-day VIX futures between May 18, 2004 and November 28, 2008. The solid line is the model-fitted price with $\kappa = 2.4208$, $\sigma_V = 0.1425$ and daily fitted θ_t. The dots are the constructed prices that are computed by using the market data of available contracts with a linear interpolation technique. The average prices of 30-, 60-, 90- and 120-day VIX futures are (17.88, 18.15, 18.36, 18.48). The roots of mean squared error between model-fitted prices and constructed prices are (1.984, 2.279, 1.783, 1.287).

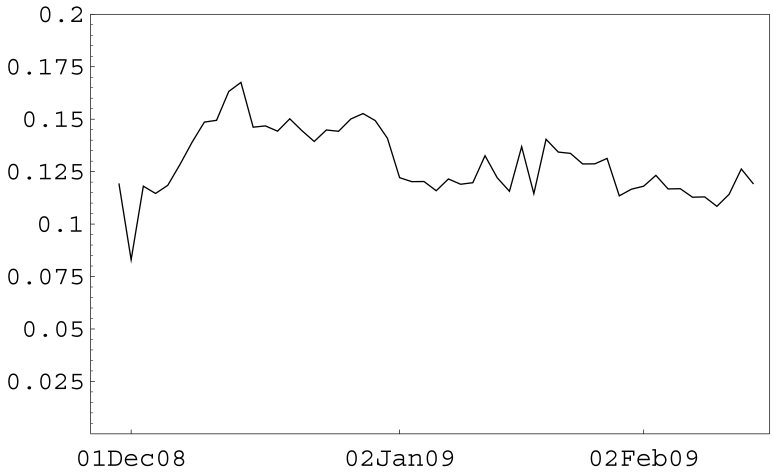
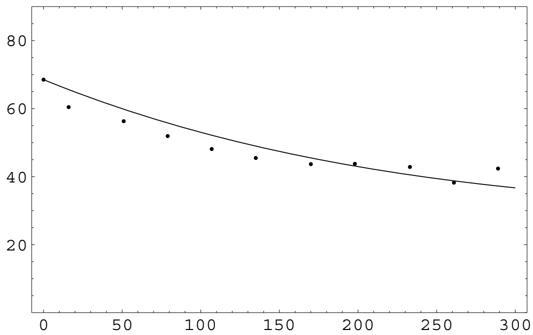
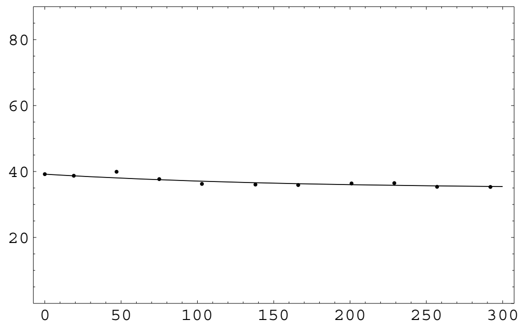


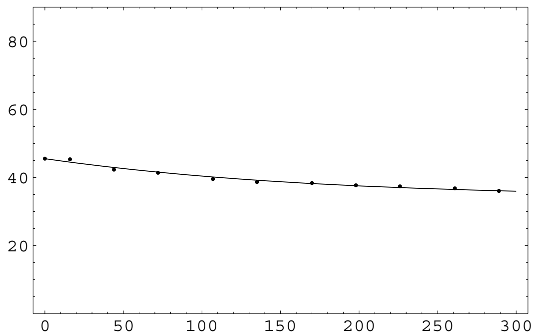
Figure 6. The process of long term mean level of variance, θ_t, calibrated daily from the market prices of VIX futures between November 28, 2008 and February 13, 2009 with $\kappa = 2.4208$ and $\sigma_V = 0.1425$. The average θ_t is 0.1294.



Dec. 1, 2008, $VIX_t = 68.51$, $\theta_t = 0.08300$



Jan. 2, 2009, $VIX_t = 39.19$, $\theta_t = 0.1221$



Feb. 2, 2009, $VIX_t = 45.52$, $\theta_t = 0.1181$

Figure 7. The term structures of VIX futures price on the first trading day in December 2008, January and February 2009. The solid line is the model-fitted price with $\kappa = 2.4208$, $\sigma_V = 0.1425$ and daily fitted θ_t. The dots are the market price.

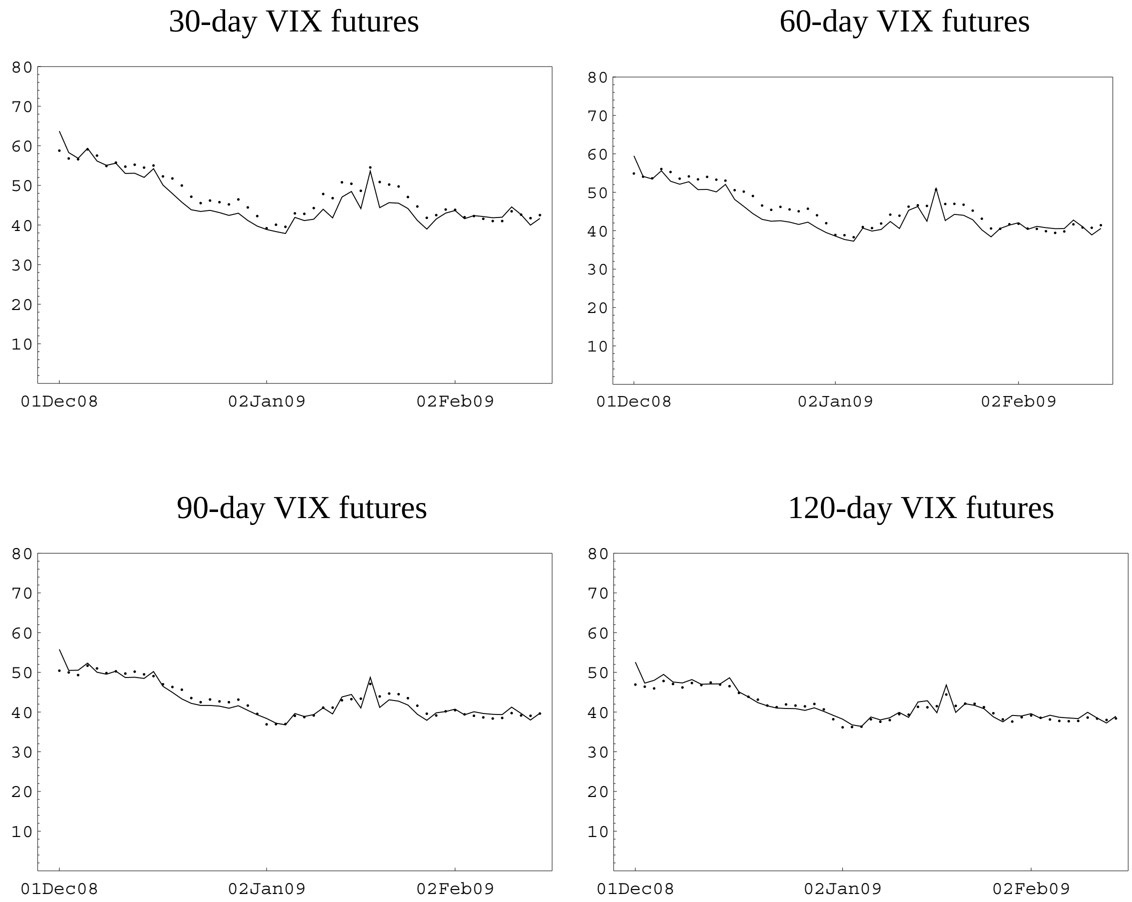


Figure 8. The time series of model-predicted prices and constructed prices of 30-, 60-, 90- and 120-day VIX futures between November 28, 2008 and February 13, 2009. The solid line is the model-predicted price with $\kappa = 2.4208$, $\sigma_V = 0.1425$ and daily fitted θ_t. The dots are the constructed prices that are computed by using the market data of available contracts with a linear interpolation technique. The average prices of 30-, 60-, 90- and 120-day VIX futures are (47.62, 45.80, 43.16, 41.33). The roots of mean squared error between model-fitted prices and constructed prices are (2.631, 2.346, 1.413, 1.270).