High Frequency Asymptotics for the Limit Order Book

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Abstract

We study the one-sided limit order book corresponding to limit sell orders and model it as a measure-valued process. Limit orders arrive to the book according to a Poisson process and are placed on the book according to a distribution which varies depending on the current best price. Market orders to buy periodically arrive to the book according to a second, independent Poisson process and remove from the book the order corresponding to the current best price. We consider the above described limit order book in a high frequency regime in which the rate of incoming limit and market orders is large and traders place their limit sell orders close to the current best price. Our first set of results provide weak limits for the unscaled price process and the properly scaled measure-valued limit order book process in the high frequency regime. In particular, we characterize the limiting measure-valued limit order book process as the solution to a measure-valued stochastic differential equation. We then provide an analysis of both the transient and long-run behavior of the limiting limit order book process.

1 Introduction

The limit order book has become one of the standard mechanisms for trading in modern electronic markets due to its ability to provide a simple and transparent set of rules for the matching of buyers and sellers of securities. Worldwide, according to Russell and Kim [22], order driven markets now account for almost half of the current stock exchanges. It is therefore of a practical as well as a theoretical interest to better understand how both the structure and the fundamental parameters of the order book affect the dynamics of price formation.

In the present paper, we study the dynamics of the order book over a time scale which is an order of magnitude larger than that of individual order driven events. We refer to this time scale as the high frequency regime and a mathematically precise definition of this regime is given in Definition 3.1 of Section 3 below. Roughly speaking, the high frequency regime which we consider is such that the arrival rate of both limit orders and market orders to the book is large and such that these arrival rates are closely matched to one another. Moreover, we also assume that traders place their individual limit orders in close proximity to the prevailing best price. This regime bears a close resemblance to the conventional heavy traffic regime commonly encountered in the queueing theory literature and indeed many of the tools and techniques of the present paper have been heavily influenced by the theory of queues.

As it turns out, several authors in the past have also been inspired to borrow techniques from queueing theory in order to apply them to studying the limit order book. Some of the earliest work in this regard is that of Mendelson [20] who models a market clearing house assuming Poisson arrivals of buy and sell ordres. Cohen et al. [6] considers a continuous time auction where buyers may only place limit orders at two prices, the best bid and the best ask. This work is then extended in Domowitz and Wang [11] to multiple price levels. Smith et al. [24] provide a mean field theory of the limit order book. Recently, Cont et al. [9] have modeled the limit order book as a finite-dimensional continuous time Markov chain. They then show how to use Laplace transform methods in order to calculate several quantities associated with the book such as the direction of the next price move and the probability of executing an order before a change in price. Cont and de Larrard [7] consider a simplified Markovian model of the limit order book in which several key quantities may be computed directly in closed form. A very closely related paper is that of Simatos [23] which elegantly treats a limit order book model similar to our own. In particular, in Simatos [23] a coupling is constructed between the limit order book process and a certain branching random walk. This coupling is then taken advantage of in order to derive necessary and sufficient conditions for the price process to drift to either 0 or $+\infty$. In [19], Maglaras et al. study a fragmented market in which traders may route their orders to one of several exchanges. Interestingly, it is shown in Maglaras et al. [19] that on a fluid scale a state-state collapse occurs between the states of the market in each of the various exchanges. Finally, we mention the work of Avellaneda et al. [2] which studies the effect of hidden liquidity on the future direction of price movement.

Unfortunately, one difficulty commonly encountered in modeling the limit order book is the high dimensionality of its state space. Multiple traders may place their limit orders at any price they wish and as a result in order to keep track of each order on the book at each moment in time one must resort to an infinite-dimensional state space. In the present paper, we confront this difficulty head on by modeling the order book as a measure-valued process. Although our approach is limited in the present paper to modeling only one side of the book, which we take to be the sell side (the buy side may treated analogously), we believe that in the future our approach may be successfully extended to a double-sided book.

We now provide a succinct description of the general setup of our model. A more precise description is given in Section 2. At each point in time $t \geq 0$, we let $\mu(t) \in \mathcal{M}_F(\mathbb{R})$ represent the state of the order book at time t, where $\mathcal{M}_F(\mathbb{R})$ denotes the set of finite, non-negative measures on $\mathbb{R}$. Specifically, for each $\mathcal{A} \in B(\mathbb{R})$, we let $\mu(t)(\mathcal{A})$ be the number of limit sell orders on the book at time t whose associated prices lie in the set $\mathcal{A}$. Then, since we are only considering the sell side of the book, the best price at each point in time is given by the lowest price on the book associated with a limit sell order. Mathematically speaking, the best price at time $t \geq 0$ is defined by setting

$$p(t) \quad = \quad \sup\{x \geq 0 : \mu(t)((-\infty, x)) = 0\}.$$

We next assume that market buy orders arrive to the book according to a Poisson process and upon arrival remove from the book the order corresponding to current best price. On the other hand, limit sell orders also arrive to the book according to a second, independent Poisson process. A limit sell order arriving to the book at time $t \geq 0$ will place its order at the price $X \cdot p(t-)$, where X is a positive random variable with distribution G. Finally, we assume the existence of a market maker who never allows the total number of orders on the book to drop below the level $a > 0$.

Our first set of results for the model described above provide limit theorems on the behavior of the unscaled price process as well as the properly scaled order book process in the high frequency regime. These results may be found in Sections 4 and 5, respectively, and are obtained under the crucial assumption that $E[\ln X] > 0$. Note that since we are dealing with the sell side of the book, this assumption implies that traders would on average prefer to sell at a higher price than the current best price. This then places upward pressure on the price and, indeed, as is shown is Theorem 5.1 below, the limiting price process which we obtain is an increasing process. One of the more interesting features of our limiting price process is that its rate of increase is inversely related to the total number of order on the book. This result is in line with the recent empirical findings of Cont et al. [8] which studied the relationship between order flow imbalance and changes in price. In particular, in Cont et al. [8] it is empirically observed that over short intervals of time, changes in price are linear with respect to changes in net order flow and inversely proportional with respect to market depth. Similarly, Hasbrouck and Saar [14] have observed a statistically significant negative relationship between price volatility and order book depth in the vicinity of the best price. In Theorem 6.1 below, we provide a limit theorem for the properly scaled measure-valued order book process. In particular, we show that in the limit of the high frequency regime, the majority of the mass of the order book is concentrated at the prevailing best price. Theorem 6.1 may be viewed as providing a link between the limiting price process itself and the limiting distribution of the orders on the book.

Our second set of results provide a transient as well as a long-term analysis of the limiting price process and the limiting properly scaled order book process. Our transient analysis may be found in Section 7 and essentially relies upon a modification of the Feynman-Kac formula to account for semi-martingales with a singular component to them. In particular, in Section 7 we show how to calculate the expectation of various functionals of the limiting price process and the limiting properly scaled order book process at each point in time. In Section 8, we conduct our long term analysis. As it turns out, the long term behavior of the the limiting price process and the limiting properly scaled order book process depends critically upon whether the incoming rate of limit orders is less than, greater than, or equal to the incoming rate of market orders. In Section 8, we provide the almost sure long-term asymptotics of our limiting processes in each of these three cases. The proofs of many of our results of the paper may be found in the Appendix.

1.1 Notation

The following notation will be used throughout this paper. We assume that all random variables are defined on a common probability space $(\Omega, \mathcal{F}, P)$. Letting $\mathcal{S}$ be a separable and complete metric space, we define $D([0,\infty), \mathcal{S})$ to be the Skorokhod space of all functions with domain $[0,\infty)$ that are right-continuous on $[0,\infty)$, with left limits everywhere in $(0,\infty)$, and taking values in $\mathcal{S}$. We equip $D([0,\infty), \mathcal{S})$ with the standard Skorokhod J_1-topology (see, for instance, Chapter 16 of Billingsley [4]) and its associated Borel σ-algebra. All stochastic processes in this paper are viewed as measurable maps from $(\Omega, \mathcal{F}, P)$ to $D([0,\infty), \mathcal{S})$, with $\mathcal{S}$ being set equal to either $\mathbb{R}$ or $\mathcal{M}_F(\mathbb{R})$, the set of all finite, non-negative measures on $\mathbb{R}$. The product of a finite number of metric spaces is endowed with the product topology. We endow $\mathcal{M}_F(\mathbb{R})$ with the topology of weak convergence. That is, a sequence of elements $\{\nu_n, n \geq 1\}$ in $\mathcal{M}_F(\mathbb{R})$ weakly converges to an element $\nu \in \mathcal{M}_F(\mathbb{R})$ if and only if $\langle \nu_n, \varphi \rangle \to \langle \nu, \varphi \rangle$ as $n \to \infty$, for each $\varphi \in C_b(\mathbb{R})$, where here we have adopted the notation

$$\langle \nu, \varphi \rangle = \int_{\mathbb{R}} \varphi(u) d\nu(u).$$

For each $\xi, \nu \in \mathcal{M}_F(\mathbb{R})$, we recall from (6.11) of Billingsley [4], the definition of the Prohorov metric π,

$$\pi(\xi,\nu) \quad = \quad \inf\{\varepsilon > 0 | \xi(\mathcal{A}) \leq \nu(\mathcal{A}^\varepsilon) + \varepsilon \text{ and } \nu(\mathcal{A}) \leq \xi(\mathcal{A}^\varepsilon) + \varepsilon \text{ for all } \mathcal{A} \in \mathcal{B}(\mathbb{R})\}.$$

We also recall the well known fact that $(\mathcal{M}_F(\mathbb{R}), \pi)$ is a separable and complete metric space and moreover (see, for instance, Theorem 6.8 of Billingsley [4]), that π induces the topology of weak convergence on $\mathcal{M}_F(\mathbb{R})$. The main results in this paper are concerned with the weak convergence of $\mathbb{R}$ or $\mathcal{M}_F(\mathbb{R})$ valued, RCLL (right continuous with left limits) processes. Finally, we denote by $e = \{t, t \geq 0\}$ the identity process on $\mathbb{R}_+$.

2 The Model

In this section, we provide the model which we consider for the remainder of the paper. As mentioned in the Introduction, our approach is to model the limit order book as a measure-valued process. For simplicity, we have chosen to only model one side of the book, which we take to be the sell side of the book. The case of the buy side of the book may be treated analogously. The details of our model are as follows.

We assume that limit sell orders arrive to the book according to a Poisson process $N_L = \{N_L(t), t \geq 0\}$ with rate λ_L. For each $i \geq 1$, we denote by

$$\tau_i^L \quad = \quad \inf\{t \geq 0 : N_L(t) \geq i\}$$

the time at which the ith limit sell order arrives to the book. Each time that a limit sell order arrives to the book, it places a single order on the book. The price associated with this order is a random quantity which may be greater than, less than or equal to the current best price. In particular, for each $t \geq 0$, let $p(t)$ denote the current best price. $p(t)$ represents the smallest price associated with a limit sell order which is present on the book at time t. Then, a limit sell order arriving to the book at time t will place its order at the price $X \cdot p(t-)$, where X is a strictly positive random variable with distribution G. Beyond requiring that X be a strictly positive random variable, the only other assumption which we place on the distribution G is that there exists a constant $1 < C < \infty$ such that

$$P(1/C < X < C) \quad = \quad 1. \tag{1}$$

Figures 1 through 3 illustrate the three possible transitions of the order book when a limit sell order arrives. They correspond, respectively, to the cases of $X > 1, X < 1$ and $X = 1$.

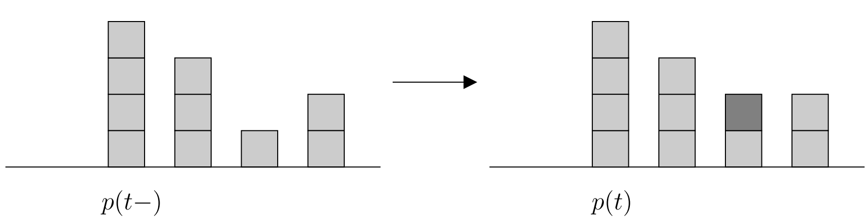

Figure 1: A limit sell order is placed on the book above the best price. The price does not change.

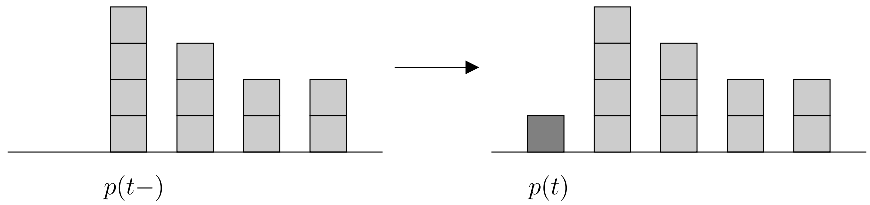

Figure 2: A limit sell order is placed on the book below the best price. The price decreases.

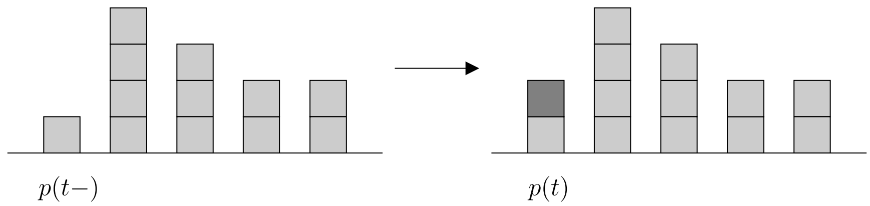

Figure 3: A limit sell order is placed on the book at the best price. The price does not change.

We next assume that market buy orders arrive to the book according to second Poisson process $N_M = \{N_M(t), t \geq 0\}$ with rate λ_M. We assume, for convenience, that N_M is independent of N_L. For each $i \geq 1$, we denote by

$$\tau_i^M \quad = \quad \inf\{t \geq 0 : N_M(t) \geq i\}$$

the time at which the the ith market buy order arrives to the book. Each time that a market buy order arrives to the book, it removes a single limit sell order corresponding the current best price. Figures 4 and 5 illustrate the two possible transitions of the order book when a market buy order arrives. The price may either increase or stay the same.

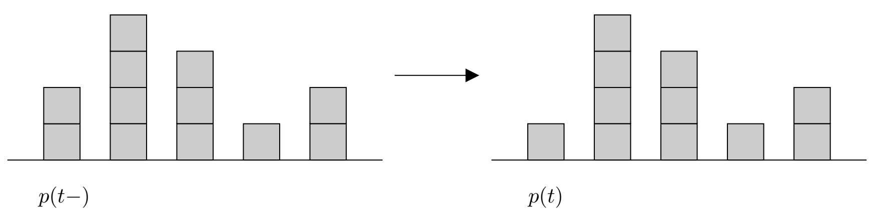

Figure 4: A market buy order arrives to the book and removes one of the limit sell orders corresponding to the best price. Since there were more than one limit sell orders at the best price, the price does not change.

We also assume the existence of a "market maker". The role of the market maker in our model is to never allow the total number of orders on the book to drop below some predetermined level $a > 0$. In particular, suppose that the total number of orders on the book is at the level a and that a market buy order arrives and removes one of the limit sell orders corresponding to the current best price. The market maker will then instantaneously place a new limit sell order on the book at

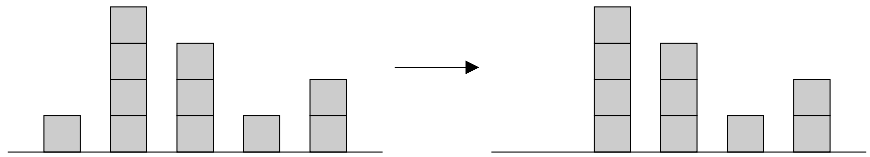

Figure 5: A market buy order arrives to the book and removes the limit sell order corresponding to the best price. This was the last remaining limit sell order at the best price and so the price increases.

the same price at which the previous limit sell order was removed. Thus, the total number of orders on the book will never drop below the level a. We let $L = \{L(t), t \geq 0\}$ denote the non-decreasing process which keeps track of the points in time at which the market maker places limit sell orders on the book. In other words, $L(t)$ represents the cumulative number of limit orders that the market maker has placed on the book by time t.

Before proceeding with a discussion of the full dynamics of the measure-valued order book, we next provide the details of the simpler process which keeps track of the total number of orders on the book. For each $t \geq 0$, let $S(t)$ denote the total number of orders on the book at time t and let $S = \{S(t), t \geq 0\}$ be the process keeping track of the total number of orders on the book. Then, according to the discussion in the preceding three paragraphs, it is straightforward to see that one has the identity

$$S(t) = S(0) + N_L(t) - N_M(t) + L(t), \quad t \geq 0, \tag{2}$$

where $S(0)$ denotes the total number of orders on the book at time 0. Moreover, using results from the standard theory of reflected random walks (see, for instance, Chapter 2 of Asmussen [1]), one may express L in terms of $S(0), N_L, N_M$ and a. In particular, we have that

$$L(t) \quad = \quad \sup_{0 \leq s \leq t} \{-(S(0) + N_L(s) - N_M(s) - a)\} \vee 0, \quad t \geq 0, \tag{3}$$

which also implies the identity

$$\int_0^\infty 1\{S(s) > a\} dL(s) \quad = \quad 0.$$

Moreover, we note that by (2) and (3) one has that the dynamics of S are identical to those of an $M/M/1$ queue with an arrival rate of λ_L and a service rate of λ_M, which has been adjusted upwards by the amount a.

At several points in the paper, and in particular in the proof of Theorem 5.1, it will be helpful to view S as a semi-martingale. Hence, for each $t \geq 0$, let $M(t) = (N_L(t) - \lambda_L t) - (N_M(t) - \lambda_M t)$ and set $M = \{M(t), t \geq 0\}$. Then, since N_M and N_L are both Poisson processes, with rates λ_M and λ_L, respectively, one immediately sees that the process M is a martingale. Moreover, by (2) we may write

$$S(t) \quad = \quad S(0) + M(t) + B(t), \quad t \geq 0, \tag{4}$$

where $B(t) = (\lambda_L - \lambda_M)t + L(t)$ and we set $B = \{B(t), t \geq 0\}$. Since B is of bounded variation, one then obtains from (4) a semi-martingale decomposition of S. This decomposition will be called upon repeatedly in our proofs.

We are now in a position to describe the measure-valued order book in detail. For each $t \geq 0$, let $\mu(t) \in \mathcal{M}_F(\mathbb{R})$ be a counting measure on $\mathbb{R}$ which represents the state of the order book at time t. In particular, we assume that for each $\mathcal{A} \in B(\mathbb{R})$, the quantity $\mu(t)(\mathcal{A})$ represents the number of limit sell orders on the book at time t whose prices lie in the set $\mathcal{A}$. We also set $\mu = \{\mu(t), t \geq 0\}$ to be the measure-valued order book process and note that since N_L and N_M are assumed to be independent Poisson processes, it may be rigorously shown that μ is in fact a Markov process. Next, for each $t \geq 0$, let

$$p(t) \quad = \quad \sup\{x \geq 0 : \mu(t)((-\infty, x)) = 0\}, \tag{5}$$

denote the leftmost point of the support of $\mu(t)$. It then follows that $p(t)$ is the smallest price associated with a limit sell order on the book at time t and hence $p(t)$ represents the current best price. Also note that since the market maker never allows the total number of order on the book to fall below the level $a > 0$, it follows that the price $p(t)$ is always well-defined. We set $p = \{p(t), t \geq 0\}$ to be the price process.

The dynamics of the measure-valued order book may now be described as follows. For each $x \in \mathbb{R}$, let δ_x denote the Dirac measure at x and let $\{X_i, i \geq 1\}$ be an i.i.d. sequence of random variables, independent of N_L and N_M, and with common distribution G. Next, let $\mu(0) \in \mathcal{M}_F(\mathbb{R})$ be a (random) counting measure representing the state of the book at time $t = 0$. We then have that, according to the preceding discussion, the state of order book at each point in time $t \geq 0$ is given by

$$\mu(t) \quad = \quad \mu(0) + \sum_{i=1}^{N_L(t)} \delta_{p(\tau_i^L-)X_i} - \int_0^t \delta_{p(s-)} dN_M(s) + \int_0^t \delta_{p(s-)} dL(s). \tag{6}$$

Note that the assumption that $G(0) = 0$ ensures that the price always stays positive and that since $\mu(0)(\mathbb{R}) = S(0)$, we must assume that $\mu(0)(\mathbb{R}) \geq a$, a.s. Next, note that using the identity (2), one has that (6) may be expressed in the more parsimonious form

$$\mu(t) \quad = \quad \mu(0) + \sum_{i=1}^{N_L(t)} (\delta_{p(\tau_i^L-)X_i} - \delta_{p(\tau_i^L-)}) + \int_0^t \delta_{p(s-)} dS(s). \tag{7}$$

The representation (7) serves as the starting point for many of our proofs that follow.

We will also often find it more convenient to analyze the measure-valued limit order book process by evaluating it using test functions φ in $C_b(\mathbb{R})$. Recall from Section 1.1 that for an arbitrary measure $\nu \in \mathcal{M}_F(\mathbb{R})$ and function $\varphi \in C_b(\mathbb{R})$, we denote the inner product of ν and φ by setting

$$\langle \nu, \varphi \rangle \quad = \quad \int_{\mathbb{R}} \varphi(x) d\nu(x). \tag{8}$$

It then follows upon integrating both sides of (7) that for each $\varphi \in C_b(\mathbb{R})$ and $t \geq 0$, we have that

$$\langle \mu(t), \varphi \rangle \quad = \quad \langle \mu(0), \varphi \rangle + \sum_{i=1}^{N_L(t)} (\varphi(p(\tau_i^L-)X_i) - \varphi(p(\tau_i^L-))) + \int_0^t \varphi(p(s-)) dS(s). \tag{9}$$

The representation (9) will also be helpful in our proofs that follow.

3 High Frequency Regime

In this paper, we are interested in obtaining asymptotic approximations to both the price process and the measure-valued limit order book process. The asymptotic regime which we will consider in order to obtain our approximations is such that the arrival rate of both limit orders and market orders to the order book is high, while, at the same time, traders place their limit sell orders relatively close to the current best price. We refer to this regime as the "high frequency" regime. The detailed mathematical setup of the high frequency regime is as follows.

We consider a sequence of limit order books indexed by $n \geq 1$, where n may be roughly thought of as the rate at which market and limit orders arrive to the nth system (see Item 1 of Definition 3.1 below). For each fixed $n \geq 1$, the system dynamics are the same as those defined in Section 2, and all quantities associated with the nth order book are denoted by a superscript n. In particular, limit sell orders arrive to the the nth order book according to the Poisson process $N_L^n = \{N_L^n(t), t \geq 0\}$, which has rate λ_L^n. For each $i \geq 1$, we denote by $\tau_i^{L,n}$ the time at which the ith limit order arrives to the nth order book. Each time that a limit sell order arrives to the nth order book, it places a single order on the book. In particular, letting $p^n(t)$ denote the best price at time $t \geq 0$ in the nth limit order book, then, a limit sell order arriving to the nth limit order book at time $\tau_i^{L,n}$ will place its order at the price $X_{n,i} p^n(\tau_i^{L,n}-)$. Our assumption on the distribution of $X_{n,i}$ is given by Item 3 of Definition 3.1 below. Market buy orders arrive to the nth order book according to the Poisson process $N_M^n = \{N_M^n(t), t \geq 0\}$ which has rate $\lambda_M^n > 0$, and, for each $i \geq 1$, we denote by $\tau_i^{M,n}$ the time at which the ith market order arrives to the nth limit order book. Each market order arriving to the book removes a single order corresponding to the current best price. Finally, we assume that in the nth limit order book, there exists a market maker who never allows the total number of orders on the book fall below the level $a^n > 0$.

The high frequency regime is now defined by the following set of four assumptions.

Definition 3.1 (High Frequency Regime)**.** *The high frequency regime is defined by assuming that*

1. $n^{-1}\lambda_L^n \to \lambda > 0$ *and* $n^{-1}\lambda_M^n \to \lambda > 0$ *as* $n \to \infty$,
2. $\sqrt{n}(n^{-1}\lambda_L^n - n^{-1}\lambda_M^n) \to \theta \in \mathbb{R}$ *as* $n \to \infty$,
3. *For each* $n \geq 1$, $X_{n,i} = X_i^{1/n^{1/2}}$ *for* $i \geq 1$*, where* $\{X_i, i \geq 1\}$ *is a sequence of i.i.d. random variable with common distribution* G*, independent of* N_L^n *and* N_M^n.
4. $n^{-1/2}a^n \to \tilde{a} > 0$ *as* $n \to \infty$.

We now comment on Items 1 through 4 in Definition 3.1 above. First note that from Item 1 it is evident that the incoming rate of limit sell and market buy orders is on the order of n in the nth limit order book, hence the term "high frequency". Moreover, note that Item 1 also implies that these rates are closely matched with one another as n grows large. In particular, Item 2 of Definition 3.1 provides a refined asymptotic on the mismatch between the supply of and demand for limit orders in the high frequency regime. This mismatch is characterized through the scalar θ. In general, θ may be positive, negative, or equal to zero. In Section 8, we study the effects of the sign of θ on the long term behavior of the limiting price process. Item 3 of Definition 3.1 implies that as n tends ∞, the distribution of the i.i.d. sequence $\{X_{n,i}, i \geq 1\}$ weakly converges to the constant 1. This in turn implies that in the high frequency regime, the majority of traders place their limit sell in close proximity to the current best price. This type of behavior has been observed empirically by Biais et al. [3] whose studied order flow dynamics at the Paris Bourse and concluded that the majority of order flow occurs close to the best quote. Moreover, note that performing a first order Taylor expansion, one has that for each $n \geq 1$,

$$X_{n,i} \;=\; X_i^{1/n^{1/2}} \;=\; 1 + \frac{\ln(X_i)}{n^{1/2}} + o(n^{-1/2}), \quad i \geq 1. \tag{10}$$

Thus, the expansion (10) suggests that the sign of $\ln(X)$ plays an important role in the direction of the price movement and indeed, as mentioned in the Introduction, our results in this paper require the assumption that $E[\ln X] > 0$. Finally, letting $\tilde{a}^n = n^{-1/2}a^n$ for each $n \geq 1$, note that Item 4 of Definition 3.1 implies that the market maker never allows the total number of orders on the nth limit order book to fall too far below the level $\sqrt{n}\tilde{a}$. This can be attributed to the fact that as a result of Item 2 of Definition 3.1, the fluctuations of the total number of orders on the book are on the order of $n^{1/2}$.

We now complete this section by describing our choice of scaling for all relevant processes associated with the nth limit order book. For each $n \geq 1$, let $S^n(t)$ denote the total number of orders on the nth limit order book at time $t \geq 0$. We then define $\tilde{S}^n(t) = n^{-1/2}S^n(t)$ to be the diffusion scaled total number of orders on the book and we define the process $\tilde{S}^n = \{\tilde{S}^n(t), t \geq 0\}$. Note that making use of the semi-martingale decomposition of S^n provided by (4) of Section 2, one may also write

$$\tilde{S}^n(t) \;=\; \tilde{S}^n(0) + \tilde{M}^n(t) + \tilde{B}^n(t), \quad t \geq 0, \tag{11}$$

where

$$\tilde{M}^n(t) \;=\; n^{-1/2}(N_L^n(t) - \lambda_L^n t) - n^{-1/2}(N_M^n(t) - \lambda_M^n t), \quad t \geq 0, \tag{12}$$

and

$$\tilde{B}^n(t) \;=\; n^{1/2}(n^{-1}\lambda_L^n - n^{-1}\lambda_M^n)t + \tilde{L}^n(t), \quad t \geq 0, \tag{13}$$

and

$$\tilde{L}^n(t) \;=\; n^{-1/2}L^n(t), \quad t \geq 0. \tag{14}$$

We also define the resulting processes $\tilde{M}^n = \{\tilde{M}^n(t), t \geq 0\}, \tilde{B}^n = \{\tilde{B}^n(t), t \geq 0\}$ and $\tilde{L}^n = \{\tilde{L}^n(t), t \geq 0\}$, and note that in the sequel it will also be convenient to set

$$\bar{N}_L^n(t) \;=\; n^{-1}N_L^n(t) \;\text{ and }\; \bar{N}_M^n(t) = n^{-1}N_M^n(t), \quad t \geq 0,$$

and to define the resulting processes $\bar{N}_L^n = \{\bar{N}_L^n(t), t \geq 0\}$ and $\bar{N}_M^n = \{\bar{N}_M^n(t), t \geq 0\}$. Next, for each $n \geq 1$, let $\mu^n(t) \in \mathcal{M}_F(\mathbb{R})$ be the state of the nth limit order book at time $t \geq 0$. As it turns out, we will require two assumptions on the initial state of the nth measure-valued order book. These assumptions may be summarized as follows.

Assumption 3.2 (Initial state of the order book)**.** *We shall assume that for each $n \geq 1$, the initial state of the nth order book, $\mu^n(0)$, satisfies the following two conditions:*

1.
$$\sup_{n\geq 1} \frac{1}{n^2} E\left[(S^n(0))^4\right] < \infty,$$

2. *If $x < y$ are consecutive orders on the nth order book at time $t = 0$, that is, $\mu^n(0)(\{x\}) > 0$, $\mu^n(0)(\{y\}) > 0$, and $\mu^n(0)((x, y)) = 0$, then $y/x \leq C^{\frac{1}{\sqrt{n}}}$.*

Note that Item 2 of Assumption 3.2 is consistent with our earlier assumption (1) and Item 3 of Definition 3.1. We next define $\tilde{\mu}^n(t) = n^{-1/2}\mu^n(t)$ to be the state of the diffusion scaled order book at time $t \geq 0$. In other words, we set

$$\tilde{\mu}^n(t)(\mathcal{A}) \quad = \quad n^{-1/2}\mu^n(t)(\mathcal{A}) \;\; \text{for each} \;\; \mathcal{A} \in B(\mathbb{R}).$$

Substituting the above scalings into the system dynamics equation (9) of Section 2, one now obtains that for each $n \geq 1$ and $\varphi \in C_b(\mathbb{R})$,

$$\begin{aligned}\langle \tilde{\mu}^n(t), \varphi \rangle \quad &= \quad \langle \tilde{\mu}^n(0), \varphi \rangle + \frac{1}{\sqrt{n}} \sum_{i=1}^{N_L^n(t)} \left(\varphi \left(p^n \left(\tau_i^{L,n} - \right) X_i^{1/n^{1/2}} \right) - \varphi \left(p^n \left(\tau_i^{L,n} - \right) \right) \right) \\ &\quad + \int_0^t \varphi(p^n(s-)) d\tilde{S}^n(s), \quad t \geq 0.\end{aligned} \tag{15}$$

Equation (15) will be helpful as we proceed throughout our proofs in the Appendix. It is also instructive to point out that clearly one must have that $\langle \tilde{\mu}^n(t), 1 \rangle = \tilde{S}^n(t)$ for each $t \geq 0$. Finally, we note in passing that the price process is not scaled in any way in the high frequency regime. Indeed, in Theorem 5.1 of Section 5 below, we obtain a non-trivial weak limit for the sequence of unscaled price processes in the high frequency regime.

4 Weak Convergence of the Total Number of Orders on the Book

In this section, we provide a result on the weak convergence of the sequence of processes $\{\tilde{S}^n, n \geq 1\}$ defined in Section 3. Recall that these processes keep track of the total number of orders on the book, properly scaled. Our main result is to show that $\{\tilde{S}^n, n \geq 1\}$ weakly converges in the high frequency regime to what is referred to as a "reflected Brownian motion". Although this result is to some extent straightforward, it is important for the remainder of the paper and so we state it here with a short proof.

We begin by providing a rigorous definition of reflected Brownian motion. Although this definition can take many forms, we feel that the following is most relevant for the remainder of the paper.

Definition 4.1. *Let $\tilde{X} = \{\tilde{X}(t), t \geq 0\}$ be a standard Brownian motion and let $\tilde{Z}(0)$ be a random variable, independent of $\tilde{X}$. Moreover, let $a, \theta, \sigma \in \Re$. Then, we say that $\tilde{Z} = \{\tilde{Z}(t), t \geq 0\}$ is a Brownian motion, reflected at the level a, with constant drift θ and infinitesimal variance σ^2, started at $\tilde{Z}(0)$, if the following three conditions are satisfied almost surely,*

1. $\tilde{Z}(t) \;=\; \tilde{Z}(0) + \theta t + \sigma \tilde{X}(t) + \tilde{Y}(t) \;\geq\; a, \quad t \geq 0,$

2. *$\tilde{Y} = \{\tilde{Y}(t), t \geq 0\}$ is a non-decreasing process with $\tilde{Y}(0) = 0$,*

3. *The pair $(\tilde{Z}, \tilde{Y})$ satisfies the complementarity condition*

$$\int_0^\infty 1\{\tilde{Z}(s) > a\} d\tilde{Y}(s) \quad = \quad 0.$$

Moreover, we refer to $\tilde{Y}$ as the local time of $\tilde{Z}$ at a.

Note that by conditions (1)-(3) above, one has that $\tilde{Z}$ behaves like a Brownain motion with drift θ and infinitesimal variance σ^2 in the open set (a, ∞), and is instantaneously reflected back into (a, ∞) upon hitting the boundary a. For a further discussion on reflected Brownian motion, one may consult the excellent text of Harrison [13].

The following is now the main result of this section.

Proposition 4.2. *If $\tilde{S}^n(0) \Rightarrow \tilde{S}(0)$ as $n \to \infty$, and Items 1-4 of Definition 3.1 hold, then the pair $(\tilde{S}^n, \tilde{L}^n) \Rightarrow (\tilde{S}, \tilde{L})$ in $D^2([0,\infty),\mathbb{R})$ as $n \to \infty$, where $\tilde{S}$ is a Brownian motion, reflected at $\tilde{a} > 0$, with constant drift θ and infinitesimal variance 2λ, started at $\tilde{S}(0)$, and with local time $\tilde{L}$ at $\tilde{a}$.*

Proof. First note that by (11)-(14) above, we have that we may write $\tilde{S}^n(t) = \tilde{S}^n(0) + \tilde{M}^n(t) + n^{1/2}(n^{-1}\lambda_L^n - n^{-1}\lambda_M^n)t + \tilde{L}^n(t)$, for each $t \geq 0$. Next, note that by the discussion in Section 2, it is clear that the pair $(\tilde{S}^n - \tilde{a}^n, \tilde{L}^n)$ almost surely satisfy conditions (6.11)-(6.13) of Chen and Yao [5]. Hence, by Theorem 6.1 of Chen and Yao [5], we may write $(\tilde{S}^n - \tilde{a}^n, \tilde{L}^n) = (\Phi,\ \Psi)((\tilde{S}^n(0) - \tilde{a}^n) + n^{1/2}(n^{-1}\lambda_L^n - n^{-1}\lambda_M^n)e + \tilde{M}^n)$, where the map $(\Phi, \Psi) : D([0,\infty),\mathbb{R}) \mapsto D^2([0,\infty),\mathbb{R})$ is continuous at $x \in C([0,\infty),\mathbb{R})$. Moreover, Items 1-2 of Definition 3.1 and the martingale central limit theorem (see, for instance, Theorem 7.1.4 of Ethier and Kurtz [12]) imply that $(\tilde{S}^n(0) - \tilde{a}^n) + n^{1/2}(n^{-1}\lambda_L^n - n^{-1}\lambda_M^n)e + \tilde{M}^n \Rightarrow (\tilde{S}(0) - \tilde{a}) + \theta e + \sqrt{2\lambda}\tilde{X}$ as $n \to \infty$, where $\tilde{X}$ is a standard Brownian motion. It therefore follows by Item 4 of Definition 3.1, the representation $(\tilde{S}^n - \tilde{a}^n, \tilde{L}^n) = (\Phi, \Psi)((\tilde{S}^n(0) - \tilde{a}^n) + n^{1/2}(n^{-1}\lambda_L^n - n^{-1}\lambda_M^n)e + \tilde{M}^n)$, the almost sure continuity of Brownian motion (see, for instance, Definition 2.1.1 of Karatzas and Shreve [17]), and the continuous mapping theorem (see, for instance, Theorem 3.4.3 of Whitt [25]), that $(\tilde{S}^n - \tilde{a}^n, \tilde{L}^n) \Rightarrow (\tilde{S} - \tilde{a}, \tilde{L}) = (\Phi, \Psi)((\tilde{S}(0) - \tilde{a}) + \theta e + \sqrt{2\lambda}\tilde{X})$ as $n \to \infty$. However, it is clear by conditions (6.11)-(6.13) of Chen and Yao [5] that this then implies that $\tilde{S}$ is a Brownian motion, reflected at the level $\tilde{a} > 0$, with constant drift θ and infinitesimal variance 2λ, started at $\tilde{S}(0)$, and with local time $\tilde{L}$ at $\tilde{a}$. This completes the proof. □

Note that Definition 4.1 and Proposition 4.2 together imply that we may write

$$\tilde{S}(t) \quad = \quad \tilde{S}(0) + \sqrt{2\lambda}\tilde{B}(t) + \theta t + \tilde{L}(t) \ \geq \ \tilde{a}, \quad t \geq 0, \tag{16}$$

where $\tilde{B} = \{\tilde{B}(t), t \geq 0\}$ is a standard Brownian motion, and $\tilde{L}$ is a non-decreasing process such that

$$\int_0^\infty 1\{\tilde{S}(s) > \tilde{a}\} d\tilde{L}(s) \quad = \quad 0. \tag{17}$$

The representation (16)-(17) will be useful as we proceed.

5 Weak Convergence of the Price Process

In this section, we provide our main result on the weak convergence of the sequence of price processes $\{p^n, n \geq 1\}$ in the high frequency regime described in Section 3, see Theorem 5.1 below. Then, in Section 6, we show how Theorem 5.1 may be used to prove that the sequence of diffusion scaled limit order books, $\{\tilde{\mu}^n, n \geq 1\}$, weakly converges in the high frequency regime. The supporting lemmas and propositions for Theorem 5.1 may be found in Sections 9.1 and 9.2 of the Appendix.

Before providing the statement of Theorem 5.1, we must first setup the following notation. Let $\ln = \ln(x)$ be the natural log function on $(0, \infty)$ and, for each $n \geq 1$, let

$$\Delta\tilde{\mu}^{l,n}(0) \quad = \quad \langle \tilde{\mu}^n(0), \ln \rangle - \tilde{S}^n(0)\ln(p^n(0)). \tag{18}$$

We then have the following.

Theorem 5.1. *If $E[\ln X] > 0$ and $E[(\Delta\tilde{\mu}^{l,n}(0))^2] \to 0$ and $(p^n(0), \tilde{S}^n(0)) \Rightarrow (p(0), \tilde{S}(0))$ as $n \to \infty$, and if Items 1-4 of Definition 3.1 hold, then $p^n \Rightarrow p$ as $n \to \infty$, where the process $p = \{p(t), t \geq 0\}$ is given by*

$$p(t) \quad = \quad p(0)\exp\left(\int_0^t \frac{E[\ln X]}{\tilde{S}(s)} d(\lambda s)\right), \text{ for } t \geq 0, \tag{19}$$

where $\tilde{S} = \{\tilde{S}(t), t \geq 0\}$ is a Brownian motion, reflected at $\tilde{a} > 0$, with constant drift θ and infinitesimal variance 2λ, started at $\tilde{S}(0)$.

Proof. First note that in the notation of Section 9.1, we may write $p = \exp(p^l)$. The proof is then a direct consequence of the fact that exp is a continuous map on $(-\infty, \infty)$, the continuous mapping theorem (see, for instance, Theorem 3.4.3 of Whitt [25]) and Proposition 9.1 of Section 9.1 of the Appendix. □

Note that Theorem 5.1 implies that in the high frequency regime, the limiting price process p is an increasing process. This, of course, is a consequence of our assumption that $E[\ln X] > 0$. Moreover, recall by Proposition 4.2 of Section 4 that the reflected Brownian motion $\tilde{S} = \{\tilde{S}(t), t \geq 0\}$ appearing in Theorem 5.1 keeps track of the total number of orders on the order book. Thus, the fact that $\tilde{S}(t)$ appears in the denominator in (19) implies that in the limit the instantaneous rate of increase of the limiting price process is inversely related to the total number of orders on the book. Intuitively speaking, this can be explained by the fact that in order for the price to increase to a new level, one must first remove all of those orders from the book which lie below that level. Moreover, as is shown in Theorem 6.1 below, for n sufficiently large most of the orders on the book lie in a close vicinity of the best price. We also recall that by Item 4 of Definition 3.1, we have that $\tilde{S}(t) \geq \tilde{a} > 0$ for each $t \geq 0$, and so the integral in (19) is well defined. In Section 7, we study the transient behavior of the limiting price process p given in Theorem 5.1 and in Section 8, we study the long term behavior of p.

6 Weak Convergence of the Order Book Process

In this section, we state Theorem 6.1, which provides a weak limit for the sequence $\{\tilde{\mu}^n, n \geq 1\}$ of diffusion scaled, measure-valued order book processes in the high frequency regime. Recall first that in the high frequency regime described in Section 3, the distance at which traders place their limit orders from the current best price weakly converges to zero. Hence, it is natural to expect that when n is large, all of the orders on the book will be contained in a small neighborhood of the current best price. Indeed, under appropriate initial conditions, it turns out that this is the case, as we now show. The supporting lemmas and propositions for Theorem 6.1 may be found in Section 9.3. We have the following.

Theorem 6.1. *If $E[\ln X] > 0$ and $E[(\Delta\tilde{\mu}_l^n(0))^2] \to 0$ and $(p^n(0), \tilde{S}^n(0)) \Rightarrow (p(0), \tilde{S}(0))$ as $n \to \infty$, and if Items 1-4 of Definition 3.1 hold, then $\tilde{\mu}^n \Rightarrow \tilde{\mu}$ as $n \to \infty$, where $\tilde{\mu} = \{\tilde{\mu}(t), t \geq 0\}$ is given by*

$$\tilde{\mu}(t) \quad = \quad \tilde{S}(t)\delta_{p(t)}, \quad t \geq 0, \tag{20}$$

where $p = \{p(t), t \geq 0\}$ and $\tilde{S} = \{\tilde{S}(t), t \geq 0\}$ are as given in Theorem 5.1.

Proof. First recall that by Proposition 9.14 of Section 9.3, the sequence $\{\tilde{\mu}^n, n \geq 1\}$ is tight in $D(\mathbb{R}_+, \mathcal{M}_F(\mathbb{R}))$. Hence, in order to complete the proof, it suffices by Theorem 13.1 of Billingsley [4] to show that the finite dimensional distributions of $\{\tilde{\mu}^n, n \geq 1\}$ converge to those of $\tilde{\mu}$ as given by (20).

Let $t_1, t_2 \geq 0$. We will show that $(\tilde{\mu}^n(t_1), \tilde{\mu}^n(t_2)) \Rightarrow (\tilde{\mu}(t_1), \tilde{\mu}(t_2))$ in $\mathcal{M}_F(\mathbb{R}) \times \mathcal{M}_F(\mathbb{R})$ as $n \to \infty$. The proof for an arbitrary number of dimensions follows similarly. Let $\mathrm{BL}^+(\mathbb{R})$ denote the set of bounded and non-negative Lipschitz functions on $\mathbb{R}$. Next, for each $\varphi_1, \varphi_2 \in \mathrm{BL}^+(\mathbb{R})$, let $F_{\varphi_1,\varphi_2} \in C_b(\mathcal{M}_F(\mathbb{R}) \times \mathcal{M}_F(\mathbb{R}))$ be defined by setting

$$F_{\varphi_1,\varphi_2}(\nu_1, \nu_2) \quad = \quad \exp\left(-\langle \nu_1, \varphi_1\rangle - \langle \nu_2, \varphi_2\rangle\right)$$

and set

$$V \quad = \quad \left\{ \sum_{i=1}^n \lambda_i F_{\varphi_{i,1},\varphi_{i,2}};\ n \in \mathbb{N},\ \lambda_1, ..., \lambda_n \in \mathbb{R},\ \ \varphi_{1,1}, \varphi_{1,2}, ..., \varphi_{n,1}, \varphi_{n,2} \in \mathrm{BL}^+(\mathbb{R}) \right\}.$$

Then, $V \subset C_b(\mathcal{M}_F(\mathbb{R}) \times \mathcal{M}_F(\mathbb{R}))$ and it is straightforward to check that V is an algebra. Moreover, since $\mathbb{F} = \{\langle\ \cdot\ , \varphi\rangle, \varphi \in \mathrm{BL}^+(\mathbb{R}_+)\}$ separates points on $\mathcal{M}_F(\mathbb{R})$, it is straightforward to show that V separates points on $\mathcal{M}_F(\mathbb{R}) \times \mathcal{M}_F(\mathbb{R})$. Now let $F \in V$. By Proposition 9.15 and the continuous mapping theorem (see, for instance, Theorem 3.4.3 of Whitt [25]), we have that

$$F(\tilde{\mu}^n(t_1), \tilde{\mu}^n(t_2)) \quad \Rightarrow \quad F(\tilde{\mu}(t_1), \tilde{\mu}(t_2)) \ \text{ as } \ n \to \infty,$$

where $\tilde{\mu}$ is as given in (20). Hence, since F is a bounded function it, follows that

$$E[F(\tilde{\mu}^n(t_1), \tilde{\mu}^n(t_2))] \quad \to \quad E[F(\tilde{\mu}(t_1), \tilde{\mu}(t_2))] \ \text{ as } \ n \to \infty.$$

It therefore follows by the tightness of the sequence $\{(\tilde{\mu}^n(t_1), \tilde{\mu}^n(t_2)), n \geq 1\}$ and Lemma 3.4.3 and Theorem 3.4.5 of Ethier and Kurtz [12], that $(\tilde{\mu}^n(t_1), \tilde{\mu}^n(t_2)) \Rightarrow (\tilde{\mu}(t_1), \tilde{\mu}(t_2))$ in $\mathcal{M}_F(\mathbb{R}) \times \mathcal{M}_F(\mathbb{R})$ as $n \to \infty$. The proof for $n > 2$ and $t_1, ..., t_n \geq 0$ follows similarly. This completes the proof. □

Note that Theorem 6.1 effectively says that in the limit of the high frequency regime, the entire mass of the order book is contained at the best price. Moreover, recall from Theorem 5.1 of Section 5 that the dynamics of the limiting price process are intricately related to process keeping track of the total number of orders on the book. Thus, taken together, Theorems 5.1 and 6.1 imply interesting behavior for various functionals of the limiting order book. In particular, let $\xi(x) = x$ be the identity function on $\mathbb{R}$ and note that $\langle \mu^n(t), \xi\rangle$ represents the total value of all of the orders on the book at time t. Although the function ξ is clearly not a member of $C_b(\mathbb{R})$, it is still natural to conjecture from Theorem 6.1 that $\langle \tilde{\mu}^n, \xi\rangle \Rightarrow \langle \tilde{\mu}, \xi\rangle$ as $n \to \infty$. In particular, by Theorem 5.1, this would imply that for each $t \geq 0$, one has the convergence

$$\langle \tilde{\mu}^n(t), \xi\rangle \quad \Rightarrow \quad \tilde{S}(t)p(0)\exp\left(\int_0^t \frac{E[\ln X]}{\tilde{S}(s)} d(\lambda s)\right) \ \text{ as } \ n \to \infty. \tag{21}$$

In Section 7, we provide a method for calculating the expected value of the quantity appearing on the righthand side of (21) at arbitrary points in time. Then, in Section 8, we analyze the long term behavior of the quantity appearing on the righthand side of (21).

We now conclude this section by providing a characterization of the limit process $\tilde{\mu}$ in Theorem 6.1 as the solution to a measure-valued stochastic differential equation. We have the following.

Proposition 6.2. *The limiting measure-valued order book process $\tilde{\mu}$ given in Theorem 6.1 may be characterized as a solution to the measure-valued SDE,*

$$\langle \tilde{\mu}(t), \varphi \rangle \quad = \quad \langle \tilde{\mu}(0), \varphi \rangle + \int_0^t \varphi(p(s)) d\tilde{S}(s) + \lambda E[\ln X] \int_0^t \varphi^{'}(p(s)) p(s) ds, \quad t \geq 0, \tag{22}$$

for each $\varphi \in C_b^2(\mathbb{R})$, where $\tilde{S} = \{\tilde{S}(t), t \geq 0\}$ is a Brownian motion, reflected at $\tilde{a} > 0$, with constant drift θ and infinitesimal variance 2λ, started at $\langle \tilde{\mu}(0), 1 \rangle$, and

$$p(t) \quad = \quad \sup\{x \geq 0 : \tilde{\mu}(t)((-\infty, x)) = 0\}.$$

Proof. The proof follows by (20) of Theorem 6.1 and a direct application of Ito's formula (see, for instance, Theorem 3.3.3 of Karatzas and Shreve [17]). □

In general, we believe that the measure-valued SDE (22) characterizes the limit process $\tilde{\mu}$ under more relaxed assumptions on the initial conditions of the order book than those contained in the statements of Theorems 5.1 and 6.1. However, under these more relaxed initial conditions, the proofs of Theorems 5.1 and 6.1, as well as the solution to (22), become significantly more complicated while adding little additional insight. We have therefore restricted ourselves in the present paper to the limiting initial condition $\tilde{\mu}(0) = \tilde{S}(0)\delta_{p(0)}$.

7 Transient Behavior of the Limiting Price Process

In this section, we study the transient behavior of the limiting price process p given in Theorem 5.1, as well as the transient behavior of the limiting order book process $\tilde{\mu}$ given in Theorem 6.1. In particular, for each point in time $t \geq 0$, we show how to calculate the expected value of the price, the expected total value of the book, and we also show how to calculate the expected value of the total amount of money which the market maker must outlay up until time t, properly discounted. We begin with the easiest case of calculating the expected value of the limiting price process at a fixed point in time.

First note that it is quite straightforward to calculate the expected value of the natural log of the limiting price process at each point in time. In particular, letting $p^l(t) = \ln p(t)$ for each $t \geq 0$, one has by Theorem 5.1 that for each $t \geq 0$,

$$E[p^l(t) \mid \tilde{S}(0) = x, p^l(0) = p^l] \quad = \quad p^l + \int_0^t E\left[\frac{\lambda E[\ln X]}{\tilde{S}(s)} \mid \tilde{S}(0) = x\right] ds, \tag{23}$$

where $\tilde{S} = \{\tilde{S}(t), t \geq 0\}$ is a Brownian motion, reflected at the level $\tilde{a} > 0$, with constant drift θ and infinitesimal variance 2λ . The transient distribution of reflected Brownian motion is known in closed form (see (4) of Section 1.9 of Harrison [13]) and so in principle the calculations of the expectations in (23) may be preformed.

The situation changes when one attempts to calculate the expected value of the actual price itself. In this case, by Theorem 5.1, one is confronted with the calculation

$$E[p(t) \mid \tilde{S}(0) = x, p(0) = p] \;=\; pE\left[\exp\left(\int_0^t \frac{\lambda E[\ln X]}{\tilde{S}(s)} ds\right) \mid \tilde{S}(0) = x\right]. \tag{24}$$

The calculation of the expectation on the righthand side above is in general difficult. Therefore, in order to ease ourselves into the situation, we first consider the case in which $\tilde{S}$ has zero drift, i.e. $\theta = 0$. In this case, it is well known that $\tilde{S}$ is equivalent in law to $\tilde{a} + \sqrt{2\lambda}|\tilde{X}|$, where $\tilde{X}$ is a standard Brownian motion. We therefore have the equality

$$E\left[\exp\left(\int_0^t \frac{\lambda E[\ln X]}{\tilde{S}(s)} ds\right) \mid \tilde{S}(0) = x\right] \;=\; E\left[\exp\left(\int_0^t \frac{\lambda E[\ln X]}{\tilde{a} + \sqrt{2\lambda}|\tilde{X}(s)|} ds\right) \mid \sqrt{2\lambda}\tilde{X}(0) = x - \tilde{a}\right].$$

The expectation on the righthand side above may now be calculated via an application of the Feynman-Kac formula (see, for instance, Theorem 4.4.2 of Karatzas and Shreve [17]). In general, however, $\tilde{S}$ has a singular component to it and so it does not fit neatly into the usual context of the Feynman-Kac formula and the same maneuver cannot be applied. Nevertheless, $\tilde{S}$ is still a semimartingale and so the formula may be appropriately modified in order to yield a partial differential equation whose solution is the desired expected value. In particular, let $k : [\tilde{a}, \infty) \mapsto (0, \lambda E[\ln X]/\tilde{a}]$ be the function defined by

$$k(x) \;=\; \frac{\lambda E[\ln X]}{x}, \tag{25}$$

and note that k is bounded. We then have the following.

Theorem 7.1. *Suppose that $v \in C^{1,2}$ is an $\mathbb{R}$-valued function on $[0,T) \times [\tilde{a}, \infty)$ satisfying the growth condition*

$$\max_{0 \le t \le T} |v(t,x)| \;\le\; Ke^{bx^2}, \; x \in [\tilde{a}, \infty), \tag{26}$$

for some constants $K > 0$ and $b > 0$, and that v satisfies

$$-\frac{\partial v}{\partial t} - kv \;=\; \lambda \frac{\partial^2 v}{\partial x^2} + \theta \frac{\partial v}{\partial x}, \; \text{on} \; [0,T) \times [\tilde{a}, \infty), \tag{27}$$

subject to

$$v(T,x) \;=\; 1, \; x \in [\tilde{a}, \infty), \tag{28}$$

$$\frac{\partial v(t,\tilde{a})}{\partial x} \;=\; 0, \; t \in [0,T). \tag{29}$$

Then, such a v is unique, and

$$v(t,x) \;=\; E\left[\exp\left(\int_0^{T-t} \frac{\lambda E[\ln X]}{\tilde{S}(s)} ds\right) \mid \tilde{S}(0) = x\right].$$

Note, of course, that from Theorem 5.1 one obtains that

$$E[p(T-t)|\tilde{S}(0) = x, p(0) = p] \;=\; pv(t,x).$$

The proof of Theorem 7.1 is as follows. For the most part, it follows the proof of Theorem 4.4.2 of of Karatzas and Shreve [17], with the proper adjustments necessary.

Proof of Theorem 7.1. Recall first that from (16)-(17) we have that $\tilde{S} = \{\tilde{S}(t), t \geq 0\}$ has the semi-martingale decomposition

$$\tilde{S}(t) = \tilde{S}(0) + \sqrt{2\lambda}\tilde{B}(t) + \theta t + \tilde{L}(t), \quad t \geq 0, \tag{30}$$

where $\tilde{B} = \{\tilde{B}(t), t \geq 0\}$ is a standard Brownian motion, and $\tilde{L} = \{\tilde{L}(t), t \geq 0\}$ is a non-decreasing process such that

$$\int_0^\infty 1\{\tilde{S}(s) > \tilde{a}\} d\tilde{L}(s) = 0. \tag{31}$$

Hence, applying Ito's rule (see, for instance, Theorem 3.3.3 of Karatzas and Shreve [17]) and making note of (27), one obtains that

$$\begin{aligned} d\left[v(t+\beta, \tilde{S}(\beta)) \exp\left(\int_0^\beta k(\tilde{S}(s))ds\right)\right] &= \exp\left(\int_0^\beta k(\tilde{S}(s))ds\right) v_x(t+\beta, \tilde{S}(\beta)) d\tilde{L}(\beta) \\ &+ \sqrt{2\lambda}\left(\int_0^\beta k(\tilde{S}(s))ds\right) v_x(t+\beta, \tilde{S}(\beta)) d\tilde{B}(\beta). \end{aligned}$$

Also note that by (29) and (31),

$$\exp\left(\int_0^\beta k(\tilde{S}(s))ds\right) v_x(t+\beta, \tilde{S}(\beta)) d\tilde{L}(\beta) = 0. \tag{32}$$

Now let $\tau_n = \inf\{t \geq 0 : |\tilde{S}_t| \geq n\}$. Letting $0 < r < T - t$ and integrating on $[0, r \wedge \tau_n]$, the stochastic integral above has zero expectation, and so we obtain that

$$v(t,x) = E\left[v(t+\tau_n, \tilde{S}(\tau_n)) \exp\left(\int_0^{\tau_n} k(\tilde{S}(s))ds\right) 1\{\tau_n \leq r\} \mid \tilde{S}(0) = x\right] \tag{33}$$

$$+E\left[v(t+r, \tilde{S}(r)) \exp\left(\int_0^{r} k(\tilde{S}(s))ds\right) 1\{\tau_n > r\} \mid \tilde{S}(0) = x\right]. \tag{34}$$

The remainder of the proof now follows similarly to the proof of Theorem 4.4.2 of Karatzas and Shreve [17]. We treat each of the terms in (33) and (34) separately.

Regarding (33), note that since k is bounded, we have by (26) that (33) is bounded by

$$\begin{aligned} &\exp\left((T-t)\frac{\lambda E[\ln X]}{\tilde{a}}\right) E\left[|v(t+\tau_n, \tilde{S}(\tau_n))| 1\{\tau_n \leq r\} \mid \tilde{S}(0) = x\right] \\ \leq\ & K e^{bn^2} P(\tau_n \leq T | \tilde{S}(0) = x). \end{aligned}$$

However, since by Lemma 13.5.1 of Whitt [25] there exists a constant $C > 0$ such that

$$\sup_{0 \leq t \leq T} |\tilde{S}(t)| \leq C \sup_{0 \leq t \leq T} |\tilde{S}(0) + \theta t + \tilde{B}(t)|,$$

it follows as in the proof of Theorem 4.4.2 of Karatzas and Shreve [17] that $e^{bn^2} P(\tau_n \leq T | \tilde{S}(0) = x) \to 0$ as $n \to \infty$.

Finally, letting $n \to \infty$ and $r \to T - t$, it follows by the dominated convergence theorem (see, for instance, Theorem 1.8 of Lieb and Loss [18]) and (28) that

$$\begin{aligned} & E\left[v(t+r, \tilde{S}(r)) \exp\left(\int_0^r k(\tilde{S}(s))ds\right) 1\{\tau_n > r\} \mid \tilde{S}(0) = x\right] \\ \to \quad & E\left[\exp\left(\int_0^{T-t} \frac{\lambda E[\ln X]}{\tilde{S}(s)} ds\right) \mid \tilde{S}(0) = x\right], \end{aligned}$$

which completes the proof. □

We next show how to informally calculate the expected total value of the book at an arbitrary point in time. Note that this is not as easy as simply calculating the expected size of the book and multiplying it by the expected price. In particular, as was already noted in Section 5, the rate of increase of the price is inversely related to the size of the book and so the price itself is determined by the sample path of the size of the book. Now let $\xi(x) = x$ be the identity function. Then, given the form of the limiting order book from Theorem 6.1, we now, informally, must calculate

$$E\left[\langle \tilde{\mu}(t), \xi \rangle \mid \tilde{S}(0) = x, p(0) = p\right] \quad = \quad E\left[\tilde{S}(t)p(t) \mid \tilde{S}(0) = x, p(0) = p\right].$$

However, by Theorem 5.1, one has that

$$E\left[\tilde{S}(t)p(t) \mid \tilde{S}(0) = x, p(0) = p\right] \quad = \quad pE\left[\tilde{S}(t) \exp\left(\int_0^t \frac{\lambda E[\ln X]}{\tilde{S}(s)} ds\right) \mid \tilde{S}(0) = x\right]. \tag{35}$$

We now proceed to calculate the expectation on the righthand side of (35). The proof of the following result is similar to the proof of Theorem 7.1 and so we omit it.

Theorem 7.2. *Suppose that $v \in C^{1,2}$ is a $\mathbb{R}$-valued function on $[0, T) \times [\tilde{a}, \infty)$ satisfying the growth condition*

$$\max_{0 \le t \le T} |v(t, x)| \quad \le \quad Ke^{bx^2}, \;\; x \in [\tilde{a}, \infty), \tag{36}$$

for some constants $K > 0$ and $b > 0$, and that v satisfies

$$-\frac{\partial v}{\partial t} - kv \quad = \quad \lambda \frac{\partial^2 v}{\partial x^2} + \theta \frac{\partial v}{\partial x}, \;\; \text{on} \;\; [0, T) \;\; \times [\tilde{a}, \infty), \tag{37}$$

subject to

$$\begin{aligned} v(T, x) \quad &= \quad x, \;\; x \in [\tilde{a}, \infty), \\ \frac{\partial v(t, \tilde{a})}{\partial x} \quad &= \quad 0, \;\; t \in [0, T). \end{aligned}$$

Then, such a v is unique, and

$$v(t, x) \quad = \quad E\left[\tilde{S}(T-t) \exp\left(\int_0^{T-t} \frac{\lambda E[\ln X]}{\tilde{S}(s)} ds\right) \mid \tilde{S}(0) = x\right].$$

Now suppose that the market maker nets a fixed proportion $0 < \alpha \leq 1$ of the price each time that he or she is asked to place a limit sell order on the the book. It is then natural to attempt to calculate the market maker's expected discounted cash flow over a finite period of time. In order to make the calculation, first recall that in the model of Section 2, the non-decreasing process $L = \{L(t), t \geq 0\}$ keeps track of the points in time at which the market maker places sell orders on the book. Moreover, recall by Proposition 4.2 that $\tilde{L}^n \Rightarrow \tilde{L}$ as $n \to \infty$, where $\tilde{L}$ is the local time of $\tilde{S}$ as $\tilde{a}$. Hence, assuming that the discount rate is given by $\gamma > 0$, then, in the limit, the natural approximation to the expected discounted cash of the market maker is given by

$$\alpha E\left[\int_0^T e^{-\gamma t} p(t) d\tilde{L}(t) \mid p(0) = p, \tilde{S}(0) = x\right]. \tag{38}$$

Now recalling the precise form of the limiting price process p given by Theorem 5.1, we then see that calculating this expectation is equivalent to determining the quantity

$$\alpha p E\left[\int_0^T \exp\left(\int_0^t \left(\frac{\lambda E[\ln X]}{\tilde{S}(s)} - \gamma\right) ds\right) d\tilde{L}(t) \mid \tilde{S}(0) = x\right]. \tag{39}$$

The following result now provides a partial differential equation whose solution yields the expected value in (39). Its proof is similar in nature to that of Theorem 7.1 and so we omit it.

Theorem 7.3. *Suppose that $v \in C^{1,2}$ is a $\mathbb{R}$-valued function on $[0,T) \times [\tilde{a}, \infty)$ satisfying the growth condition*

$$\max_{0 \leq t \leq T} |v(t,x)| \quad \leq \quad K e^{bx^2}, \;\; x \in [\tilde{a}, \infty), \tag{40}$$

for some constants $K > 0$ and $b > 0$, and that v satisfies

$$-\frac{\partial v}{\partial t} - (k - \gamma) v \quad = \quad \lambda \frac{\partial^2 v}{\partial x^2} + \theta \frac{\partial v}{\partial x}, \;\; on \;\; [0,T) \;\; \times [\tilde{a}, \infty), \tag{41}$$

subject to

$$\begin{aligned} v(T,x) &= 0, \;\; x \in [\tilde{a}, \infty), \\ \frac{\partial v(t,\tilde{a})}{\partial x} &= -1, \;\; t \in [0,T). \end{aligned}$$

Then, such a v is unique, and

$$v(t,x) \quad = \quad E\left[\int_0^{T-t} \exp\left(\int_0^s \left(\frac{\lambda E[\ln X]}{\tilde{S}(u)} - \gamma\right) du\right) d\tilde{L}(s) \mid \tilde{S}(0) = x\right].$$

Note, of course, that if v satisfies the partial differential equation given by Theorem 7.3, then, by (38) and (39), the market maker's expected discounted payoff is given by

$$\alpha E\left[\int_0^{T-t} e^{-\gamma s} p(s) d\tilde{L}(s) \mid p(0) = p, \tilde{S}(0) = x\right] \quad = \quad \alpha p v(t,x).$$

This completes our analysis of the transient behavior of p and $\tilde{\mu}$.

8 Long Term Behavior of the Limiting Price and Order Book Processes

While the results of Section 7 may be thought of as describing the behavior of p and $\tilde{\mu}$ over short to medium periods of time, in the current section we focus on the long term behavior of the limiting price process p of Theorem 5.1, and the limiting order book process $\tilde{\mu}$ of Theorem 6.1. Our results will be divided into three cases, depending on whether $\theta < 0, \theta > 0$ or $\theta = 0$. Note that by (2), these three cases loosely correspond to the arrival rate of limit orders being less than, greater than, or equal to the arrival rate of market orders, respectively. Interestingly, as we will see, it turns out that the long term behavior of p is very different in these three cases. We begin with the case of $\theta < 0$.

8.1 Long Term Behavior for $\theta < 0$

The case of $\theta < 0$ corresponds, loosely speaking, to the arrival rate of limit orders being strictly less than the arrival rate of market orders. Specifically, we have that $\lambda_L^n < \lambda_M^n$ for sufficiently large n. In this case, for sufficiently large n, the process S^n which keeps track of the total number of orders on the book is identical in distribution to $a^n + Q^n$, where Q^n is the number in system process for an $M/M/1$ queue with a utilization of $\rho^n = \lambda_L^n/\lambda_M^n < 1$. One therefore has that the total number of orders on the book never grows too large without returning back down its minimum level a^n. This fact will turn out to play a critical role in determining the long term behavior of p and distinguishing it from the subsequent cases of $\theta > 0$ and $\theta = 0$. Our first result is to describe the long term behavior $p^l = \ln p$, the natural log of the limiting price process. Note that is straightforward to see from Theorem 5.1 that the precise form of p^l is given by

$$p^l(t) = p^l(0) + \int_0^t \frac{E[\ln X]}{\tilde{S}(s)} d(\lambda s), \quad t \geq 0, \tag{42}$$

where $\tilde{S} = \{\tilde{S}(t), t \geq 0\}$ is a Brownian motion, reflected at $\tilde{a} > 0$, with constant drift θ and infinitesimal variance 2λ, started at $\tilde{S}(0)$. Now let

$$E_1(x) = \int_x^\infty \frac{1}{y} e^{-y} dy, \tag{43}$$

denote the exponential integral function. We then have the following result.

Proposition 8.1. *If $\theta < 0$, then*

$$P\left(\lim_{t\to\infty} \frac{p^l(t)}{t} = -\theta E[\ln X] e^{-\theta\tilde{a}/\lambda} E_1(-\theta\tilde{a}/\lambda) \right) = 1.$$

Proof. In the case of $\theta < 0$, it is well known (see Section 5.4 of Harrison [13]) that $\tilde{S} = \{\tilde{S}(t), t \geq 0\}$ may be viewed as a regenerative process with finite expected regeneration times. One also has that $\tilde{S}(t) \Rightarrow \tilde{S}(\infty)$ as $t \to \infty$, where $\tilde{S}(\infty)$ is equal in distribution to $\tilde{a} + \exp(-\theta/\lambda)$, where $\exp(-\theta/\lambda)$ is an exponential random variable with rate $-\theta/\lambda$. Hence, $1/\tilde{S} = \{1/\tilde{S}(t), t \geq 0\}$ is also regenerative with finite expected regeneration time, and, by the continuous mapping theorem (see, for instance, Theorem 3.4.3 of Whitt [25]), $1/\tilde{S}(t) \Rightarrow 1/\tilde{S}(\infty)$. Hence, by Theorems 6.1.2 and 6.3.1 of Asmussen [1],

$$P\left(\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{\lambda E[\ln X]}{\tilde{S}(s)}ds = E\left[\frac{\lambda E[\ln X]}{\tilde{S}(\infty)}\right]\right) = 1.$$

Direct calculations and (43) then yield that

$$E\left[\frac{1}{\tilde{S}(\infty)}\right] = (-\theta/\lambda)e^{(-\theta/\lambda)\tilde{a}}E_1((-\theta/\lambda)\tilde{a}),$$

which, by (42), yields the result. □

We now make a few remarks in regard to Proposition 8.1. First note that the proposition implies that almost surely when $\theta < 0$,

$$p^l(t) = p^l(0) - \theta E[\ln X]e^{-\theta\tilde{a}/\lambda}E_1(-\theta\tilde{a}/\lambda)t + o(t), \tag{44}$$

which, since the actual limiting price $p(t) = \exp(p^l(t))$, implies that

$$p(t) = p(0)\exp\left(-\theta E[\ln X]e^{-\theta\tilde{a}/\lambda}E_1(-\theta\tilde{a}/\lambda)t\right)\exp(o(t)). \tag{45}$$

Next, we comment on the role that the market maker plays in our model. On an intuitive level, in the presence of a market maker who allows the size of the book to drop to low levels, or, in the extreme case, in the absence of any market maker at all, it stands to reason that the number of limit sell orders on the book will drop to low levels and, consequently, incoming market buy orders will tend to push the price upwards at a high rate. The market maker therefore plays an important role in moderating the rate of change of the price. It turns out that one may actually quantify the impact of the market maker in our model by first recalling the well-known asymptotic,

$$\lim_{x\to 0^+} \frac{e^{-2E_1(x)}}{x^2} = e^{2\gamma}, \tag{46}$$

where $\gamma = 0.57721...$ is the Euler-Mascheroni constant. By Proposition 8.1 above, (46) then implies that almost surely,

$$\lim_{\tilde{a}\to 0}\left(\frac{1}{-\theta E[\ln X]}\lim_{t\to\infty}\frac{p^l(t)}{t} + \exp(-\theta\tilde{a}/\lambda)\ln\left(-\frac{\theta\tilde{a}}{\lambda}\right)\right) = -\gamma. \tag{47}$$

Hence, one sees that the rate at which the rate of increase of the natural log of the price process increases with respect to the minimum market maker level $\tilde{a}$ is on the order of $\ln(-\theta\tilde{a}/\lambda)$.

We now move on towards obtaining second order refinements to (44) and (45). Let P_π denote the probability measure under which $\tilde{S}(0)$ is distributed according to the limiting distribution π of $\tilde{S}$, namely $\tilde{a} + \exp(-\theta/\lambda)$, where $\exp(-\theta/\lambda)$ is an exponential random variable with rate $-\theta/\lambda$. And, for $t \geq 0$, let $\mathrm{Cov}_\pi(1/\tilde{S}(0), 1/\tilde{S}(t))$ denote the covariance of $1/\tilde{S}(0)$ and $1/\tilde{S}(t)$ under P_π. Finally, set

$$\sigma^2 = 2\lambda^2(E[\ln X])^2\int_0^\infty \mathrm{Cov}_\pi(1/\tilde{S}(0), 1/\tilde{S}(u))du < \infty.$$

We then have the following result.

Proposition 8.2. *If $\theta < 0$, then*

$$\frac{p^l(t) - (-\theta E[\ln X] e^{-\theta\tilde{a}/\lambda} E_1(-\theta\tilde{a}/\lambda)t)}{t^{1/2}} \quad \Rightarrow \quad \sigma\mathcal{N}(0,1),$$

as $t \to \infty$, where $\mathcal{N}(0,1)$ is a standard normal random variable.

Proof. Let $\tau_{0,1} = \inf\{t \geq 0 : \tilde{S}(t) \geq 1\}$ and $\tau_{1,0} = \inf\{t \geq \tau_{0,1} : \tilde{S}(t) = 0\}$. Then, note that similar to as in Section 5.4 of Harrison [13], one has that started from $\tilde{S}(0) = 0$, the process $\tilde{S}(t) = \{\tilde{S}(t), t \geq 0\}$ may be viewed as a regenerative process with regeneration points distributed according to $\tau(1,0)$. Hence, $1/\tilde{S} = \{1/\tilde{S}(t), t \geq 0\}$ may be viewed as a regenerative process as well. Moreover, by Theorem 5.7.13 of Whitt [25], one has that both $E[\tau_{1,0}|\tilde{S}(0)] < \infty$ and $E[\tau_{1,0}^2|\tilde{S}(0)] < \infty$. The result then follows from Theorem 6.3.2 of Asmussen [1] and the fact that $1/\tilde{S}(t) \leq \tilde{a}$ for $t \geq 0$. □

Note that as far as the actual price process itself is concerned, Proposition 8.1 implies that

$$p(t) \quad = \quad p^l(0) \exp\left(-\theta E[\ln X] e^{-\theta\tilde{a}/\lambda} E_1(-\theta\tilde{a}/\lambda)t\right) \exp(\sigma\mathcal{N}(0,1)t^{1/2}) \exp(o(t^{1/2})).$$

This concludes our discussion of the case $\theta < 0$, we now move on to the case of $\theta > 0$.

8.2 Long Term Behavior for $\theta > 0$

In the case of $\theta > 0$, one has that for sufficiently large n, the arrival rate of limit orders to the book is larger than the arrival rate of market orders to the book. Thus, in this case, for sufficiently large n, the process S^n which keeps track of the total number of orders on the book is identical in distribution to $a^n + Q^n$, where Q^n is the number in system process for an $M/M/1$ queue with a utilization of $\rho^n = \lambda_L^n/\lambda_M^n > 1$. However, since an $M/M/1$ queue with a utilization of greater than one is transient and, since by Theorem 5.1 the rate of increase of the limiting prices process p is inversely related to the size of the book, this implies that the limiting price process will grow more slowly in the case of $\theta > 0$ than in the case of $\theta < 0$. Indeed, the following result bears this point out.

Proposition 8.3. *If $\theta > 0$, then*

$$P\left(\lim_{t\to\infty} \frac{p^l(t)}{\ln t} = \frac{\lambda E[\ln X]}{\theta}\right) \quad = \quad 1.$$

Proof. Recall from (42) that we may write

$$p^l(t) \quad = \quad p^l(0) + \int_0^t \frac{\lambda E[\ln X]}{\tilde{S}(s)} ds, \quad t \geq 0. \tag{48}$$

However, by Ito's Lemma (see, for instance, Theorem 3.3.3 of Karatzas and Shreve [17]) and by (16),

$$\ln(\tilde{S}(t)) \quad = \quad \ln(\tilde{S}(0)) + \int_0^t \frac{1}{\tilde{S}(s)} d(\sqrt{2\lambda}\tilde{B}(s)) + \int_0^t \frac{\theta}{\tilde{S}(s)} ds + \int_0^t \frac{1}{\tilde{S}(s)} d\tilde{L}(s)$$

$$-\int_0^t \frac{\lambda}{(\tilde{S}(s))^2} ds,$$

and hence,

$$\begin{aligned}\int_0^t \frac{\theta}{\tilde{S}(s)} ds &= \ln(\tilde{S}(t)) - \ln(\tilde{S}(0)) - \int_0^t \frac{1}{\tilde{S}(s)} d(\sqrt{2\lambda}\tilde{B}(s)) - \int_0^t \frac{1}{\tilde{S}(s)} d\tilde{L}(s) \\ &\quad + \int_0^t \frac{\lambda}{(\tilde{S}(s))^2} ds.\end{aligned} \tag{49}$$

We therefore obtain from (48) and (49) that

$$\begin{aligned}\frac{\theta}{\lambda E[\ln X]} \frac{p^l(t)}{\ln(t)} &= \frac{\theta}{\lambda E[\ln X]} \frac{p^l(0)}{\ln(t)} + \frac{\ln(\tilde{S}(t))}{\ln(t)} - \frac{\ln(\tilde{S}(0))}{\ln(t)} - \frac{1}{\ln(t)} \int_0^t \frac{1}{\tilde{S}(s)} d(\sqrt{2\lambda}\tilde{B}(s)) \\ &\quad - \frac{1}{\ln(t)} \int_0^t \frac{1}{\tilde{S}(s)} d\tilde{L}(s) + \frac{1}{\ln(t)} \int_0^t \frac{\lambda}{(\tilde{S}(s))^2} ds.\end{aligned} \tag{50}$$

Next recall that if $\theta > 0$, then $P(\tilde{L}(\infty) < \infty \mid \tilde{S}(0) = x) = 1$, for all $x \geq \tilde{a}$. Hence, by the strong law of large numbers for Brownian motion (see, for instance, problem 2.9.3 of Karatzas and Shreve [17]), it follows that given that $\tilde{S}(0) = x \geq \tilde{a}$, one has that almost surely,

$$\lim_{t\to\infty} \frac{\tilde{S}(t)}{t} = \lim_{t\to\infty} \left(\frac{x}{t} + \theta + \frac{\tilde{L}(t)}{t} + \sqrt{2\lambda} \frac{\tilde{B}(t)}{t} \right) = \theta > 0. \tag{51}$$

By (51), one has that almost surely both

$$\frac{\theta}{\lambda E[\ln X]} \frac{p^l(0)}{\ln(t)} + \frac{\ln(\tilde{S}(t))}{\ln(t)} - \frac{\ln(\tilde{S}(0))}{\ln(t)} \to 1 \tag{52}$$

and

$$\frac{1}{\ln(t)} \int_0^t \frac{\lambda}{(\tilde{S}(s))^2} ds \to 0, \tag{53}$$

as $t \to \infty$. Next, recall that $P(L(\infty) < \infty) = 1$, and so

$$\frac{1}{\ln(t)} \int_0^t \frac{1}{\tilde{S}(s)} d\tilde{L}(s) \to 0, \tag{54}$$

almost surely. For the completion of the proof, by (50), (52), (53), and (54) it is sufficient to show that

$$\lim_{t\to\infty} \int_0^t \frac{1}{\tilde{S}(u)} d\tilde{B}(u) \tag{55}$$

almost surely exists and is finite. Note that

$$\left\{ \int_0^t \frac{1}{\tilde{S}(s)} d(\sqrt{2\lambda}\tilde{B}(s)), t \geq 0 \right\}$$

is a martingale with quadratic variation process

$$\left\{ \int_0^t \frac{1}{(\tilde{S}(s))^2} d(2\lambda s), t \geq 0 \right\}.$$

By Revuz and Yor [21], Chapter V, Proposition 1.8, the limit in the expression (55) exists and is finite if and only if

$$\int_0^\infty \frac{1}{(\tilde{S}(u))^2} du \quad < \quad \infty, \quad \text{P-a.s.} \tag{56}$$

However, (56) follows from (51). This completes the proof. □

Recalling that $p^l(t) = \ln p(t)$ for $t \geq 0$, and comparing Propositions 8.1 and 8.3 side-by-side, one may now clearly see that the rate of increase of the limiting price process p is faster when $\theta < 0$, than when $\theta > 0$. In fact, p grows like $\exp(t)$ when $\theta < 0$, while only growing according to $t^{\lambda E[\ln X]/\theta}$ when $\theta > 0$. However, Proposition 8.3 does not identify the specific factor in front of the $\mathcal{O}(t^{\lambda E[\ln X]/\theta})$ growth term when $\theta > 0$. Nevertheless, upon closer inspection, one may actually see that the proof of Proposition 8.3 yields the following sharper result on the long term behavior of the limiting price process itself when $\theta > 0$.

Proposition 8.4. *If $\theta > 0$, then*

$$P\left(\lim_{t\to\infty} \frac{p(t)}{t^{\lambda E[\ln X]/\theta}} = p(0) \left(\frac{\theta}{\tilde{S}(0)}\right)^{\lambda E[\ln X]/\theta} R \right) \quad = \quad 1,$$

where the random variable R is given by

$$R \quad = \quad \exp\left(-\frac{\lambda E[\ln X]}{\theta} \left(\int_0^\infty \frac{1}{\tilde{S}(s)} d(\sqrt{2\lambda}\tilde{B}(s)) - \int_0^\infty \frac{\lambda}{(\tilde{S}(s))^2} ds + \int_0^\infty \frac{1}{\tilde{S}(s)} d\tilde{L}(s) \right) \right).$$

Note that in above result, there could be some dependence between R and the factor lying in front it.

Proof. Recalling that $p(t) = \exp(p^l(t))$ and using (50) one obtains that

$$\begin{aligned} p(t) \quad &= \quad p(0) \left(\frac{\tilde{S}(t)}{\tilde{S}(0)} \right)^{\lambda E[\ln X]/\theta} \\ &\quad \exp\left(-\frac{\lambda E[\ln X]}{\theta} \left(\int_0^t \frac{1}{\tilde{S}(s)} d(\sqrt{2\lambda}\tilde{B}(s)) - \int_0^t \frac{\lambda}{(\tilde{S}(s))^2} ds + \int_0^t \frac{1}{\tilde{S}(s)} d\tilde{L}(s) \right) \right). \end{aligned}$$

Dividing the above by $t^{\lambda E[\ln X]/\theta}$, recalling from (51) that $\tilde{S}(t)/t \to \theta$ almost surely and proceeding as in the proof of Proposition 8.3 yields the result. □

For our final result of this section, we characterize the long term behavior of the total value of the limiting order book. In particular, recall that in the limit, the total value of the book is given by $p(t)\tilde{S}(t)$. Hence, essentially the same proof as that of Proposition 8.4 yields the following.

Proposition 8.5. *If $\theta > 0$, then*

$$P\left(\lim_{t\to\infty} \frac{p(t)\tilde{S}(t)}{t^{\lambda E[\ln X]/\theta+1}} = p(0)\theta \left(\frac{\theta}{\tilde{S}(0)}\right)^{\lambda E[\ln X]/\theta} R \right) \quad = \quad 1,$$

where R is as given in Proposition 8.4.

8.3 Long Term Behavior for $\theta = 0$

In the case when $\theta = 0$, it is more difficult to obtain the exact asymptotics of the long term behavior of the limiting price process p, as compared to the previous two cases of $\theta < 0$ and $\theta > 0$. However, we are still able to obtain tight upper and lower bounds on the rate of growth of the limiting price process. The proof of our main result in this subsection relies on the following generalization of the l'Hopital rule. The proof of this generalization is quite similar to the proof of the l'Hopital rule itself and so we omit the details.

Lemma 8.6. *Let $f, g \in C^1(\mathbb{R})$ be such that for every $t \in (0, \infty)$, we have that $g(t) \neq 0$ and $g'(t) \neq 0$ and such that $\lim_{t\to\infty} |f(t)| = \lim_{t\to\infty} |g(t)| = \infty$. Then*

$$\liminf_{t\to\infty} \frac{f'(t)}{g'(t)} \;\leq\; \liminf_{t\to\infty} \frac{f(t)}{g(t)} \;\leq\; \limsup_{t\to\infty} \frac{f(t)}{g(t)} \;\leq\; \limsup_{t\to\infty} \frac{f'(t)}{g'(t)}.$$

The following is now the main result of this subsection.

Proposition 8.7. *If $\theta = 0$, then for every $\gamma > 0$,*

$$P\left(\lim_{t\to\infty} \frac{p^l(t)}{t^{\frac{1}{2}+\gamma}} = 0\right) \;=\; 1 \tag{57}$$

and

$$P\left(\lim_{t\to\infty} \frac{p^l(t)}{t^{\frac{1}{2}-\gamma}} = \infty\right) \;=\; 1. \tag{58}$$

Proof. We start with proving (58). For each $t \geq 0$, let

$$l(t) \;=\; \sqrt{2t \ln \ln t}.$$

Then, by Lemma 8.6, (42), and the Law of Iterated Logarithm (see, for instance, Theorem 2.9.23 of Karatzas and Shreve [17]), we have that P-a.s.,

$$\liminf_{t\to\infty} \frac{p^l(t)}{t^{\frac{1}{2}-\gamma}} \;\geq\; \liminf_{t\to\infty} \frac{\lambda E[\ln X]}{\frac{1}{2}-\gamma} \frac{t^{\frac{1}{2}+\gamma}}{\tilde{S}(t)} \;\geq\; \frac{\lambda E[\ln X]}{\frac{1}{2}-\gamma} \liminf_{t\to\infty} \frac{l(t)}{\tilde{S}(t)} \times \liminf_{t\to\infty} \frac{t^{\frac{1}{2}+\gamma}}{l(t)} \;=\; \infty.$$

This completes our proof of (58).

Next, we proceed to prove (57). Let $f(x) = x\left((\ln x) - 1\right)$, and hence $f'(x) = \ln x$ and $f''(x) = 1/x$. Then, recalling the representation of $\tilde{S}(t)$ given in (16), substituting $\theta = 0$ and using Ito's rule (see, for instance, Theorem 3.3.3 of Karatzas and Shreve [17]), we have that for each $t \geq 0$,

$$f(\tilde{S}(t)) = f(\tilde{S}(0)) + \int_0^t \ln \tilde{S}(u) d\tilde{L}(u) + \int_0^t \sqrt{2\lambda} \ln \tilde{S}(u) d\tilde{B}(u) + \lambda \int_0^t \frac{1}{\tilde{S}(u)} du.$$

Taking (17) into consideration, this becomes

$$f(\tilde{S}(t)) = f(\tilde{S}(0)) + (\ln \tilde{a})\tilde{L}(t) + \int_0^t \sqrt{2\lambda} \ln \tilde{S}(u) d\tilde{B}(u) + \lambda \int_0^t \frac{1}{\tilde{S}(u)} du,$$

which may be rearranged to yield

$$\int_0^t \frac{1}{\tilde{S}(u)}du = \frac{1}{\lambda}\left[f(\tilde{S}(t)) - f(\tilde{S}(0)) - (\ln \tilde{a})\tilde{L}(t)\right] - \sqrt{\frac{2}{\lambda}}\int_0^t \ln \tilde{S}(u)d\tilde{B}(u). \tag{59}$$

Formulas (42) and (59) now imply that

$$p^l(t) = p^l(0) + E[\ln X]\left[f(\tilde{S}(t)) - f(\tilde{S}(0)) - (\ln \tilde{a})\tilde{L}(t) - \sqrt{2\lambda}\int_0^t \ln \tilde{S}(u)d\tilde{B}(u)\right]. \tag{60}$$

Now, let $w = \{w(t), t \geq 0\}$ be (on a possibly different probability space) a standard Brownian motion starting at

$$w(0) = \frac{\tilde{S}(0) - \tilde{a}}{\sqrt{2\lambda}} > 0.$$

Let $L = \{L(t), t \geq 0\}$ be the local time at zero corresponding to this Brownian motion, and set

$$\bar{B}(t) = \int_0^t \text{sign}\,(w(s))\,dw(s), \quad t \geq 0.$$

Then, notice that the three dimensional processes

$$\left\{\left(\frac{\tilde{S}(t) - \tilde{a}}{\sqrt{2\lambda}}, \frac{\tilde{L}(t)}{\sqrt{2\lambda}}, \tilde{B}(t)\right),\ t \geq 0\right\} \quad \text{and} \quad \left\{\left(|w(t)|, L(t), \bar{B}(t)\right),\ t \geq 0\right\} \tag{61}$$

have the same law. In order to see this, we cast the representation of $\tilde{S}(t)$ in (16) in the form

$$\frac{\tilde{S}(t) - \tilde{a}}{\sqrt{2\lambda}} = \frac{\tilde{S}(0) - \tilde{a}}{\sqrt{2\lambda}} + \tilde{B}(t) + \frac{1}{\sqrt{2\lambda}}\tilde{L}(t) \tag{62}$$

and also consider the Tanaka formula (see, for instance, Proposition 3.6.8 of Karatzas and Shreve [17])

$$|w(t)| = \frac{\tilde{S}(0) - \tilde{a}}{\sqrt{2\lambda}} + \bar{B}(t) + L(t), \quad t \geq 0. \tag{63}$$

The identity of the laws of the two processes in (61) follows from (62), (63), and Lemma 3.6.14 of Karatzas and Shreve [17]. Thus $\{p^l(t),\ t \geq 0\}$ has the same law as $\{\bar{p}^l(t),\ t \geq 0\}$, where

$$\bar{p}^l(t) = p^l(0) + E[\ln X]\Big(f(\sqrt{2\lambda}|w(t)| + \tilde{a}) - f(\tilde{S}(0)) - (\ln \tilde{a})\sqrt{2\lambda}L(t) - \tag{64}$$
$$\sqrt{2\lambda}\int_0^t \ln(\sqrt{2\lambda}|w(u)| + \tilde{a})\text{sign}\,(w(u))\,dw(u)\Big), \quad t \geq 0.$$

For brevity, we now introduce the notations

$$U(t) = \sqrt{2\lambda}|w(u)| + \tilde{a}, \quad t \geq 0,$$

and

$$M(t) = \int_0^t \ln(\sqrt{2\lambda}|w(u)| + \tilde{a})\text{sign}\,(w(u))\,dw(u), \quad t \geq 0, \tag{65}$$

and we note that the process M is a continuous local martingale with quadratic variation

$$\langle M \rangle_t \;=\; \int_0^t (\ln U(s))^2 ds, \quad t \geq 0.$$

Using (64), in order to prove (57) it is sufficient to show that P-a.s.,

$$\lim_{t\to\infty} \frac{L(t)}{t^{1/2+\gamma}} \;=\; 0, \tag{66}$$

$$\lim_{t\to\infty} \frac{f(U(t))}{t^{1/2+\gamma}} \;=\; 0 \tag{67}$$

and

$$\lim_{t\to\infty} \frac{M(t)}{t^{1/2+\gamma}} \;=\; 0. \tag{68}$$

We proceed in order. Formula (66) follows from (63) and the Law of Iterated Logarithm (see, for instance, Theorem 2.9.23 of Karatzas and Shreve [17]). In order to show (67), first we show that P-a.s.,

$$\limsup_{t\to\infty} \left| \frac{\ln U(t)}{\ln l(t)} \right| \;\leq\; 1. \tag{69}$$

Indeed, by the Law of Iterated Logarithm (see, for instance, Theorem 2.9.23 of Karatzas and Shreve [17]),

$$\limsup_{t\to\infty} \frac{U(t)}{l(t)} \;=\; \sqrt{2\lambda}$$

and hence

$$\limsup_{t\to\infty} \ln l(t) \left(\frac{\ln U(t)}{\ln l(t)} - 1 \right) \;=\; \ln \sqrt{2\lambda},$$

which implies that

$$\limsup_{t\to\infty} \frac{\ln U(t)}{\ln l(t)} \;\leq\; 1. \tag{70}$$

Inequality (70) and the obvious $\liminf_{t\to\infty} \ln(U(t))/\ln l(t) = 0$ now imply (69). Now we show (67). By the definition of $f(x)$, we have for each $t \geq 0$,

$$\frac{|f(U(t))|}{t^{\gamma+1/2}} \;=\; \frac{U(t)}{t^{(\gamma+1)/2}} \frac{|\ln U(t) - 1|}{t^{\gamma/2}} \;\leq\; \frac{U(t)}{t^{(\gamma+1)/2}} \left(\frac{|\ln U(t)|}{|\ln l(t)|} \times \frac{|\ln l(t)|}{t^{\gamma/2}} + \frac{1}{t^{\gamma/2}} \right). \tag{71}$$

However, by the Law of Iterated Logarithm (see, for instance, Theorem 2.9.23 of Karatzas and Shreve [17]), we have that P-a.s.,

$$\lim_{t\to\infty} \frac{U(t)}{t^{(\gamma+1)/2}} \;=\; 0.$$

Together with (69), this now implies that the right-hand side of (71) converges to zero as $t \to \infty$, and so (67) now follows.

In order to complete the proof, it now remains to show that (68) holds. Let

$$h(t) = \int_3^t (\ln l(s))^2 \, ds, \quad t \geq 0.$$

By Exercise 1.15 of Chapter V of Revuz and Yor [21], Lemma 8.6 and (69), we have that for every $\delta > 0$,

$$\begin{aligned}\limsup_{t\to\infty} \frac{|M(t)|}{(h(t))^{\delta+\frac{1}{2}}} &= \limsup_{t\to\infty} \frac{|M(t)|}{l(\langle M\rangle_t)} \times \frac{l(\langle M\rangle_t)}{(h(t))^{\delta+\frac{1}{2}}} \leq \limsup_{t\to\infty} \frac{l(\langle M\rangle_t)}{(h(t))^{\delta+\frac{1}{2}}} \\ &\leq \sqrt{2}\left(\limsup_{t\to\infty} \frac{\langle M\rangle_t}{h(t)}\right)^{\frac{1}{2}+\delta} \leq \sqrt{2}\left(\limsup_{t\to\infty} \frac{(\ln U(t))^2}{(\ln l(t))^2}\right)^{\frac{1}{2}+\delta} \leq \sqrt{2}. \end{aligned} \tag{72}$$

Then, note that for each $t \geq 0$, we may write

$$\frac{|M(t)|}{t^{\frac{1}{2}+\gamma}} = \frac{|M(t)|}{(h(t))^{\frac{(1+\gamma)}{2}}} \times \left(\frac{h(t)}{t^{1+\epsilon}}\right)^{\frac{1+\gamma}{2}}, \tag{73}$$

where $\epsilon = \frac{\gamma}{\gamma+1}$. Moreover, using the l'Hopital rule we have that

$$\lim_{t\to\infty} \frac{h(t)}{t^{1+\epsilon}} = \lim_{t\to\infty} \frac{(\ln l(t))^2}{(1+\epsilon)t^{\epsilon}} = 0. \tag{74}$$

(68) now follows from (72), (73) and (74). This completes the proof of (57), which completes the proof.

□

9 Appendix

In the Appendix, we provide the proofs of several key results needed for the proofs of Theorems 5.1 and 6.1 found in the main body of the paper. In particular, in Subsection 9.1 we provide the proof of Proposition 9.1, which is then used in the proof of Theorem 5.1 of Section 5. Next, in Subsection 9.2, we provide the proof of Lemma 9.2, which is used as a supporting lemma in the proof of Proposition 9.1. Finally, in Subsection 9.3, we provide the proof of Proposition 9.14, which is then used in the proof of Theorem 6.1 of Section 6.

As it turns out, we will find it convenient at many points throughout the Appendix to work with the natural log of the measure-valued order book process and the natural log of the price process. Specifically, for each $t \geq 0$, let $\mu^l(t) \in \mathcal{M}_F(\mathbb{R})$ be the measure defined by setting

$$\mu^l(t)((-\infty, b]) = \mu(t)((-\infty, e^b]), \quad b \in \mathbb{R}_+.$$

We then define $\mu^l = \{\mu^l(t), t \geq 0\}$ to be the natural log of the measure-valued order book process. In a similar manner, setting

$$p^l(t) = \sup\{x \in \mathbb{R} : \mu^l(t)((-\infty, x)) = 0\}, \quad t \geq 0, \tag{75}$$

we define $p^l = \{p^l(t), t \geq 0\}$ to be the natural log of the price process. It is then straightforward using (9) of Section 2 to show that for each $\varphi \in C_b(\mathbb{R})$, one has that μ^l and p^l obey the dynamics

$$\langle \mu^l(t), \varphi \rangle = \langle \mu^l(0), \varphi \rangle + \sum_{i=1}^{N_L(t)} (\varphi(p^l(\tau_i^L-) + \ln(X_i)) - \varphi(p^l(\tau_i^L-))) + \int_0^t \varphi(p^l(s-))dS(s), \quad (76)$$

for $t \geq 0$. The system equation (76) will be heavily relied upon throughout the Appendix. We also note that all quantities associated with μ^l and p^l in the nth system of the high frequency regime will be denoted by a superscript n. For instance, $p^{l,n}(0)$ represents the natural log of the price at time $t = 0$ in the nth system. Moreover, the diffusion scaled natural log of the limit order book in the nth system at time $t \geq 0$ is given by $\tilde{\mu}^{l,n}(t)(\mathcal{A}) = n^{-1/2}\mu^{l,n}(t)(\mathcal{A})$ for each $\mathcal{A} \in \mathcal{B}(\mathbb{R})$.

9.1 Proof of Proposition 9.1

As mentioned above, our main goal in this subsection is prove Proposition 9.1 below. In order to state this proposition, and also in preparation for our supporting lemmas in this Subsection, we first need to setup some additional notation. First recall from Section 7 the definition of $\xi(x) = x$ as the identity function on $\mathbb{R}$. Then, for each $n \geq 1$, set

$$\Delta\tilde{\mu}^{l,n}(t) = \langle \tilde{\mu}^{l,n}(t), \xi \rangle - \tilde{S}^n(t)p^{l,n}(t), \quad t \geq 0, \quad (77)$$

and define the process $\Delta\tilde{\mu}^{l,n} = \{\Delta\tilde{\mu}^{l,n}(t), t \geq 0\}$. Note, in particular, that $\Delta\tilde{\mu}^{l,n}(0)$ as defined in (77) above coincides with $\Delta\tilde{\mu}^{l,n}(0)$ as defined in (18) of Section 5. We then have the following result.

Proposition 9.1. *If $E[\ln X] > 0$ and $E[(\Delta\tilde{\mu}^{l,n}(0))^2] \to 0$ and $(p^{l,n}(0), \tilde{S}^n(0)) \Rightarrow (p^l(0), \tilde{S}(0))$ as $n \to \infty$, and if Items 1-4 of Definition 3.1 hold, then $p^{l,n} \Rightarrow p^l$ as $n \to \infty$, where the process $p^l = \{p^l(t), t \geq 0\}$ is given by*

$$p^l(t) = p^l(0) + \int_0^t \frac{E[\ln X]}{\tilde{S}(s)} d(\lambda s), \quad t \geq 0, \quad (78)$$

and $\tilde{S} = \{\tilde{S}(t), t \geq 0\}$ is a Brownian motion, reflected at $\tilde{a} > 0$, with constant drift θ and infinitesimal variance 2λ, started at $\tilde{S}(0)$.

We now begin to prove Proposition 9.1 by stating a lemma which is of critical importance. For the remainder of this section, until otherwise noted, we assume that $T \geq 0$ is fixed. Then, for each $n \geq 1$ and $\varepsilon > 0$, define

$$\nu_\varepsilon^n = \inf\{t \geq 0 : |p^{l,n}(t) - p^{l,n}(0)| \geq \varepsilon\} \wedge T \quad (79)$$

to be the first time that the absolute value of the difference between the natural log of the price at time $t \geq 0$ and the natural log of the price at time $t = 0$ is at least as great as ε (or T, whichever is smaller). Note that ν_ε^n is a stopping time with respect to the price process and that almost surely one has that $\nu_\varepsilon^n \leq \nu_{\varepsilon+\delta}^n$ for each $\delta \geq 0$. We now have the following result. Its proof is rather long and technical and consequently has been deferred until Subsection 9.2.

Lemma 9.2. *If $E[\ln X] > 0$ and $E[(\Delta\tilde{\mu}^{l,n}(0))^2] \to 0$ as $n \to \infty$, and if Items 1-4 of Definition 3.1 hold, then, for each $\varepsilon > 0$,*

$$E\left[\sup_{0\le t\le \nu_\varepsilon^n}(\Delta\tilde{\mu}^{l,n}(t))^2\right] \;\to\; 0 \;\; \textit{as} \;\; n\to\infty.$$

Proof. See Subsection 9.2 of the Appendix. □

We next provide a quick result which is needed in preparation for the proof of the upcoming Proposition 9.5. We have the following.

Lemma 9.3. *If Items 1-4 of Definition 3.1 hold, then, for each $T \geq 0$,*

$$E\left[\sup_{0\le t\le T}\left(\frac{1}{n}\sum_{i=1}^{N_L^n(t)}(\ln(X_i)-E[\ln X])\right)^2\right] \;\to\; 0 \;\; \textit{as} \;\; n\to\infty.$$

Proof. First note that the process

$$\left\{\frac{1}{n}\sum_{i=1}^{N_L^n(t)}(\ln(X_i)-E[\ln X]), t\geq 0\right\}$$

is a martingale. Hence, by Doob's martingale inequality (see, for instance, Theorem 1.3.8 of Karatzas and Shreve [17]), we have that for each $T \geq 0$,

$$E\left[\sup_{0\le t\le T}\left(\frac{1}{n}\sum_{i=1}^{N_L^n(t)}(\ln(X_i)-E[\ln X])\right)^2\right] \;\le\; 4E\left[\left(\frac{1}{n}\sum_{i=1}^{N_L^n(T)}(\ln(X_i)-E[\ln X])\right)^2\right].$$

However, since the sequence $\{\ln X_i, i \geq 1\}$ is by assumption i.i.d. with finite mean, and independent of N_L^n, it follows that

$$E\left[\left(\frac{1}{n}\sum_{i=1}^{N_L^n(T)}(\ln(X_i)-E[\ln X])\right)^2 |N_L^n(T)\right] \;=\; \frac{1}{n^2}N_L^n(T)\mathrm{Var}(\ln X).$$

This then implies that

$$\begin{aligned} E\left[\left(\frac{1}{n}\sum_{i=1}^{N_L^n(T)}(\ln(X_i)-E[\ln X])\right)^2\right] \;&=\; \frac{1}{n^2}\lambda_L^n T\mathrm{Var}(\ln X) \\ &\to\; 0 \;\; \text{as} \;\; n\to\infty, \end{aligned}$$

where the final convergence follows from Item 1 of Definition 3.1. This complete the proof. □

Now, for each $n \geq 1$, define the centered natural log of the price by setting

$$\Delta p^{l,n}(t) \;=\; p^{l,n}(t) - \left(p^{l,n}(0) + \int_0^t \frac{E[\ln X]}{\tilde{S}^n(s-)}d\bar{N}_L^n(s)\right), \;\; t\geq 0, \tag{80}$$

and define the process $\Delta p^{l,n} = \{\Delta p^{l,n}(t), t \geq 0\}$. Proposition 9.5 below essentially shows that for each $\varepsilon > 0$, up until the stopping time ν_ε^n, the process $\Delta p^{l,n}$ weakly converges to 0 as n tends to ∞. The significance of this convergence can be attributed to the following result.

Proposition 9.4. *If $(p^{l,n}(0), \tilde{S}^n(0)) \Rightarrow (p^l(0), \tilde{S}(0))$ as $n \to \infty$, and if Items 1-4 of Definition 3.1 hold, then*

$$\left(p^{l,n}(0) + \int_0^e \frac{E[\ln X]}{\tilde{S}^n(s-)} d\bar{N}_L^n(s), \tilde{S}^n\right) \quad \Rightarrow \quad (p^l, \tilde{S}) \quad \textit{in} \quad D^2([0,\infty), \mathbb{R}) \quad \textit{as} \quad n \to \infty,$$

where $\tilde{S} = \{\tilde{S}(t), t \geq 0\}$ is a Brownian motion, reflected at $\tilde{a} > 0$, with constant drift θ and infinitesimal variance 2λ, started at $\tilde{S}(0)$.

Proof. First note that for each $n \geq 1$, we may write

$$(p^{l,n}(0), \tilde{S}^n - \tilde{a}^n) = \Upsilon(p^{l,n}(0), (\tilde{S}^n(0) - \tilde{a}^n) + n^{1/2}(n^{-1}\lambda_L^n - n^{-1}\lambda_M^n)e + \tilde{M}^n), \quad n \geq 1, \tag{81}$$

where, for $(x, z) \in \mathbb{R} \times D([0,\infty), \mathbb{R})$, the map $\Upsilon : \mathbb{R} \times D([0,\infty), \mathbb{R}) \mapsto \mathbb{R} \times D([0,\infty), \mathbb{R})$ is defined by setting $\Upsilon(x, z) = (x, \Phi(z))$, where the map Φ is as defined in Theorem 6.1 of Chen and Yao [5]. Now recall that the map Φ is continuous at $z \in C([0,\infty), \mathbb{R})$. Hence, it follows that Υ is a continuous map at $(x, z) \in \mathbb{R} \times C([0,\infty), \mathbb{R})$. Next, note that since $\tilde{M}^n$ is independent of $(p^{l,n}(0), \tilde{S}^n(0))$, it follows as in the proof of Proposition 4.2 that $(p^{l,n}(0), (\tilde{S}^n(0) - \tilde{a}^n) + n^{1/2}(n^{-1}\lambda_L^n - n^{-1}\lambda_M^n)e + \tilde{M}^n) \Rightarrow (p^l(0), (\tilde{S}(0) - \tilde{a}) + \theta e + \sqrt{2\lambda}\tilde{X})$ as $n \to \infty$, where $\tilde{X}$ is a standard Brownian motion. Hence, by (81) and the continuous mapping theorem (see, for instance, Theorem 3.4.3 of Whitt [25]), we now obtain that $(p^{l,n}(0), \tilde{S}^n - \tilde{a}^n) \Rightarrow (p^l(0), \Phi((\tilde{S}(0) - \tilde{a}) + \theta e + \sqrt{2\lambda}\tilde{X}))$ as $n \to \infty$. However, since by Item 4 of Definition 3.1 we have that $\tilde{a}^n \to \tilde{a}$ as $n \to \infty$, this now implies that

$$(p^{l,n}(0), \tilde{S}^n) \quad \Rightarrow \quad (p^l(0), \tilde{S}) \quad \text{in} \quad \mathbb{R} \times D([0,\infty), \mathbb{R}) \quad \text{as} \quad n \to \infty, \tag{82}$$

where $\tilde{S} = \{\tilde{S}(t), t \geq 0\}$ is a Brownian motion, reflected at $\tilde{a} > 0$, with constant drift θ and infinitesimal variance 2λ, started at $\tilde{S}(0)$. Next, note that by Theorem 5.10 of Chen and Yao [5], the random time change theorem (see, for instance, page 151 of Billingsley [4]) and Item 1 of Definition 3.1, it follows that $\bar{N}_L^n \Rightarrow \lambda e$ as $n \to \infty$. Hence, by (82), we now have that

$$(p^{l,n}(0), \tilde{S}^n, \bar{N}_L^n) \quad \Rightarrow \quad (p^l(0), \tilde{S}, \lambda e) \quad \text{in} \quad \mathbb{R} \times D^2([0,\infty), \mathbb{R}) \quad \text{as} \quad n \to \infty.$$

Now, by the Skorokhod representation theorem (see, for instance, Theorem 6.7 of Billingsley [4]), there exists a sequence of processes

$$\{(\hat{p}^{l,n}(0), \hat{S}^n, \hat{N}_L^n), n \geq 1\},$$

defined on an alternative probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$, such that

$$(p^{l,n}(0), \tilde{S}^n, \bar{N}_L^n) \quad \stackrel{d}{=} \quad (\hat{p}^{l,n}(0), \hat{S}^n, \hat{N}_L^n), \quad n \geq 1, \tag{83}$$

and

$$(\hat{p}^{l,n}(0), \hat{S}^n, \hat{N}_L^n) \quad \to \quad (\hat{p}^l(0), \hat{S}, \lambda e) \quad \text{as} \quad n \to \infty, \quad \hat{P}\text{−a.s.}, \tag{84}$$

where

$$(\hat{p}^l(0), \hat{S}, \lambda e) \quad \stackrel{d}{=} \quad (p^l(0), \tilde{S}, \lambda e). \tag{85}$$

Next, recall that for each $n \geq 1$, we have that $\tilde{S}^n(t) \geq \tilde{a}^n > 0$ for each $t \geq 0$, and, that by Item 4 of Definition 3.1, we have that $\tilde{a}^n \to \tilde{a} > 0$ as $n \to \infty$. Hence, by (84) and Lemma 8.3 of Dai and Dai [10], it follows that

$$\left(\hat{p}^{l,n}(0) + \int_0^e \frac{E[\ln X]}{\hat{S}^n(s-)} d\hat{N}_L^n(s), \hat{S}^n\right) \to \left(\hat{p}^l(0) + \int_0^e \frac{E[\ln X]}{\hat{S}(s)} d(\lambda s), \hat{S}\right), \quad \hat{P}-a.s.,$$

as $n \to \infty$. However, by (83),

$$\left(p^{l,n}(0) + \int_0^e \frac{E[\ln X]}{\tilde{S}^n(s-)} d\bar{N}_L^n(s), \tilde{S}^n\right) \stackrel{d}{=} \left(\hat{p}^{l,n}(0) + \int_0^e \frac{E[\ln X]}{\hat{S}^n(s-)} d\hat{N}_L^n(s), \hat{S}^n\right), \quad n \geq 1,$$

and, similarly, by (85),

$$\left(p^l(0) + \int_0^e \frac{E[\ln X]}{\tilde{S}(s)} d(\lambda s), \tilde{S}\right) \stackrel{d}{=} \left(\hat{p}^l(0) + \int_0^e \frac{E[\ln X]}{\hat{S}(s)} d(\lambda s), \hat{S}\right).$$

This completes the proof. □

We are now in a position to prove Proposition 9.5, which actually turns out to be very close to our main result of this Subsection, Proposition 9.1. We have the following.

Proposition 9.5. *If $E[\ln X] > 0$ and $E[(\Delta\tilde{\mu}^{l,n}(0))^2] \to 0$ as $n \to \infty$, and if Items 1-4 of Definition 3.1 hold, then, for each $\varepsilon > 0$,*

$$E\left[\sup_{0 \leq t \leq \nu_\varepsilon^n} (\Delta p^{l,n}(t))^2\right] \to 0 \quad as \quad n \to \infty.$$

Proof. First note that by (76), one has that for each $n \geq 1$,

$$\langle \tilde{\mu}^{l,n}(t), \xi \rangle = \langle \tilde{\mu}^{l,n}(0), \xi \rangle + \frac{1}{n} \sum_{i=1}^{N_L^n(t)} \ln(X_i) + \int_0^t p^{l,n}(s-) d\tilde{S}^n(s), \quad t \geq 0. \tag{86}$$

Next, note that integrating-by-parts, one may write

$$\begin{aligned} \tilde{S}^n(t) \int_0^t \frac{E[\ln X]}{\tilde{S}^n(s-)} d\bar{N}_L^n(s) &= E[\ln X]\bar{N}_L^n(t) + \int_0^t \left(\int_0^{s-} \frac{E[\ln X]}{\tilde{S}^n(u-)} d\bar{N}_L^n(u)\right) d\tilde{S}^n(s) \\ &\quad + \frac{1}{n^{1/2}} \int_0^t \frac{E[\ln X]}{\tilde{S}^n(s-)} d\bar{N}_L^n(s). \end{aligned} \tag{87}$$

Now subtracting (87) from (86) and using the definition of $\Delta\tilde{\mu}^{l,n}(t)$ in (77) and $\Delta p^{l,n}(t)$ in (80), one may then verify after some algebra the remarkably simple relation

$$\begin{aligned} \tilde{S}^n(t)\Delta p^{l,n}(t) &= \Delta\tilde{\mu}^{l,n}(0) - \Delta\tilde{\mu}^{l,n}(t) + \frac{1}{n} \sum_{i=1}^{N_L^n(t)} (\ln(X_i) - E[\ln X]) \\ &\quad - \frac{1}{n^{1/2}} \int_0^t \frac{E[\ln X]}{\tilde{S}^n(s-)} d\bar{N}_L^n(s) + \int_0^t \Delta p^{l,n}(s-) d\tilde{S}^n(s). \end{aligned} \tag{88}$$

Now, for each $n \geq 1$, let

$$\begin{aligned} & \tilde{Z}^n(t) \quad (89) \\ = & \frac{1}{\tilde{S}^n(t)} \left(\Delta\tilde{\mu}^{l,n}(0) - \Delta\tilde{\mu}^{l,n}(t) + \frac{1}{n} \sum_{i=1}^{N_L^n(t)} (\ln(X_i) - E[\ln X]) - \frac{1}{n^{1/2}} \int_0^t \frac{E[\ln X]}{\tilde{S}^n(s-)} d\bar{N}_L^n(s) \right), \end{aligned}$$

for $t \geq 0$, and note that by (88) we may write

$$\Delta p^{l,n}(t) = \tilde{Z}^n(t) + \frac{1}{\tilde{S}^n(t)} \int_0^t \Delta p^{l,n}(s-) d\tilde{S}^n(s). \quad (90)$$

We now use (90) in order to prove our main result. At a high level, our approach is to use an iterative scheme which is fairly standard. Nevertheless, we provide the details below.

We begin by defining a sequence of stopping times which will be useful in several localization arguments throughout the proof. Recall by (11) that for each $n \geq 1$, we have the decomposition

$$\tilde{S}^n(t) = \tilde{S}^n(0) + \tilde{M}^n(t) + \tilde{B}^n(t), \quad t \geq 0. \quad (91)$$

Next, let $||\tilde{B}^n||_{TV}(t)$ denote the total variation of the process $\tilde{B}^n = \{\tilde{B}^n(t), t \geq 0\}$ over the interval $[0, t]$, and note that by (13) we may write

$$||\tilde{B}^n||_{TV}(t) = \sqrt{n}|n^{-1}\lambda_L^n - n^{-1}\lambda_M^n|t + \tilde{L}^n(t).$$

Now, for each $\delta > 0$, we introduce the stopping time

$$\tau_\delta^n = \inf\{t \geq 0 : ||\tilde{B}^n||_{TV}(t) + \bar{N}_L^n(t) + \bar{N}_M^n(t)) \geq \delta\} \wedge T,$$

and note that since

$$||\tilde{B}^n||_{TV}(t) - ||\tilde{B}^n||_{TV}(t-) \leq n^{-1/2},$$

and

$$(\bar{N}_L^n(t) + \bar{N}_M^n(t)) - (\bar{N}_L^n(t-) + \bar{N}_M^n(t-)) \leq n^{-1},$$

it follows that

$$||\tilde{B}^n||_{TV}(\tau_\delta^n) + (\bar{N}_L^n(\tau_\delta^n) + \bar{N}_M^n(\tau_\delta^n)) - \delta \leq \frac{2}{\sqrt{n}}. \quad (92)$$

The inequality (92) will be useful as we proceed.

Let $\varepsilon > 0$ be fixed for the remainder of the proof. We now proceed to show that there exists a $\delta_0 > 0$, which is independent of both n and ε, and such that for each $0 \leq \delta < \delta_0$, we have the convergence

$$E\left[\sup_{0 \leq t \leq \tau_\delta^n \wedge \nu_\varepsilon^n} \Delta(p^{l,n}(t))^2 \right] \to 0 \quad \text{as} \quad n \to \infty. \quad (93)$$

The convergence (93) will then be used as the base case in an induction argument in order to prove that for each $k \geq 1$, we have for $0 \leq \delta < \delta_0$, the convergence

$$E\left[\sup_{0\leq t\leq \tau^n_{k\delta}\wedge\nu^n_\varepsilon} \Delta(p^{l,n}(t))^2\right] \;\to\; 0 \;\text{ as }\; n\to\infty. \tag{94}$$

We proceed as follows.

First note that since for each $n \geq 1$ and $t \geq 0$, we have the inequality $\tilde{S}^n(t) \geq \tilde{a}^n > 0$, it follows by (89) and the identity $(x_1 + ... + x_m)^2 \leq m(x_1^2 + ... + x_m^2)$, $m \geq 1$, that

$$\begin{aligned}(\tilde{Z}^n(t))^2 \;\leq\; & \frac{4}{(\tilde{a}^n)^2}\left((\Delta\tilde{\mu}^{l,n}(0))^2 + (\Delta\tilde{\mu}^{l,n}(t))^2 + \left(\frac{1}{n}\sum_{i=1}^{N^n_L(t)}(\ln(X_i) - E[\ln X])\right)^2\right. \\ & \left. + \left(\frac{1}{n^{1/2}}\int_0^t \frac{E[\ln X]}{\tilde{S}^n(s-)}d\bar{N}^n_L(s)\right)^2\right).\end{aligned} \tag{95}$$

Now let $\delta \geq 0$ be arbitrary. It then follows by the assumption that $E[(\Delta\tilde{\mu}^{l,n}(0))^2] \to 0$ as $n \to \infty$, Lemmas 9.2 and 9.3 above, Proposition 9.4, and Item 4 of Definition 3.1, that

$$E\left[\sup_{0\leq t\leq \tau^n_{\delta}\wedge\nu^n_\varepsilon} (\tilde{Z}^n(t))^2\right] \;\to\; 0 \;\text{ as }\; n\to\infty. \tag{96}$$

Next, let $\gamma > 0$ be arbitrary and note that by (90), (96), the identity $(x_1 + ... + x_m)^2 \leq m(x_1^2 + ... + x_m^2)$, $m \geq 1$, and Item 4 of Definition 3.1, it follows that there exists an $n_\gamma \geq 1$, which is independent of ε and δ, and such that for $n \geq n_\gamma$, one may write

$$E\left[\sup_{0\leq t\leq \tau^n_{\delta}\wedge\nu^n_\varepsilon} (\Delta p^{l,n}(t))^2\right] \;\leq\; \gamma + \frac{2}{\tilde{a}_n^2}E\left[\sup_{0\leq t\leq \tau^n_{\delta}\wedge\nu^n_\varepsilon}\left(\int_0^t \Delta p^{l,n}(s-)d\tilde{S}^n(s)\right)^2\right]. \tag{97}$$

Moreover, using the identity $(x_1 + ... + x_m)^2 \leq m(x_1^2 + ... + x_m^2)$ and the semi-martingale decomposition (11), one has that

$$E\left[\sup_{0\leq t\leq \tau^n_{\delta}\wedge\nu^n_\varepsilon}\left(\int_0^t \Delta p^{l,n}(s-)d\tilde{S}^n(s)\right)^2\right] \tag{98}$$

$$= E\left[\sup_{0\leq t\leq \tau^n_{\delta}\wedge\nu^n_\varepsilon}\left(\int_0^t \Delta p^{l,n}(s-)d(\tilde{M}^n(s)+\tilde{B}^n(s))\right)^2\right]$$

$$\leq 2E\left[\sup_{0\leq t\leq \tau^n_{\delta}\wedge\nu^n_\varepsilon}\left(\int_0^t \Delta p^{l,n}(s-)d\tilde{M}^n(s)\right)^2\right] \tag{99}$$

$$+2E\left[\sup_{0\leq t\leq \tau^n_{\delta}\wedge\nu^n_\varepsilon}\left(\int_0^t \Delta p^{l,n}(s-)d\tilde{B}^n(s)\right)^2\right]. \tag{100}$$

We now proceed to bound the terms in (99) and (100). We begin with (99).

Recall that by the Burkholder-Davis-Gundy inequality (see, for instance, Theorem 3.3.28 of Karatzas and Shreve [17]), there exists a constant $K_1 \geq 0$, which is independent of the martingale $\tilde{M}^n$ and the stopping time $\tau_\delta^n \wedge \nu_\varepsilon^n$, and uniform across all $n \geq 1$, such that

$$E\left[\sup_{0\leq t\leq \tau_\delta^n\wedge\nu_\varepsilon^n}\left(\int_0^t \Delta p^{l,n}(s-)d\tilde{M}^n(s)\right)^2\right] \leq K_1 E\left[\int_0^{\tau_\delta^n\wedge\nu_\varepsilon^n}(\Delta p^{l,n}(s-))^2 d[\tilde{M}^n,\tilde{M}^n](s)\right]. \quad (101)$$

However, note from (12) that

$$[\tilde{M}^n,\tilde{M}^n](t) = \bar{N}_L^n(t)+\bar{N}_M^n(t), \quad t\geq 0,$$

and so from the inequality (92) and (101) we obtain that

$$\begin{aligned} & E\left[\sup_{0\leq t\leq \tau_\delta^n\wedge\nu_\varepsilon^n}\left(\int_0^t \Delta p^{l,n}(s-)d\tilde{M}^n(s)\right)^2\right] \quad (102)\\ \leq\ & K_1 E\left[\int_0^{\tau_\delta^n\wedge\nu_\varepsilon^n}(\Delta p^{l,n}(s-))^2 d\left(\bar{N}_L^n(s)+\bar{N}_M^n(s)\right)\right] \\ \leq\ & K_1\left(\delta+\frac{2}{\sqrt{n}}\right)E\left[\sup_{0\leq t\leq \tau_\delta^n\wedge\nu_\varepsilon^n}(\Delta p^{l,n}(t))^2\right]. \end{aligned}$$

Next, we proceed to (100). Note first by (92) that for each $n \geq 1$ and $t \geq 0$, we have that

$$\begin{aligned} \sup_{0\leq t\leq \tau_\delta^n\wedge\nu_\varepsilon^n}\left(\int_0^t \Delta p^{l,n}(s-)d\tilde{B}^n(s)\right)^2 &\leq \left(\int_0^{\tau_\delta^n\wedge\nu_\varepsilon^n}|\Delta p^{l,n}(s-)|d||\tilde{B}^n||_{TV}(s)\right)^2 \\ &\leq \left(\delta+\frac{2}{\sqrt{n}}\right)^2\sup_{0\leq t\leq \tau_\delta^n\wedge\nu_\varepsilon^n}(\Delta p^{l,n}(t))^2. \end{aligned}$$

Hence,

$$E\left[\sup_{0\leq t\leq \tau_\delta^n\wedge\nu_\varepsilon^n}\left(\int_0^t \Delta p^{l,n}(s-)d\tilde{B}^n(s)\right)^2\right] \leq \left(\delta+\frac{2}{\sqrt{n}}\right)^2 E\left[\sup_{0\leq t\leq \tau_\delta^n\wedge\nu_\varepsilon^n}(\Delta p^{l,n}(t))^2\right]. \quad (103)$$

We now obtain from (97),(98), (102), (103), and Item 4 of Definition 3.1, that there exists a constant $K_2 \geq 0$, which is independent of n, ε and δ, and such that for each $n \geq n_\gamma$,

$$E\left[\sup_{0\leq t\leq \tau_\delta^n\wedge\nu_\varepsilon^n}\Delta(p^{l,n}(t))^2\right] \leq \gamma + K_2\left(\left(\delta+\tfrac{2}{\sqrt{n}}\right)+\left(\delta+\tfrac{2}{\sqrt{n}}\right)^2\right)E\left[\sup_{0\leq t\leq \tau_\delta^n\wedge\nu_\varepsilon^n}\Delta(p^{l,n}(t))^2\right] \quad (104)$$

Next, note that there exists a $\delta_0 > 0$, which is independent ε, and such that if $0 \leq \delta < \delta_0$, then there exists an $n_\delta \geq 1$ such that

$$K_2\left(\left(\delta+\frac{2}{\sqrt{n}}\right)+\left(\delta+\frac{2}{\sqrt{n}}\right)^2\right) < \frac{1}{2}, \quad n \geq n_\delta. \quad (105)$$

Hence, for such $0 \le \delta < \delta_0$ and $n \ge n_\delta \wedge n_\gamma$, we obtain from (104), after some algebra, that

$$E\left[\sup_{0\le t\le \tau^n_\delta \wedge \nu^n_\varepsilon} \Delta(p^{l,n}(t))^2\right] \le 2\gamma.$$

However, since the choice of $\gamma > 0$ was arbitrary, this implies that for $0 \le \delta < \delta_0$,

$$E\left[\sup_{0\le t\le \tau^n_\delta \wedge \nu^n_\varepsilon} \Delta(p^{l,n}(t))^2\right] \to 0 \text{ as } n \to \infty, \tag{106}$$

thus proving (93).

We now proceed to prove (94) for $k \ge 1$. As mentioned at the outset, our proof is by induction on k. It is clear by (93) that (94) holds for the base case of $k = 1$. Suppose now that (94) holds for $k-1$ for some $k \ge 2$. That is, suppose that for $0 \le \delta < \delta_0$,

$$E\left[\sup_{0\le t\le \tau^n_{(k-1)\delta} \wedge \nu^n_\varepsilon} \Delta(p^{l,n}(t))^2\right] \to 0 \text{ as } n \to \infty. \tag{107}$$

We now show that (107) holds for k as well.

Note first by (90) that for each $t \ge \tau^n_{(k-1)\delta}$, we may write

$$\Delta p^{l,n}(t) = \tilde{Z}^n_{(k-1)\delta}(t) + \frac{1}{\tilde{S}^n(t)} \int_{\tau^n_{(k-1)\delta}\wedge \nu^n_\varepsilon}^t \Delta p^{l,n}(s-) d\tilde{S}^n(s), \tag{108}$$

where

$$\tilde{Z}^n_{(k-1)\delta}(t) = \tilde{Z}^n(t) + \frac{1}{\tilde{S}^n(t)} \int_0^{\tau^n_{(k-1)\delta}\wedge \nu^n_\varepsilon} \Delta p^{l,n}(s-) d\tilde{S}^n(s). \tag{109}$$

We now claim that

$$E\left[\sup_{0\le t\le \tau^n_{k\delta} \wedge \nu^n_\varepsilon} (\tilde{Z}^n_{(k-1)\delta}(t))^2\right] \to 0 \text{ as } n \to \infty. \tag{110}$$

In order to prove (110), first note that by the basic identity $(x_1 + ... + x_m)^2 \le m(x_1^2 + ... + x_m^2)$, $m \ge 1$, and the fact that for each $n \ge 1$, we have that $\tilde{S}^n(t) \ge \tilde{a}^n > 0$ for $t \ge 0$, it follows by (109) that

$$(\tilde{Z}^n_{(k-1)\delta}(t))^2 \le 2(\tilde{Z}^n(t))^2 + \frac{2}{(\tilde{a}^n)^2}\left(\int_0^{\tau^n_{(k-1)\delta}\wedge \nu^n_\varepsilon} \Delta p^{l,n}(s-) d\tilde{S}^n(s)\right)^2. \tag{111}$$

Next, by (95), the assumption that $E[(\Delta\tilde{\mu}^{l,n}(0))^2] \to 0$ as $n \to \infty$, Lemmas 9.2 and 9.3 above, Proposition 9.4, and Item 4 of Definition 3.1, one has that

$$E\left[\sup_{0\le t\le \tau^n_{k\delta} \wedge \nu^n_\varepsilon} (\tilde{Z}^n(t))^2\right] \to 0 \text{ as } n \to \infty. \tag{112}$$

Hence, in order to show (110), it suffices by (111), (112) and Item 4 of Definition 3.1, to show that

$$E\left[\left(\int_0^{\tau^n_{(k-1)\delta}\wedge \nu^n_\varepsilon} \Delta p^{l,n}(s-) d\tilde{S}^n(s)\right)^2\right] \to 0 \text{ as } n \to \infty.$$

However, proceeding in a manner similar to earlier in the proof, one may show that there exists a constant $K_3 \geq 0$, which is independent of n, ε and δ, and such that

$$E\left[\left(\int_0^{\tau^n_{(k-1)\delta}\wedge\nu^n_\varepsilon} \Delta p^{l,n}(s-)d\tilde{S}^n(s)\right)^2\right] \leq K_3 \cdot E\left[\sup_{0\leq t\leq \tau^n_{(k-1)\delta}\wedge\nu^n_\varepsilon} (\Delta p^{l,n}(t))^2\right]. \tag{113}$$

The details of the proof of (113) are omitted. Moreover, by the induction hypothesis, we have that

$$E\left[\sup_{0\leq t\leq \tau^n_{(k-1)\delta}\wedge\nu^n_\varepsilon} (\Delta p^{l,n}(t))^2\right] \to 0 \text{ as } n\to\infty. \tag{114}$$

Hence, (109), (112), (113) and (114) now imply (110).

We now use (108) and (110) in order to prove (107) with $k-1$ replaced by k. Note first that by (108) and the basic identity $(x_1+...+x_m)^2 \leq m(x_1^2+...+x_m^2)$, $m\geq 1$, we have that for each $n\geq 1$ and $t\geq \tau^n_{(k-1)\delta}$,

$$(\Delta p^{l,n}(t))^2 \leq 2(\tilde{Z}^n_{(k-1)\delta}(t))^2 + \frac{2}{(\tilde{a^n})^2}\left(\int_{\tau^n_{(k-1)\delta}\wedge\nu^n_\varepsilon}^t \Delta p^{l,n}(s-)d\tilde{S}^n(s)\right)^2. \tag{115}$$

Now proceeding in a manner similar to the proof of the base case $k=1$, one may show that there exists a constant $K_4 \geq 0$ such that

$$\begin{aligned}&\frac{2}{(\tilde{a}^n)^2}E\left[\sup_{\tau^n_{(k-1)\delta}\wedge\nu^n_\varepsilon < t\leq \tau^n_{k\delta}\wedge\nu^n_\varepsilon}\left(\int_{\tau^n_{(k-1)\delta}}^t \Delta p^{l,n}(s-)d\tilde{S}^n(s)\right)^2\right] \\ &\leq K_4\left(\left(\delta+\frac{2}{\sqrt{n}}\right)+\left(\delta+\frac{2}{\sqrt{n}}\right)^2\right)E\left[\sup_{\tau^n_{(k-1)\delta}\wedge\nu^n_\varepsilon < t\leq \tau^n_{k\delta}\wedge\nu^n_\varepsilon}(\Delta p^{l,n}(t))^2\right].\end{aligned} \tag{116}$$

In particular, the constant K_4 in (116) is the same constant K_2 used in the proof of (104) above. The details of the proof of (116) are omitted. Hence, using the same δ_0 and n_δ as in (105) above, we obtain that for $0\leq\delta<\delta_0$, we have the bound

$$K_4\left(\left(\delta+\frac{2}{\sqrt{n}}\right)+\left(\delta+\frac{2}{\sqrt{n}}\right)^2\right) < \frac{1}{2}, \quad n\geq n_\delta,$$

and so from (115), after some algebra, it follows that for such $0<\delta<\delta_0$ and $n\geq n_\delta$,

$$E\left[\sup_{\tau^n_{(k-1)\delta}\wedge\nu^n_\varepsilon < t\leq \tau^n_{k\delta}\wedge\nu^n_\varepsilon}(\Delta p^{l,n}(t))^2\right] \leq 4E\left[\sup_{\tau^n_{(k-1)\delta}\wedge\nu^n_\varepsilon < t\leq \tau^n_{k\delta}\wedge\nu^n_\varepsilon}(\tilde{Z}^n_{(k-1)\delta}(t))^2\right]. \tag{117}$$

However, by (110),

$$E\left[\sup_{\tau^n_{(k-1)\delta}\wedge\nu^n_\varepsilon < t\leq \tau^n_{k\delta}\wedge\nu^n_\varepsilon}(\tilde{Z}^n_{(k-1)\delta}(t))^2\right] \leq E\left[\sup_{0\leq t\leq \tau^n_{k\delta}\wedge\nu^n_\varepsilon}(\tilde{Z}^n_{(k-1)\delta}(t))^2\right] \to 0 \text{ as } n\to\infty, \tag{118}$$

and so by (117),

$$E\left[\sup_{\tau^n_{(k-1)\delta\wedge\nu^n_\varepsilon}<t\leq\tau^n_{k\delta}\wedge\nu^n_\varepsilon}(\Delta p^{l,n}(t))^2\right] \;\to\; 0 \;\text{ as }\; n\to\infty. \tag{119}$$

Thus, by the induction hypothesis and (119), we obtain that

$$\begin{aligned}
& E\left[\sup_{0\leq t\leq\tau^n_{k\delta}\wedge\nu^n_\varepsilon}(\Delta p^{l,n}(t))^2\right] \qquad (120)\\
\leq\;& E\left[\sup_{0\leq t\leq\tau^n_{(k-1)\delta}\wedge\nu^n_\varepsilon}(\Delta p^{l,n}(t))^2\right] + E\left[\sup_{\tau^n_{(k-1)\delta}\wedge\nu^n_\varepsilon<t\leq\tau^n_{k\delta}\wedge\nu^n_\varepsilon}(\Delta p^{l,n}(t))^2\right] \;\to\; 0 \;\text{ as }\; n\to\infty,
\end{aligned}$$

as desired. This proves (94).

We now complete the proof by removing the stopping times $\tau^n_{k\delta}$ in (120). First note that for each $n\geq 1, k\geq 1$ and $0\leq\delta<\delta_0$, we may write

$$\begin{aligned}
& E\left[\sup_{0\leq t\leq\nu^n_\varepsilon}(\Delta p^{l,n}(t))^2\right]\\
=\;& E\left[1\{\tau^n_{k\delta}\geq T\}\sup_{0\leq t\leq\tau^n_{k\delta}\wedge\nu^n_\varepsilon}(\Delta p^{l,n}(t))^2\right] + E\left[1\{\tau^n_{k\delta}<T\}\sup_{0\leq t\leq\nu^n_\varepsilon}(\Delta p^{l,n}(t))^2\right]. \qquad (121)
\end{aligned}$$

Now note that in order to show that the right-hand side above converges to 0 as $n\to\infty$, it is sufficient to show that the first term in (121) converges to 0 for every fixed $k\geq 1$ as $n\to\infty$, and that the second term converges to 0 as $k\to\infty$, uniformly in n. By (120), for each $k\geq 1$,

$$E\left[1\{\tau^n_{k\delta}\geq T\}\sup_{0\leq t\leq\tau^n_{k\delta}\wedge\nu^n_\varepsilon}(\Delta p^{l,n}(t))^2\right] \;\leq\; E\left[\sup_{0\leq t\leq\tau^n_{k\delta}\wedge\nu^n_\varepsilon}(\Delta p^{l,n}(t))^2\right] \;\to\; 0, \tag{122}$$

as $n\to\infty$. Thus, in order to complete the proof, it remains to consider the second term in (121).

Note that by (1), Item 3 of Definition 3.1, and Item 2 of Assumption 3.2

$$|p^{n,l}(t)-p^{n,l}(t-)|\leq\frac{\ln C}{\sqrt{n}},\quad t\geq 0,$$

thus by the definition of ν^n_ε in (79) we have that

$$|p^{l,n}(t)| \;\leq\; p^{l,n}(0)+\varepsilon+\frac{\ln C}{\sqrt{n}}. \tag{123}$$

From inequality (123), the definition of $\Delta p^{l,n}(t)$ in (80), the basic identity $(x_1+...+x_m)^2\leq m(x_1^2+...+x_m^2)$, $m\geq 1$, the fact that for each $n\geq 1$, we have that $\tilde{S}^n(t)\geq\tilde{a}^n>0$ for $t\geq 0$, and the fact that $E[\ln X]>0$, it follows that

$$\sup_{0\leq t\leq\nu^n_\varepsilon}(\Delta p^{l,n}(t))^2 \;=\; \sup_{0\leq t\leq\nu^n_\varepsilon}\left(p^{l,n}(t)-\left(p^{l,n}(0)+\int_0^t\frac{E[\ln X]}{\tilde{S}^n(s-)}d\bar{N}^n_L(s)\right)\right)^2 \tag{124}$$

$$
\begin{aligned}
&\leq 2\left(\sup_{0\leq t\leq \nu_\varepsilon^n} (p^{l,n}(t))^2\right) \\
&\quad +2\left(\sup_{0\leq t\leq \nu_\varepsilon^n}\left(\int_0^t \frac{E[\ln X]}{\tilde{S}^n(s)} d\bar{N}_L^n(s)\right)^2\right) \\
&\leq 2\left(|p^{l,n}(0)|+\varepsilon+\frac{\ln C}{\sqrt{n}}\right)^2 + 2\left(\int_0^T \frac{E[\ln X]}{\tilde{S}^n(s)} d\bar{N}_L^n(s)\right)^2 \\
&\leq 2\left(|p^{l,n}(0)|+\varepsilon+\frac{\ln C}{\sqrt{n}}\right)^2 + 2\left(\frac{E[\ln X]}{\tilde{a}^n}(\bar{N}_L^n(T))\right)^2 \\
&\leq 2\left(|p^{l,n}(0)|+\varepsilon+\frac{\ln C}{\sqrt{n}}\right)^2 + 2\left(\frac{E[\ln X]}{\tilde{a}^n}\right)^2 (\bar{N}_L^n(T))^2.
\end{aligned}
$$

It therefore follows by (124) and the Cauchy-Schwarz inequality that

$$
\begin{aligned}
&\left(E\left[1\{\tau_{k\delta}^n < T\} \sup_{0\leq t\leq \nu_\varepsilon^n} (\Delta p^{l,n}(t))^2\right]\right)^2 \qquad (125)\\
&\leq P(\tau_{k\delta}^n < T)\cdot E\left[4\left(\left(|p^{l,n}(0)|+\varepsilon+\frac{\ln C}{\sqrt{n}}\right)^2 + \left(\frac{E[\ln X]}{\tilde{a}^n}\right)^2 (\bar{N}_L^n(T))^2\right)^2\right] \\
&\leq 8P(\tau_{k\delta}^n < T)\cdot E\left[\left(|p^{l,n}(0)|+\varepsilon+\frac{\ln C}{\sqrt{n}}\right)^4 + \left(\frac{E[\ln X]}{\tilde{a}^n}\right)^4 (\bar{N}_L^n(T))^4\right].
\end{aligned}
$$

Now, using the properties of the Poisson distribution, it is straightforward to show that there exists some $K_5 > 0$ such that

$$
E\left[\left(\bar{N}_L^n(T)\right)^4\right] = \frac{1}{n^4}E\left[\left(N_L^n(T)\right)^4\right] \leq \frac{1}{n^4}K_5(1+\lambda_L^n T)^4 \to K_5(\lambda T)^4 \text{ as } n\to\infty,
$$

where the final convergence follows by Item 1 of Definition 3.1. Thus, using (1) and Item 4 of Definition 3.1, it follows that

$$
\sup_{n\geq 1} E\left[\left(|p^{l,n}(0)|+\varepsilon+\frac{\ln C}{\sqrt{n}}\right)^4 + \left(\frac{E[\ln X]}{\tilde{a}^n}\right)^4 (\bar{N}_L^n(T))^4\right] < \infty. \qquad (126)
$$

On the other hand, note that we may write

$$
\begin{aligned}
P(\tau_{k\delta}^n < T) &\leq P\left(||\tilde{B}^n||_{TV}(T) + \bar{N}_L^n(T) + \bar{N}_M^n(T) \geq k\delta\right) \qquad (127)\\
&\leq P\left(||\tilde{B}^n|||_{TV}(T) \geq \frac{k\delta}{3}\right) + P\left(\bar{N}_L^n(T) \geq \frac{k\delta}{3}\right) + P\left(\bar{N}_M^n(T) \geq \frac{k\delta}{3}\right).
\end{aligned}
$$

However, by Proposition 4.2 we have that $\tilde{L}^n(T) \Rightarrow \tilde{L}(T)$ as $n\to\infty$, and by Item 1 of Definition 3.1 it follows that both $\bar{N}_L^n(T) \Rightarrow \lambda T$ and $\bar{N}_M^n(T) \Rightarrow \lambda T$ as $n\to\infty$. It therefore follows that the sequences of random variables $\{\tilde{L}^n(T), n\geq 1\}$, $\{\bar{N}_L^n(T), n\geq 1\}$, and $\{\bar{N}_M^n(T), n\geq 1\}$ are tight. Hence, the expression in (127) converges to 0 as $k\to\infty$ uniformly in n, which implies that

$$
\lim_{k\to\infty}\sup_{n\geq 1} P(\tau_{k\delta}^n < T) = 0. \qquad (128)
$$

From (125), (126), and (128) it now follows that

$$\lim_{k\to\infty}\sup_{n\geq 1} E\left[1\{\tau^n_{k\delta} < T\}\sup_{0\leq t\leq \nu^n_\varepsilon}(\Delta p^{l,n}(t))^2\right] \quad = \quad 0, \tag{129}$$

which completes the proof. □

Now note that given Proposition 9.5 above, the only remaining obstacle in our way to proving Proposition 9.1 is to show that ν^n_ε weakly converges to T for ε sufficiently large. However, in this regard, we have the following result.

Proposition 9.6. *If $E[\ln X] > 0$ and $(p^{l,n}(0), \tilde{S}^n(0)) \Rightarrow (p^l(0), \tilde{S}(0))$ as $n \to \infty$, and if Items 1-4 of Definition 3.1 hold, then, for each $\delta > 0$, there exists an $\varepsilon > 0$ such that*

$$\limsup_{n\geq 1} P(\nu^n_\varepsilon < T) \quad < \quad \delta. \tag{130}$$

Proof. The proof is by contradiction. Suppose that (130) does not hold. Then, for some $\delta > 0$ and all $\varepsilon > 0$,

$$\limsup_{n\geq 1} P(\nu^n_\varepsilon < T) \quad \geq \quad \delta. \tag{131}$$

We now show that (131) will result in a contradiction with Proposition 9.5 above, thereby yielding (130).

For each $n \geq 1$, let

$$\tilde{A}^n(t) \quad = \quad \int_0^t \frac{E[\ln X]}{\tilde{S}^n(s)} d\bar{N}^n_L(s), \quad t \geq 0. \tag{132}$$

We first show that if $\varepsilon > 0$ is sufficiently large, then

$$\lim_{n\to\infty} P(\tilde{A}^n(T) > \varepsilon/2) \quad = \quad 0. \tag{133}$$

Indeed, note that since by assumption $E[\ln X] > 0$, and since by Item 4 of Definition 3.1, we have that for each $t \geq 0$, $\tilde{S}^n(t) \geq \tilde{a}^n \to \tilde{a} > 0$ as $n \to \infty$, it follows that for n sufficiently large,

$$\begin{aligned} P(\tilde{A}^n(T) > \varepsilon/2) \quad &= \quad P\left(\int_0^T \frac{E[\ln X]}{\tilde{S}^n(s)} d\bar{N}^n_L(s) > \frac{\varepsilon}{2}\right) \\ &\leq \quad P\left(\frac{E[\ln X]}{\tilde{a}/2}\bar{N}^n_L(T) > \frac{\varepsilon}{2}\right). \end{aligned}$$

However, since by Theorem 5.10 of Chen and Yao [5], the random time change theorem (see, for instance, page 151 of Billnigsley [4]) and Item 1 of Definition 3.1, we have that $\bar{N}^n_L \Rightarrow \lambda e$ as $n \to \infty$, it follows that the final expression on the righthand side above converges to 0 as $n \to \infty$ whenever

$$\frac{E[\ln X]}{\tilde{a}/2}\lambda T \quad < \quad \frac{\varepsilon}{2}.$$

Thus, (133) is proven. Moreover, note that for fixed n, the quantity $P(\tilde{A}^n(T) > \varepsilon/2)$ is decreasing in ε, thus for each $\delta > 0$, we have that for sufficiently large ε and n,

$$P(\tilde{A}^n(T) > \varepsilon/2) \quad < \quad \frac{\delta}{2}. \tag{134}$$

Next we show that on the event $\{\nu_\varepsilon^n < T, \tilde{A}^n(T) \le \varepsilon/2\}$, we have that

$$\sup_{0 \le t \le \nu_\varepsilon^n} \left(p^{l,n}(t) - \tilde{A}^n(t)\right)^2 \quad \ge \quad \kappa \tag{135}$$

where

$$\kappa = \max\left\{\left(\frac{\varepsilon}{2} + \ln c_1\right)^2,\ (-\varepsilon + \ln C_1)^2\right\}, \tag{136}$$

whenever

$$\varepsilon > \max\left\{-2\ln c_1,\ \ln C_1\right\}. \tag{137}$$

First note that

$$\sup_{0 \le t \le \nu_\varepsilon^n} \left(p^{l,n}(t) - \tilde{A}^n(t)\right)^2 \quad \ge \quad \left(p^{l,n}\left(\nu_\varepsilon^n\right) - \tilde{A}^n\left(\nu_\varepsilon^n\right)\right)^2. \tag{138}$$

By the definition of ν_ε^n in (79) on the event $\{\nu_\varepsilon^n < T, \tilde{A}^n(T) \le \varepsilon/2\}$ we have either $p^{l,n}\left(\nu_\varepsilon^n\right) - p^{l,n}(0) \ge \varepsilon$ or $p^{l,n}\left(\nu_\varepsilon^n\right) - p^{l,n}(0) \le -\varepsilon$. By (137) and Item 2 of Assumption 3.2 on the event $\{\nu_\varepsilon^n < T, \tilde{A}^n(T) \le \varepsilon/2\}$ we have

$$\left(p^{l,n}\left(\nu_\varepsilon^n\right) - \tilde{A}^n\left(\nu_\varepsilon^n\right)\right)^2 \ge \left(\frac{\varepsilon}{2} + \ln c_1\right)^2,$$

and in the second case

$$\left(p^{l,n}\left(\nu_\varepsilon^n\right) - \tilde{A}^n\left(\nu_\varepsilon^n\right)\right)^2 \ge (-\varepsilon + \ln C_1)^2,$$

and now (135) follows.

Hence, by (131) and (134), it follows for infinitely many n,

$$\begin{aligned} & E\left[\sup_{0 \le t \le \nu_\varepsilon^n} \left(p^{l,n}(t) - \tilde{A}^n(t)\right)^2\right] \\ \ge \quad & \kappa P(\nu_\varepsilon^n < T, \tilde{A}^n(T) \le \varepsilon/2) \\ = \quad & \kappa(P(\nu_\varepsilon^n < T) - P(\nu_\varepsilon^n < T, \tilde{A}^n(T) > \varepsilon/2)) \\ \ge \quad & \kappa\left(P(\nu_\varepsilon^n < T) - P(\tilde{A}^n(T) > \varepsilon/2)\right) \\ \ge \quad & \frac{\delta}{2} \cdot \kappa. \end{aligned} \tag{139}$$

However, (139) contradicts Proposition 9.5, which completes the proof of (130).

□

Given Propositions 9.4, 9.5 and 9.6 above, we are now in a position to present the proof of Proposition 9.1. We have the following.

Proof of Proposition 9.1. Note that by (80) and Proposition 9.4, in order to complete the proof it suffices to show that $\Delta p^{l,n} \Rightarrow 0$ as $n \to \infty$. In particular, it suffices to show that for each fixed $T > 0$,

$$\sup_{0 \le t \le T} |\Delta p^{l,n}(t)| \;\; \to \;\; 0 \;\text{ in probability }\; \text{as} \;\; n \to \infty. \tag{140}$$

Let $\delta > 0$ be arbitrary and note that by Proposition 9.6, we may select an $\varepsilon > 0$ such that for sufficiently large n we have that $P(\nu_\varepsilon^n < T) \le \delta$. Now let $K > 0$ be arbitrary and note that by Proposition 9.5, we have that for sufficiently large n,

$$\begin{aligned} & P\left(\sup_{0 \le t \le T} |\Delta p^{l,n}(t)| > K\right) \qquad (141) \\ = \;\; & P\left(\sup_{0 \le t \le T} \left(\Delta p^{l,n}(t)\right)^2 > K^2\right) \\ \le \;\; & P\left(\sup_{0 \le t \le T} \left(\Delta p^{l,n}(t)\right)^2 1\{\nu_\varepsilon^n < T\} > \frac{K^2}{2}\right) + P\left(\sup_{0 \le t \le \nu_\varepsilon^n} \left(\Delta p^{l,n}(t)\right)^2 1\{\nu_\varepsilon^n = T\} > \frac{K^2}{2}\right) \\ \le \;\; & P\left(0 \le \nu_\varepsilon^n < T\right) + P\left(\sup_{0 \le t \le \nu_\varepsilon^n} \left(\Delta p^{l,n}(t)\right)^2 > \frac{K^2}{2}\right) \\ < \;\; & 2\delta. \end{aligned}$$

(141) now implies (140), which completes the proof.

□

9.2 Proof of Lemma 9.2

In this subsection, we provide the proof of Lemma 9.2 from Section 9.1. The proof is rather long technical and consequently, we divide the subsection into three stages. First, in Subsubsection 9.2.1, we provide stochastic bounds on certain key quantities which will be used later in the proof of Lemma 9.2. Next, in Subsubsection 9.2.2, we prove two lemmas which will be useful in our proof of Lemma 9.2. Finally, in Subsubsection 9.2.3, we provide the proof of Lemma 9.2.

9.2.1 Stochastic Bounds

In this subsubsection, we provide two stochastic bounds which will be useful in the proof of Lemma 9.2. We begin with the following.

Lemma 9.7. *If Items 1-4 of Definition 3.1 hold, then, for each* $T \ge 0$,

$$\sup_{n \ge 1} E\left[(\tilde{L}^n(T))^4\right] \;\; < \;\; \infty \quad \textit{and} \quad \sup_{n \ge 1} E\left[\sup_{0 \le t \le T} (\tilde{S}^n(t))^4\right] \;\; < \;\; \infty.$$

Proof. Let $T \ge 0$. We first prove that if Items 1,2 and 4 of Definition 3.1 hold, then,

$$\sup_{n \ge 1} E\left[(\tilde{L}^n(T))^4\right] \;\; < \;\; \infty. \tag{142}$$

In order to see that this is the case, note that by (3) of Section 2 and the basic inequality $(x_1 + ... + x_m)^4 \leq m^3(x_1^4 + ... + x_m^4), m \geq 1$, one has that for each $n \geq 1$,

$$\begin{aligned}(L^n(T))^4 &= \left(\sup_{0\leq t\leq T}\{-(S^n(0) + N_L^n(t) - N_M^n(t) - a^n)\} \vee 0\right)^4 \\ &\leq \left(S^n(0) + a^n + |\lambda_L^n - \lambda_M^n|T + \sup_{0\leq t\leq T}|(N_L^n(t) - \lambda_L^n t) - (N_M^n(t) - \lambda_M^n t)|\right)^4 \\ &\leq 4^3\left((S^n(0))^4 + (a^n)^4 + (\lambda_L^n - \lambda_M^n)^4 T^4 + \sup_{0\leq t\leq T}((N_L^n(t) - \lambda_L^n t) - (N_M^n(t) - \lambda_M^n t))^4\right).\end{aligned}$$

Now dividing the above inequality through by n^2, it follows that

$$4^{-3}(\tilde{L}^n(T))^4 \leq (\tilde{S}^n(0))^4 + (\tilde{a}^n)^4 + (\sqrt{n}|n^{-1}\lambda_L^n - n^{-1}\lambda_M^n|)^4 T^4 + \sup_{0\leq t\leq T}(\tilde{M}^n(t))^4.$$

However, by Items 2 and 4 of Definition 3.1, as well as Item 1 of Assumption 3.2, one has that

$$\sup_{n\geq 1} E[(\tilde{S}^n(0))^4 + (\tilde{a}^n)^4 + (\sqrt{n}|n^{-1}\lambda_L^n - n^{-1}\lambda_M^n|)^4 T^4] < \infty. \tag{143}$$

Hence, in order to prove (142), it remains to show that

$$\sup_{n\geq 1} E\left[\sup_{0\leq t\leq T}(\tilde{M}^n(t))^4\right] < \infty. \tag{144}$$

We will use Doob's martingale inequality (see, for instance, Theorem 1.3.8 of Karatzas and Shreve [17]). In particular, recall from the discussion in Section 2 that the process $\tilde{M}^n = \{\tilde{M}^n(t), t \geq 0\}$ is a martingale. Moreover, using the basic identity $(x_1 + ... + x_m)^4 \leq m^3(x_1^4 + ... + x_m^4), m \geq 1$, we have that

$$\begin{aligned}E\left[(\tilde{M}^n(T))^4\right] &\leq 2^3 n^{-2}(E[(N_L^n(T) - \lambda_L^n T)^4] + E[(N_M^n(T) - \lambda_M^n T)^4]) \\ &= 2^3 n^{-2}(\lambda_L^n(1 + 3\lambda_L^n) + \lambda_M^n(1 + 3\lambda_M^n)) \\ &\to 2^4(3)(\lambda^2) \text{ as } n \to \infty,\end{aligned} \tag{145}$$

where the final convergence follows from Item 1 of Definition 3.1. Thus, the result (144) now follows as a result of (145) and Doob's martingale inequality (see, for instance, Theorem 1.3.8 of Karatzas and Shreve [17]). This proves (142).

We next prove that

$$\sup_{n\geq 1} E\left[\sup_{0\leq t\leq T}(\tilde{S}^n(t))^4\right] < \infty. \tag{146}$$

Recall from (11) and (13) of Section 3 that for each $0 \leq t \leq T$, we may write

$$\tilde{S}^n(t) = \tilde{S}^n(0) + \tilde{M}^n(t) + \sqrt{n}(n^{-1}\lambda_L^n - n^{-1}\lambda_M^n)t + \tilde{L}^n(t).$$

However, since $\{\tilde{L}^n(t), t \geq 0\}$ is a non-decreasing process, it follows by the basic identity $(x_1 + ... + x_m)^4 \leq m^3(x_1^4 + ... + x_m^4), m \geq 1$, that

$$4^{-3} \sup_{0\leq t\leq T} (\tilde{S}^n(t))^4 \leq (\tilde{S}^n(0))^4 + \sup_{0\leq t\leq T} (\tilde{M}^n(t))^4 + (\sqrt{n}(n^{-1}\lambda_L^n - n^{-1}\lambda_M^n))^4 T^4 + (\tilde{L}^n(T))^4,$$

which implies that

$$\begin{aligned} 4^{-3} E\left[\sup_{0\leq t\leq T} (\tilde{S}^n(t))^4\right] &\leq E\left[(\tilde{S}^n(0))^4\right] + E\left[\sup_{0\leq t\leq T} (\tilde{M}^n(t))^4\right] + (\sqrt{n}(n^{-1}\lambda_L^n - n^{-1}\lambda_M^n))^4 T^4 \\ &\quad + E\left[(\tilde{L}^n(T))^4\right]. \end{aligned}$$

(146) now follows as a result of the inequality above and (142), (143) and (144). This completes the proof. ∎

Next, we have the following result which is our final result of this subsubsection. Let $\varphi \in C_b^2(\mathbb{R})$ be an arbitrary function with bounded first and second derivatives and for each $n \geq 1$ and $t \geq 0$, define

$$\begin{aligned} &\Delta\varphi^{l,n}(t) \qquad (147)\\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^{N_L^n(t)} \left\{\varphi\left(p^{l,n}\left(\tau_i^{L,n}-\right) + \ln X_i^{1/n^{1/2}}\right) - \varphi\left(p^{l,n}\left(\tau_i^{L,n}-\right)\right)\right\} - \lambda E[\ln X] \int_0^t \varphi'\left(p^{l,n}(s-)\right) ds. \end{aligned}$$

Also, define the process $\Delta\varphi^{l,n} = \{\Delta\varphi^{l,n}(t), t \geq 0\}$. We then have the following.

Lemma 9.8. *If Items 1-4 of Definition 3.1 hold, then, for each $\varphi \in C_b^2(\mathbb{R})$ and $T \geq 0$,*

$$E\left[\sup_{0\leq t\leq T} |\Delta\varphi^{l,n}(t)|\right] \to 0 \quad as \quad n \to \infty.$$

Proof. First note by (147) that for each $n \geq 1, t \geq 0$ and $\varphi \in C_b^2(\mathbb{R})$, we have the decomposition

$$\Delta\varphi^{l,n}(t) =$$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{N_L^n(t)} \left\{\varphi\left(p^{l,n}\left(\tau_i^{L,n}-\right) + \ln X_i^{1/n^{1/2}}\right) - \varphi\left(p^{l,n}\left(\tau_i^{L,n}-\right)\right) - \frac{\ln(X_i)}{\sqrt{n}}\varphi'\left(p^{l,n}\left(\tau_i^{L,n}-\right)\right)\right\} \qquad (148)$$

$$+ \frac{1}{n} \sum_{i=1}^{N_L^n(t)} \left\{\ln(X_i) - E[\ln X])\varphi'\left(p^{l,n}\left(\tau_i^{L,n}-\right)\right)\right\} \qquad (149)$$

$$+ E[\ln X] \int_0^t \varphi'\left(p^{l,n}(s-)\right) d\left(\bar{N}_L^n(s) - \lambda s\right), \qquad (150)$$

where $\bar{N}_L^n(t)$ is as defined in Section 3 above. Let $T \geq 0$. We now treat each of the terms on the righthand side above separately, showing that the expected value supremum of their absolute value over the interval $[0,T]$ converges to zero in expectation as n goes to ∞. This will then complete the proof.

We begin with (148). First note that since $\varphi \in C_b^2(\mathbb{R})$, we have by Taylor's Theorem that we may write

$$\varphi(p^{l,n}(\tau_i^{L,n}-) + \ln X_i^{1/n^{1/2}}) - \varphi(p^{l,n}(\tau_i^{L,n}-)) \quad = \quad \frac{\ln(X_i)}{\sqrt{n}}\varphi^{'}(p^{l,n}(\tau_i^{L,n}-)) + \mathcal{O}\left(\frac{1}{n}\right),$$

where, by (1) and $\varphi \in C_b^2(\mathbb{R})$, the $\mathcal{O}\left(\frac{1}{n}\right)$ term above is uniformly bounded, i.e. $|\mathcal{O}\left(\frac{1}{n}\right)| \leq K_1/n$ for some constant $K_1 > 0$. We therefore obtain that for each $t \geq 0$, we may write

$$\begin{aligned}
&\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{N_L^n(t)}\left\{\varphi\left(p^{l,n}\left(\tau_i^{L,n}-\right) + \ln X_i^{1/n^{1/2}}\right) - \varphi\left(p^{l,n}\left(\tau_i^{L,n}-\right)\right) - \frac{\ln(X_i)}{\sqrt{n}}\varphi^{'}\left(p^{l,n}\left(\tau_i^{L,n}-\right)\right)\right\}\right| \\
\leq \;& \frac{1}{\sqrt{n}}N_L^n(t)\frac{K_1}{n} = \frac{K_1}{\sqrt{n}}\bar{N}_L^n(t).
\end{aligned}$$

Thus, since $\{\bar{N}_L^n(t), t \geq 0\}$ is a non-decreasing process, it follows that

$$\begin{aligned}
& E\left[\sup_{0\leq t\leq T}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{N_L^n(t)}\left\{\varphi(p^{l,n}(\tau_i^{L,n}-) + \ln X_i^{1/n^{1/2}}) - \varphi(p^{l,n}(\tau_i^{L,n}-)) - \frac{\ln(X_i)}{\sqrt{n}}\varphi^{'}(p^{l,n}(\tau_i^{L,n}-))\right\}\right|\right] \\
\leq \;& \frac{K_1}{\sqrt{n}}E[\bar{N}_L^n(T)] \\
= \;& \frac{K_1 T}{n^{3/2}}\lambda_L^n \\
\to \;& 0 \quad \text{as} \quad n \to \infty, \qquad (151)
\end{aligned}$$

where the final convergence follows by Item 1 of Definition 3.1. This completes our treatment of (148).

The expected value of the supremum over the interval $[0,T]$ of the absolute value of (149) converges to 0 as $n \to \infty$ by Lemma 9.3 and the boundedness of φ.

We now complete the proof by showing that the supremum of (150) over the interval $[0,T]$ converges to zero in expectation as n goes to ∞. First note that for each $t \geq 0$, we may write

$$\begin{aligned}
& E[\ln X]\int_0^t \varphi^{'}(p^{l,n}(s-))d(\bar{N}_L^n(s) - \lambda s) \\
= \;& E[\ln X]\int_0^t \varphi^{'}(p^{l,n}(s-))d(\bar{N}_L^n(s) - n^{-1}\lambda_L^n s) \qquad (152) \\
& + E[\ln X]\int_0^t \varphi^{'}(p^{l,n}(s-))d((n^{-1}\lambda_L^n - \lambda)s). \qquad (153)
\end{aligned}$$

We now treat each of the terms (152) and (153) separately. We begin with (152). First note that the process

$$\left\{E[\ln X]\int_0^t \varphi^{'}(p^{l,n}(s-))d(\bar{N}_L^n(s) - n^{-1}\lambda_L^n s), t \geq 0\right\}$$

is a martingale. Hence, by Doob's martingale inequality (see, for instance, Theorem 1.3.8 of Karatzas and Shreve [17]), we obtain that

$$\begin{aligned} & E\left[\sup_{0\le t\le T}\left(E[\ln X]\int_0^t \varphi^{'}(p^{l,n}(s-))d(\bar{N}_L^n(s)-n^{-1}\lambda_L^n s)\right)^2\right] \\ \le\ & 4(E[\ln X])^2 E\left[\left(\int_0^T \varphi^{'}(p^{l,n}(s-))d(\bar{N}_L^n(s)-n^{-1}\lambda_L^n s)\right)^2\right]. \end{aligned} \tag{154}$$

However, note that

$$\begin{aligned} & E\left[\left(\int_0^T \varphi^{'}(p^{l,n}(s-))d(\bar{N}_L^n(s)-n^{-1}\lambda_L^n s)\right)^2\right] \\ =\ & \frac{1}{n}E\left[\int_0^T (\varphi^{'}(p^{l,n}(s-)))^2 d(n^{-1}\lambda_L^n s)\right] \\ \le\ & n^{-2}\lambda_L^n\cdot T\cdot\sup_{x\in\mathbb{R}}|\varphi^{'}(x)| \\ \to\ & 0 \ \text{ as } \ n\to\infty, \end{aligned} \tag{155}$$

where the final convergence follows by Item 1 of Definition 3.1. Thus, combining (154) and (155) together, we obtain that

$$E\left[\sup_{0\le t\le T}\left(E[\ln X]\int_0^t \varphi^{'}(p^{l,n}(s-))d(\bar{N}_L^n(s)-n^{-1}\lambda_L^n s)\right)^2\right] \ \to\ 0 \ \text{ as } \ n\to\infty, \tag{156}$$

which completes our treatment of (152). Next, regarding (153), note that

$$\begin{aligned} & E\left[\sup_{0\le t\le T}\left|E[\ln X]\int_0^t \varphi^{'}(p^{l,n}(s-))d((n^{-1}\lambda_L^n-\lambda)s)\right|\right] \\ \le\ & |n^{-1}\lambda_L^n-\lambda|\cdot E[\ln X]\cdot E\left[\int_0^T |\varphi^{'}(p^{l,n}(s-))|ds\right] \\ \le\ & |n^{-1}\lambda_L^n-\lambda|\cdot E[\ln X]\cdot T\cdot\sup_{x\in\mathbb{R}}|\varphi^{'}(x)| \\ \to\ & 0 \ \text{ as } \ n\to\infty, \end{aligned}$$

where the final convergence follows by Item 1 of Definition 3.1. This completes our treatment of (153), which completes the proof. □

9.2.2 Two Lemmas

In this subsubsection, we provide two lemmas, both of which play a useful and important role in the proofs in Section 9.2.3 below. Each of these lemmas are of a sample path nature. We begin with the following result.

Lemma 9.9. *For each $n\ge 1$, if $0\le t_1\le t_2$ and $p^{l,n}(t_2)<p^{l,n}(t_1)$, then for all $p^{l,n}(t_2)\le x_1<x_2\le p^{l,n}(t_1)$,*

$$\tilde{\mu}^{l,n}(t_2)\left([x_1,x_2]\right)+\frac{1}{\sqrt{n}}\ge (x_2-x_1)/C.$$

Proof. Let

$$y_1 = \max\left\{p^{l,n}(t):\ t\in[t_1,t_2]\right\},$$

$$T_1 = \sup\left\{t\in[t_1,t_2]:\ p^{l,n}(t-)=y_1\right\}$$

and

$$S_1 = \inf\left\{t\in[t_1,T_1):\ p^{l,n}(\cdot)\text{ has no jump in }[t,T_1)\right\}.$$

Hence $[S_1,T_1)$ is the interval on which $p^{l,n}(\cdot)$ achieves its maximum y_1, and if there are several such intervals, then $[S_1,T_1)$ is the largest of those intervals. We shall now define the intervals $[S_i,T_i)$ recursively. If $i\geq 2$ and $T_{i-1}<t_2$, then

$$y_i = \max\left\{p^{l,n}(t):\ t\in[T_{i-1},t_2]\right\},$$

$$T_i = \max\left\{t\in[T_{i-1},t_2]:\ p^{l,n}(t-)=y_i\right\}$$

and

$$S_i = \inf\left\{t\in[T_{i-i},T_i):\ p^{l,n}(\cdot)\text{ has no jump in }[t,T_i)\right\}.$$

Let k be the first index such that $T_k=t_2$; we stop the recursion at this point. A particular scenario using the above notation is shown in Figure 6. We now have $t_1\leq S_1<T_1\leq S_2<T_2\cdots\leq S_k<T_k=t_2$ and $y_1>y_2>\cdots>y_k=p^{l,n}(t_2)$, and $y_1\geq p^{l,n}(t_1)$. All the points $y_1,\ldots,y_k$ are in the book at time t_2, i.e., $\mu^{l,n}(t_2)(\{y_i\})>0$ for every $i=1,\ldots,k$. Indeed, the only way an order y_i could have been removed from the book is if a market order arrived at the time when y_i was the prevailing price, but this could not have happened because that would imply an upward jump of the price at that time. Next we note that $N^n_L(t_2)-N^n_L(t_1)>0$ because $p^{l,n}(\cdot)$ decreased between times t_1 and t_2, and so we can define the non-empty set

$$J=\{N^n_L(t_1)+1,\ldots,N^n_L(t_2)\}.$$

We shall now show that

$$y_{i+1}-y_i \ \leq\ \max\left\{\frac{1}{\sqrt{n}}|\ln X_j|:\ j\in J\right\}. \tag{157}$$

By the definition of T_{i+1}, we must have $p^{l,n}(T_i)\leq y_{i+1}$, which implies that $y_i-y_{i+1}\leq y_i-p^{l,n}(T_i)=p^{l,n}(T_i-)-p^{l,n}(T_i)$. This is a downward jump of $p^{l,n}(\cdot)$ between times t_1 and t_2, hence $p^{l,n}(T_i-)-p^{l,n}(T_i)\in\left\{\frac{1}{\sqrt{n}}|\ln X_j|:\ j\in J\right\}$, which implies (157). All the y's are in the book at time t_2, and hence

$$\left(\mu^{l,n}(t_2)([x_1,x_2])+1\right)\max\left\{\frac{1}{\sqrt{n}}|\ln X_j|:\ j\in J\right\}\geq x_2-x_1,$$

and the statement of the lemma follows.

□

Our next lemma completes this section. We have the following.

Lemma 9.10. *For each $n\geq 1$, if $0\leq s\leq t$, then*

$$\int_s^t 1\{p^{l,n}(u-)<p^{l,n}(s)\}dL^n(u) \ = \ 0.$$

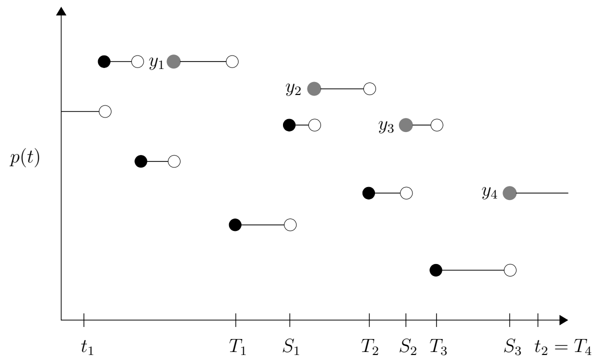

Figure 6: The shaded circles represent those limit orders remaining on the book at time t_2. They are also the successive maxima of the reversed price process.

Proof. First note that it suffices to prove that $\tilde{\mu}^{l,n}(u)([p^{l,n}(s), \infty)) \geq \tilde{a}^n$ for all $u \geq s$ such that $p^{l,n}(u-) < p^{l,n}(s)$. Indeed, let $y = \sup\{p^{l,n}(t) : \ t \in [s, u]\}$ and $T = \inf\left\{t \in [s, u] : \ p^{l,n}(t) = y\right\}$. Clearly $s \leq T < u$ (this holds even if $p^{l,n}$ has a jump at u). It follows that every order that was on the book at time T is still on the book at time u. So we have

$$\tilde{a}^n \leq \tilde{\mu}^{l,n}(T)([p^{l,n}(T), \infty)) \leq \tilde{\mu}^{l,n}(u)([p^{l,n}(T), \infty)) \leq \tilde{\mu}^{l,n}(u)([p^{l,n}(s), \infty)).$$

This completes the proof. □

9.2.3 Proof of Lemma 9.2

In this subsubsection, we use the results of Sections 9.2.1 and 9.2.2 above in order to prove Lemma 9.2. We begin by setting up some additional notation. For each $\varepsilon > 0$ and $T \geq 0$, let

$$\varrho^n_\varepsilon \quad = \quad \inf\{t \geq 0 : (p^{l,n}(t) - p^{l,n}(0)) \geq \varepsilon\} \wedge T \tag{158}$$

be the first time that the natural log of the price is greater than or equal to $p^{l,n}(0) + \varepsilon$ (or T, whichever is smaller). Note also the subtle difference between ϱ^n_ε as defined in (158) above and ν^n_ε as defined in (79) in Section 5. Figure 7 will orientate the reader for the following Proposition.

Proposition 9.11. *If $E[\ln X] > 0$ and if Items 1-4 of Definition 3.1 hold, then, for each $\varepsilon, \delta, \vartheta > 0$,*

$$P\left(\inf_{\varrho^n_\varepsilon \leq u \leq \varrho^n_{\varepsilon+\delta}} (p^{l,n}(u) - p^{l,n}(0)) \leq \varepsilon - \vartheta\right) \quad \to \quad 0 \quad as \quad n \to \infty. \tag{159}$$

Proof. We first show that without any loss of generality, we can assume in this proof that $p^{l,n}(0) = 0$, a.s., for all $n \geq 1$. Define the sequence of limit order books corresponding to the sequence of measure-valued processes $\left\{\hat{\mu}^{l,n}, \ n \geq 1\right\}$ with the (random) initial measure

$$\hat{\mu}^{l,n}(0)\left(\{x\}\right) \quad = \quad \mu^{l,n}(0)\left(\{x + p^{l,n}(0)\}\right), \quad x \in \mathbb{R},$$

keeping all other random quantities $(X_{n,i}, N_L^n, N_M^n)$ the same. Next, let $\hat{\mu}^{l,n}(t)$ be the random measure corresponding to the nth book at time $t \geq 0$ with the above initial measure, and let

$$\hat{p}^{l,n}(t) \quad = \quad \sup\left\{x \in \mathbb{R} : \ \hat{\mu}^{l,n}(t)\left((-\infty, x)\right) = 0\right\}. \tag{160}$$

Then, clearly we have that

$$\hat{p}^{l,n}(t) \quad = \quad p^{l,n}(t) - p^{l,n}(0) \tag{161}$$

and

$$\hat{\mu}^{l,n}(t)\left(\{x\}\right) \quad = \quad \mu^{l,n}(t)\left(\{x + p^{l,n}(0)\}\right), \quad x \in \mathbb{R}. \tag{162}$$

In particular,

$$\hat{p}^{l,n}(0) \quad = \quad 0. \tag{163}$$

Now, for each $\varepsilon > 0$, let

$$\hat{\varrho}^n_\varepsilon \quad = \quad \inf\{t \geq 0 : \hat{p}^{l,n}(t) \geq \varepsilon\} \wedge T, \tag{164}$$

so that we have that $\hat{\varrho}^n_\varepsilon = \varrho^n_\varepsilon$. It then follows from (161) that for each $0 < \vartheta < \varepsilon$,

$$P\left(\inf_{\varrho^n_\varepsilon \leq u \leq \varrho^n_{\varepsilon+\delta}} (p^{l,n}(u) - p^{l,n}(0)) \leq \varepsilon - \vartheta\right) \quad = \quad P\left(\inf_{\hat{\varrho}^n_\varepsilon \leq u \leq \hat{\varrho}^n_{\varepsilon+\delta}} \left(\hat{p}^{l,n}(u)\right) \leq \varepsilon - \vartheta\right).$$

Hence, in order to complete the proof, it suffices to show that the right-hand side of the above equality converges to zero as n goes to ∞. However, by (160), (162) and (163), this implies that for the remainder of the proof it is sufficient to assume that $p^{l,n}(0) = 0$, a.s., for all $n \geq 1$. We proceed as follows.

The probability in the statement of the lemma is a non-increasing function of $\vartheta > 0$, so without any loss of generality we can and shall assume that $0 < \vartheta < \varepsilon$. Next, for each $n \geq 1$, set

$$\zeta^\varepsilon_\vartheta \quad = \quad \inf\{t \geq \varrho^n_\varepsilon : p^{l,n}(t) \leq \varepsilon - \vartheta\} \wedge T \tag{165}$$

to be the first time after crossing the threshold ε that the price process drops back down below the level $\varepsilon - \vartheta$. Note that in the above we have suppressed the dependence of $\zeta^\varepsilon_\vartheta$ on n. Next, note that clearly $\zeta^\varepsilon_\vartheta$ is a stopping time. Moreover, by definition, $\varrho^n_\varepsilon \leq \zeta^\varepsilon_\vartheta$. Now using (76) along with the definition of $\Delta\varphi^{l,n}$ in (147), one obtains that for arbitrary $\varphi \in C^2_b(\mathbb{R})$ and $\delta > 0$,

$$\begin{aligned}\langle \tilde{\mu}^{l,n}(\zeta^\varepsilon_\vartheta \wedge \varrho^n_{\varepsilon+\delta}), \varphi \rangle \quad = \quad & \langle \tilde{\mu}^{l,n}(\varrho^n_\varepsilon), \varphi \rangle + \Delta\varphi^{l,n}(\zeta^\varepsilon_\vartheta \wedge \varrho^n_{\varepsilon+\delta}) - \Delta\varphi^{l,n}(\varrho^n_\varepsilon) \\ & + \lambda E[\ln X] \int_{\varrho^n_\varepsilon}^{\zeta^\varepsilon_\vartheta \wedge \varrho^n_{\varepsilon+\delta}} \varphi'(p^{l,n}(s-))ds \\ & + \int_{\varrho^n_\varepsilon}^{\zeta^\varepsilon_\vartheta \wedge \varrho^n_{\varepsilon+\delta}} \varphi(p^{l,n}(s-))d\tilde{S}^n(s).\end{aligned} \tag{166}$$

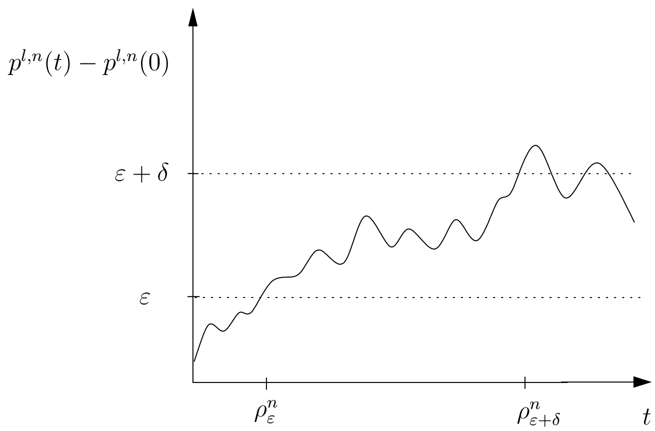

Figure 7: Stopping times ρ^n_ε and $\rho^n_{\varepsilon+\delta}$.

Moreover, since ϱ^n_ε and $\zeta^\varepsilon_\vartheta \wedge \varrho^n_{\varepsilon+\delta}$ are both stopping times, it follows by the semi-martingale decomposition (11) that

$$E\left[\int_{\varrho^n_\varepsilon}^{\zeta^\varepsilon_\vartheta \wedge \varrho^n_{\varepsilon+\delta}} \varphi(p^{l,n}(s-))d\tilde{S}^n(s)\right] \quad = \quad E\left[\int_{\varrho^n_\varepsilon}^{\zeta^\varepsilon_\vartheta \wedge \varrho^n_{\varepsilon+\delta}} \varphi(p^{l,n}(s-))d\tilde{B}^n(s)\right]. \tag{167}$$

Thus, taking expectations in (166), one obtains that

$$\begin{aligned} E[\langle \tilde{\mu}^{l,n}(\zeta^\varepsilon_\vartheta \wedge \varrho^n_{\varepsilon+\delta}), \varphi\rangle] \quad &= \quad E[\langle \tilde{\mu}^{l,n}(\varrho^n_\varepsilon), \varphi\rangle] + E[\Delta\varphi^{l,n}(\zeta^\varepsilon_\vartheta \wedge \varrho^n_{\varepsilon+\delta}) - \Delta\varphi^{l,n}(\varrho^n_\varepsilon)] \\ &\quad + \lambda E[\ln X] E\left[\int_{\varrho^n_\varepsilon}^{\zeta^\varepsilon_\vartheta \wedge \varrho^n_{\varepsilon+\delta}} \varphi^{'}(p^{l,n}(s-))ds\right] \\ &\quad + E\left[\int_{\varrho^n_\varepsilon}^{\zeta^\varepsilon_\vartheta \wedge \varrho^n_{\varepsilon+\delta}} \varphi(p^{l,n}(s-))d\tilde{B}^n(s)\right]. \end{aligned} \tag{168}$$

Our next step is to substitute a class of bounded, non-negative functions $\{\varphi^\eta_\varepsilon \in C^2_b(\mathbb{R}),\ \eta > 0\}$ into (168) which also satisfy the following additional conditions:

$$\varphi^\eta_\varepsilon \quad = \quad (x-\varepsilon)^2, \quad x \in \mathbb{R},\ \eta > 0, \tag{169}$$

$$\sup_{x \geq \varepsilon} \varphi^\eta_\varepsilon(x) \quad < \quad \frac{1}{\eta} \quad \text{and} \quad \sup_{x \geq \varepsilon} (\varphi^\eta_\varepsilon)^{'}(x) \quad < \quad \frac{1}{\eta}, \quad \eta > 0. \tag{170}$$

We will then use the equality (168) together with our specially chosen functions and the fact that $E[\ln X] > 0$ in order to ultimately prove (159). In order to be concrete, we define our functions $\varphi^\eta_\varepsilon \in C^2_b(\mathbb{R}),\ \eta > 0$, as follows. For each $\eta > 0$, we let φ^η_ε be the non-negative function defined by

$$\varphi^\eta_\varepsilon(x) \quad = \quad \begin{cases} c_1 \arctan\left(-\left(\epsilon + c_2 x\right)\right) + c_3 & \text{if } x \leq 0, \\ (x-\varepsilon)^2 & \text{if } 0 \leq x \leq \varepsilon, \\ 2\left(\frac{1}{\eta^2} - \left(\frac{(x-\varepsilon)}{\eta} + \frac{1}{\eta^2}\right) e^{-\eta(x-\varepsilon)}\right) & \text{if } \varepsilon \leq x, \end{cases}$$

where we set

$$c_1 = 4\epsilon^3, \quad c_2 = \frac{1+\epsilon^2}{2\epsilon^2} \quad \text{and} \quad c_3 = 4\epsilon^3 (\arctan \epsilon) + \epsilon^2.$$

We now show how the sequence of functions $\{\varphi_\varepsilon^\eta, \eta > 0\}$ may be used in order to obtain a sequence of upper bounds on the probability $P(\zeta_\vartheta^\varepsilon < \varrho_{\varepsilon+\delta}^n)$. Specifically, note that by the definition of ϱ_ε^n in (158) and of $\zeta_\vartheta^\varepsilon$ in (165), we have that in order to complete the proof, it suffices to show that for each $\varepsilon, \delta > 0$ and $0 < \vartheta < \varepsilon$, we have that

$$P(\zeta_\vartheta^\varepsilon < \varrho_{\varepsilon+\delta}^n) \quad \to \quad 0 \quad \text{as} \quad n \to \infty. \tag{171}$$

We proceed as follows. First note that for each $\eta > 0$, we have the decomposition

$$\begin{aligned} E[\langle \tilde{\mu}^{l,n}(\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n), \varphi_\varepsilon^\eta \rangle] \quad &= \quad E[\langle \tilde{\mu}^{l,n}(\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n), \varphi_\varepsilon^\eta \rangle 1\{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n = \zeta_\vartheta^\varepsilon < T\}] \\ &\quad + E[\langle \tilde{\mu}^{l,n}(\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n), \varphi_\varepsilon^\eta \rangle 1\{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n = \varrho_{\varepsilon+\delta}^n < T\}] \\ &\quad + E[\langle \tilde{\mu}^{l,n}(\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n), \varphi_\varepsilon^\eta \rangle 1\{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n = T\}]. \end{aligned}$$

Next, note that on the event $\{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n = \zeta_\vartheta^\varepsilon < T\}$, one has that $p^{l,n}(\zeta_\vartheta^\varepsilon) < p^{l,n}(\varrho_\varepsilon^n)$ and that, also on the event $\{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n = \zeta_\vartheta^\varepsilon < T\}$, it follows by (158) and (165) that

$$p^{l,n}(\zeta_\vartheta^\varepsilon) \;\leq\; \varepsilon - \vartheta \;<\; \varepsilon \;\leq\; p^{l,n}(\varrho_\varepsilon^n).$$

Hence, by Lemma 9.9 of Section 9.2.2, the fact that by assumption $0 < \vartheta < \varepsilon$, and the fact that φ_ε^η is decreasing on $[0, \varepsilon]$, we have that

$$\begin{aligned} & E[\langle \tilde{\mu}^{l,n}(\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n), \varphi_\varepsilon^\eta \rangle 1\{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n = \zeta_\vartheta^\varepsilon < T\}] \\ = \;& E[\langle \tilde{\mu}^{l,n}(\zeta_\vartheta^\varepsilon), \varphi_\varepsilon^\eta \rangle 1\{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n = \zeta_\vartheta^\varepsilon < T\}] \\ \geq \;& E[\varphi_\varepsilon^\eta(\varepsilon - \vartheta/2) \tilde{\mu}^{l,n}(\zeta_\vartheta^\varepsilon)([\varepsilon - \vartheta, \varepsilon - \vartheta/2]) 1\{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n = \zeta_\vartheta^\varepsilon < T\}] \\ = \;& \frac{\vartheta^2}{4} E[\tilde{\mu}^{l,n}(\zeta_\vartheta^\varepsilon)([\varepsilon - \vartheta, \varepsilon - \vartheta/2]) 1\{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n = \zeta_\vartheta^\varepsilon < T\}] \\ \geq \;& \frac{\vartheta^2}{4} \left(\frac{\vartheta}{2C} - \frac{1}{\sqrt{n}} \right) E[1\{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n = \zeta_\vartheta^\varepsilon < T\}] \\ = \;& \frac{\vartheta^2}{4} \left(\frac{\vartheta}{2C} - \frac{1}{\sqrt{n}} \right) P(\zeta_\vartheta^\varepsilon < \varrho_{\varepsilon+\delta}^n). \end{aligned}$$

Moreover, note that for sufficiently large n we have that

$$\frac{\vartheta^2}{4} \left(\frac{\vartheta}{2C} - \frac{1}{\sqrt{n}} \right) P(\zeta_\vartheta^\varepsilon < \varrho_{\varepsilon+\delta}^n) \quad \geq \quad \frac{\vartheta^3}{16C} P(\zeta_\vartheta^\varepsilon < \varrho_{\varepsilon+\delta}^n),$$

and that by the non-negativity of φ_ε^η, it follows that both

$$E[\langle \tilde{\mu}^{l,n}(\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n), \varphi_\varepsilon^\eta \rangle 1\{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n = \varrho_{\varepsilon+\delta}^n < T\}] \quad \geq \quad 0$$

and

$$E[\langle \tilde{\mu}^{l,n}(\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n), \varphi_\varepsilon^\eta \rangle 1\{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n = T\}] \quad \geq \quad 0.$$

Hence, by (168) with $\varphi = \varphi_\varepsilon^\eta$, one obtains that for sufficiently large n,

$$
\begin{aligned}
(\vartheta^3/16C)P(\zeta_\vartheta^\varepsilon < \varrho_{\varepsilon+\delta}^n) \quad \leq \quad & E[\langle \tilde{\mu}^{l,n}(\varrho_\varepsilon^n), \varphi_\varepsilon^\eta \rangle] + E[\Delta(\varphi_\varepsilon^\eta)^{l,n}(\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n) - \Delta(\varphi_\varepsilon^\eta)^{l,n}(\varrho_\varepsilon^n)] \quad (172) \\
& + \lambda E[\ln X] E\left[\int_{\varrho_\varepsilon^n}^{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n} (\varphi_\varepsilon^\eta)'(p^{l,n}(s-))ds\right] \\
& + E\left[\int_{\varrho_\varepsilon^n}^{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n} \varphi_\varepsilon^\eta(p^{l,n}(s-))d\tilde{B}^n(s)\right],
\end{aligned}
$$

which provides our first desired upper bound on $P(\zeta_\vartheta^\varepsilon < \varrho_{\varepsilon+\delta}^n)$.

We next proceed to bound the terms on the righthand side of (172). In particular, note that we may write

$$
\begin{aligned}
& \lambda E[\ln X] E\left[\int_{\varrho_\varepsilon^n}^{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n} (\varphi_\varepsilon^\eta)'(p^{l,n}(s-))ds\right] + E\left[\int_{\varrho_\varepsilon^n}^{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n} \varphi_\varepsilon^\eta(p^{l,n}(s-))d\tilde{B}^n(s)\right] \\
= \; & \lambda E[\ln X] E\left[\int_{\varrho_\varepsilon^n}^{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n} 1\{p^{l,n}(s-) < \varepsilon\}(\varphi_\varepsilon^\eta)'(p^{l,n}(s-))ds\right] \quad (173) \\
& + E\left[\int_{\varrho_\varepsilon^n}^{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n} 1\{p^{l,n}(s-) < \varepsilon\}\varphi_\varepsilon^\eta(p^{l,n}(s-))d\tilde{B}^n(s)\right] \\
& + \lambda E[\ln X] E\left[\int_{\varrho_\varepsilon^n}^{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n} 1\{p^{l,n}(s-) \geq \varepsilon\}(\varphi_\varepsilon^\eta)'(p^{l,n}(s-))ds\right] \quad (174) \\
& + E\left[\int_{\varrho_\varepsilon^n}^{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n} 1\{p^{l,n}(s-) \geq \varepsilon\}\varphi_\varepsilon^\eta(p^{l,n}(s-))d\tilde{B}^n(s)\right].
\end{aligned}
$$

Therefore, our next step is to analyze the terms (173) and (174), providing convenient upper bounds for each. We begin with (173).

First note that by the definition of $\tilde{B}^n$ in (13) and Item 2 of Definition 3.1, we have that for sufficiently large n,

$$
\begin{aligned}
& \lambda E[\ln X] E\left[\int_{\varrho_\varepsilon^n}^{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n} 1\{p^{l,n}(s-) < \varepsilon\}(\varphi_\varepsilon^\eta)'(p^{l,n}(s-))ds\right] \quad (175) \\
& + E\left[\int_{\varrho_\varepsilon^n}^{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n} 1\{p^{l,n}(s-) < \varepsilon\}\varphi_\varepsilon^\eta(p^{l,n}(s-))d\tilde{B}^n(s)\right] \\
\leq \; & E\left[\int_{\varrho_\varepsilon^n}^{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n} 1\{p^{l,n}(s-) < \varepsilon\}(\lambda E[\ln X](\varphi_\varepsilon^\eta)'(p^{l,n}(s-)) + 2|\theta|\varphi_\varepsilon^\eta(p^{l,n}(s-)))ds\right] \\
& + E\left[\int_{\varrho_\varepsilon^n}^{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n} 1\{p^{l,n}(s-) < \varepsilon\}\varphi_\varepsilon^\eta(p^{l,n}(s-))d\tilde{L}^n(s)\right].
\end{aligned}
$$

Next, note that since $\lambda > 0$ and $E[\ln X] > 0$, it follows by the definition of φ_ε^η above that for ϑ sufficiently small (and independent of η of and n) and $\varepsilon - \vartheta \leq x \leq \varepsilon$,

$$
\lambda E[\ln X](\varphi_\varepsilon^\eta)'(x) + 2|\theta|\varphi_\varepsilon^\eta(x) \quad \leq \quad 0.
$$

Hence,

$$1\{p^{l,n}(s-) < \varepsilon\}(\lambda E[\ln X](\varphi_\varepsilon^\eta)'(p^{l,n}(s-)) + 2|\theta|\varphi_\varepsilon^\eta(p^{l,n}(s-))) \leq 0 \tag{176}$$

for $\varrho_\varepsilon^n \leq s < \zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n$ and ϑ sufficiently small. Moreover, recall by Lemma 9.10 of Section 9.2.2 that

$$\int_{\varrho_\varepsilon^n}^{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n} 1\{p^{l,n}(s-) < \varepsilon\} d\tilde{L}^n(s) = 0. \tag{177}$$

Now combining (176) and (177), we obtain from (175) that for n sufficiently large and ϑ sufficiently small (how small ϑ should be does not depend on n),

$$\begin{aligned} &\lambda E[\ln X] E\left[\int_{\varrho_\varepsilon^n}^{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n} 1\{p^{l,n}(s-) < \varepsilon\}(\varphi_\varepsilon^\eta)'(p^{l,n}(s-))ds\right] \\ &+E\left[\int_{\varrho_\varepsilon^n}^{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n} 1\{p^{l,n}(s-) < \varepsilon\}\varphi_\varepsilon^\eta(p^{l,n}(s-))d\tilde{B}^n(s)\right] \\ \leq\ & 0. \end{aligned} \tag{178}$$

This provides our upper bound for (173)

We next proceed to bound (174). First recall by (170) that for η sufficiently large we have that

$$1\{p^{l,n}(s-) \geq \varepsilon\}(\varphi_\varepsilon^\eta)'(p^{l,n}(s-)) < \frac{1}{\eta}$$

and

$$1\{p^{l,n}(s-) \geq \varepsilon\}\varphi_\varepsilon^\eta(p^{l,n}(s-)) < \frac{1}{\eta}.$$

Hence, for η sufficiently large, it follows that for n sufficiently large (independent of η) we have that

$$\begin{aligned} &\lambda E[\ln X] E\left[\int_{\varrho_\varepsilon^n}^{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n} 1\{p^{l,n}(s-) \geq \varepsilon\}(\varphi_\varepsilon^\eta)'(p^{l,n}(s-))ds\right] \\ &+E\left[\int_{\varrho_\varepsilon^n}^{\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n} 1\{p^{l,n}(s-) \geq \varepsilon\}\varphi_\varepsilon^\eta(p^{l,n}(s-))d\tilde{B}^n(s)\right] \\ \leq\ & (1/\eta)((\lambda E[\ln X] + 2|\theta|)T + E[\tilde{L}^n(T)]) \\ \leq\ & K_1/\eta, \end{aligned} \tag{179}$$

where the constant $K_1 > 0$ follows by the fact that by (1) we have that $0 < E[\ln X] < \infty$ and since by Lemma 9.7 of Section 9.2.1, we have that the sequence

$$\{(\lambda E[\ln(X_1)] + 2|\theta|)T + E[\tilde{L}^n(T)], n \geq 1\}$$

is bounded. (179) now provides our desired upper bound for (174).

Finally, note that by Lemma 9.8 of Section 9.2.1, we have that

$$E[\Delta(\varphi_\varepsilon^\eta)^{l,n}(\zeta_\vartheta^\varepsilon \wedge \varrho_{\varepsilon+\delta}^n) - \Delta(\varphi_\varepsilon^\eta)^{l,n}(\varrho_\varepsilon^n)] \leq 2E\left[\sup_{0\leq t\leq T} |\Delta(\varphi_\varepsilon^\eta)^{l,n}(t)|\right] \to 0, \tag{180}$$

as $n \to \infty$. Moreover, note that since $p^{l,n}(\varrho_\varepsilon^n) \geq \varepsilon$, by (170) we have that for η sufficiently large (and independent of n),

$$E[\langle \tilde{\mu}^{l,n}(\varrho_\varepsilon^n), \varphi_\varepsilon^\eta \rangle] \;\; \leq \;\; \frac{1}{\eta} E[\tilde{S}^n(\varrho_\varepsilon^n)] \;\; \leq \;\; \frac{1}{\eta} E\left[\sup_{0 \leq t \leq T} \tilde{S}^n(t) \right] \;\; \leq \;\; \frac{K_2}{\eta}, \tag{181}$$

where the constant $K_2 > 0$ follows since by Lemma 9.7 of Section 9.2.1, we have that the sequence $\{E\left[\sup_{0 \leq t \leq T} \tilde{S}^n(t)\right], n \geq 1\}$ is bounded.

Thus, from (172), (178), (179), (180) and (181), we obtain that for each $\eta > 0$ sufficiently large and $\vartheta > 0$ sufficiently small and $\varsigma > 0$, we have that for sufficiently large n,

$$(\vartheta^3/16C) P(\zeta_\vartheta^\varepsilon < \varrho_{\varepsilon+\delta}^n) \;\; \leq \;\; (K_1 + K_2)/\eta + \varsigma.$$

This then implies that

$$P(\zeta_\vartheta^\varepsilon < \varrho_{\varepsilon+\delta}^n) \;\; \leq \;\; (16C/\vartheta^3)((K_1 + K_2)/\eta + \varsigma).$$

However, since η may be made arbitrarily large and ς arbitrarily small, the above inequality implies (171), which completes proof. $\square$

We now show that Proposition 9.11 implies the following somewhat stronger result.

Proposition 9.12. *If $E[\ln X] > 0$ and if Items 1-4 of Definition 3.1 hold, then, for each $\varepsilon > 0$,*

$$E\left[\left(\sup_{0 \leq t \leq \varrho_\varepsilon^n} \left| (p^{l,n}(t) - p^{l,n}(0)) - \sup_{0 \leq s \leq t} (p^{l,n}(s) - p^{l,n}(0)) \right| \right)^4 \right] \;\; \to \;\; 0 \;\; as \;\; n \to \infty.$$

Proof. Let $\varepsilon > 0$ and, for each $0 < \delta < \varepsilon$, let $\{[t_i, t_{i+1}), i = 0, ..., \lceil 2\varepsilon/\delta \rceil - 1\}$ be a partition of $[0, \delta\lceil 2\varepsilon/\delta \rceil)$, where $t_i = i\delta$. Note of course that $\delta\lceil 2\varepsilon/\delta \rceil > \varepsilon$. Also note that by the definition of ϱ_ε^n in (158), one has that $\varrho_{t_i}^n \leq \varrho_{t_{i+1}}^n$ for $i = 0, ..., \lceil 2\varepsilon/\delta \rceil - 1$, and hence $\{[\varrho_{t_i}^n, \varrho_{t_{i+1}}^n), i = 0, ..., \lceil 2\varepsilon/\delta \rceil - 1\}$ forms a natural partition of $[0, \varrho_{\delta\lceil 2\varepsilon/\delta \rceil}^n)$. In the above partition, if $\varrho_{t_i}^n = \varrho_{t_{i+1}}^n$, then $[\varrho_{t_i}^n, \varrho_{t_{i+1}}^n) = \emptyset$ and we may simply ignore that interval. Now note that we may write

$$\begin{aligned} & \sup_{0 \leq t < \varrho_\varepsilon^n} \left| (p^{l,n}(t) - p^{l,n}(0)) - \sup_{0 \leq s \leq t} (p^{l,n}(s) - p^{l,n}(0)) \right| \\ \leq & \sup_{0 \leq t < \varrho_{\delta\lceil 2\varepsilon/\delta \rceil}^n} \left| (p^{l,n}(t) - p^{l,n}(0)) - \sup_{0 \leq s \leq t} (p^{l,n}(s) - p^{l,n}(0)) \right| \\ = & \; U_\varepsilon^n(\delta), \end{aligned} \tag{182}$$

where

$$U_\varepsilon^n(\delta) \;\; = \;\; \max_{i=0,...,\lceil 2\varepsilon/\delta \rceil - 1} \sup_{\varrho_{t_i}^n \leq t < \varrho_{t_{i+1}}^n} \left| (p^{l,n}(t) - p^{l,n}(0)) - \sup_{0 \leq s \leq t} (p^{l,n}(s) - p^{l,n}(0)) \right|. \tag{183}$$

Our goal then is to show that for each $0 < \delta < \varepsilon$, we have that

$$\limsup_{n \geq 1} E\left[(U_\varepsilon^n(\delta))^4 \right] \;\; \leq \;\; \delta^4,$$

which, by (182), will complete the proof.

First note that by the definition of ϱ^n_ε in (158), one has that for $\varrho^n_{t_i} \le t < \varrho^n_{t_{i+1}}$,

$$i\delta \;\le\; \sup_{0\le s\le t} (p^{l,n}(s) - p^{l,n}(0)) \;\le\; (i+1)\delta.$$

Moreover, by Proposition 9.11, for each $\vartheta > 0$,

$$P\left(i\delta - \vartheta \le (p^{l,n}(t) - p^{l,n}(0)) \le (i+1)\delta \;\text{ for }\; \varrho^n_{t_i} \le t < \varrho^n_{t_{i+1}}\right) \;\to\; 1 \;\text{ as }\; n \to \infty.$$

Hence, for each $\vartheta > 0$,

$$P\left(\sup_{\varrho^n_{t_i} \le t < \varrho^n_{t_{i+1}}} \left|(p^{l,n}(t) - p^{l,n}(0)) - \sup_{0\le s\le t}(p^{l,n}(s) - p^{l,n}(0))\right| \le \delta + \vartheta\right) \;\to\; 1 \;\text{ as } n \to \infty. \tag{184}$$

Next, note we may write the identity

$$E\left[(U^n_\varepsilon(\delta))^4\right] \;=\; E\left[1\{U^n_\varepsilon(\delta) \le \delta + \vartheta\}(U^n_\varepsilon(\delta))^4\right] + E\left[1\{U^n_\varepsilon(\delta) > \delta + \vartheta\}(U^n_\varepsilon(\delta))^4\right]. \tag{185}$$

Moreover, clearly

$$E\left[1\{U^n_\varepsilon(\delta) \le \delta + \vartheta\}(U^n_\varepsilon(\delta))^4\right] \;\le\; (\delta + \vartheta)^4. \tag{186}$$

On the other hand, by the definition of ρ^n_ε in (158), one has that

$$U^n_\varepsilon(\delta) \;\le\; \delta\lceil 2\varepsilon/\delta\rceil,$$

and so by (183) and (184), it follows that

$$\begin{aligned} E\left[1\{U^n_\varepsilon(\delta) > \delta + \vartheta\}(U^n_\varepsilon(\delta))^4\right] \;&\le\; \delta^4(\lceil 2\varepsilon/\delta\rceil)^4 P(U^n_\varepsilon(\delta) > \delta + \vartheta) \\ &\to\; 0 \;\text{ as }\; n \to \infty. \end{aligned} \tag{187}$$

Thus, since $\vartheta > 0$ was arbitrary, it follows by (185), (186) and (187) that

$$\limsup_{n\ge 1} E\left[(U^n_\varepsilon(\delta))^4\right] \;\le\; \delta^4,$$

which completes the proof. □

Now note that using the definitions (79) and (158) of ν^n_ε and ϱ^n_ε, respectively, it follows immediately that $\nu^n_\varepsilon \le \varrho^n_\varepsilon$. Hence, we obtain that

$$\sup_{0\le t\le \nu^n_\varepsilon} \left|(p^{l,n}(t) - p^{l,n}(0)) - \sup_{0\le s\le t}(p^{l,n}(s) - p^{l,n}(0))\right| \;\le\; \sup_{0\le t\le \varrho^n_\varepsilon} \left|(p^{l,n}(t) - p^{l,n}(0)) - \sup_{0\le s\le t}(p^{l,n}(s) - p^{l,n}(0))\right|.$$

This then immediately implies the following as a result of Proposition 9.12.

Proposition 9.13. *If $E[\ln X] > 0$ and if Items 1-4 of Definition 3.1 hold, then, for each $\varepsilon > 0$,*

$$E\left[\left(\sup_{0\le t\le \nu^n_\varepsilon} \left|(p^{l,n}(t) - p^{l,n}(0)) - \sup_{0\le s\le t}(p^{l,n}(s) - p^{l,n}(0))\right|\right)^4\right] \;\to\; 0 \;\textit{ as }\; n \to \infty.$$

We are now in a position to prove Lemma 9.2. We have the following.

Proof of Lemma 9.2. Note that by the definition of $\Delta\tilde{\mu}^{l,n}(t)$ in (77), we may write the simple identity

$$\Delta\tilde{\mu}^{l,n}(t) = \langle\tilde{\mu}^{l,n}(t),\xi\rangle - \tilde{S}^n(t)\sup_{0\le s\le t} p^{l,n}(s) + \tilde{S}^n(t)\sup_{0\le s\le t}(p^{l,n}(s)-p^{l,n}(t)),$$

Thus

$$(\Delta\tilde{\mu}^{l,n}(t))^2 \le 2\left(\langle\tilde{\mu}^{l,n}(t),\xi\rangle - \tilde{S}^n(t)\sup_{0\le s\le t} p^{l,n}(s)\right)^2 + 2\left(\tilde{S}^n(t)\sup_{0\le s\le t}(p^{l,n}(s)-p^{l,n}(t))\right)^2 \quad (188)$$

Let $\varepsilon > 0$. We now show that

$$E\left[\sup_{0\le t\le\nu_\varepsilon^n}\left(\langle\tilde{\mu}^{l,n}(t),\xi\rangle - \tilde{S}^n(t)\sup_{0\le s\le t} p^{l,n}(s)\right)^2\right] \to 0 \quad (189)$$

and

$$E\left[\sup_{0\le t\le\nu_\varepsilon^n}\left(\tilde{S}^n(t)\sup_{0\le s\le t}(p^{l,n}(s)-p^{l,n}(t))\right)^2\right] \to 0, \quad (190)$$

as $n\to\infty$, which, by (188), will complete the proof. We proceed in order.

We begin with (189). First note that after some straightforward calculations, it may be shown that

$$\begin{aligned}&\langle\tilde{\mu}^{l,n}(t),\xi\rangle - \tilde{S}^n(t)\sup_{0\le s\le t} p^{l,n}(s) \quad (191)\\ \le\ & \tilde{S}^n(t)\max_{i=1,\dots,N_L^n(t)}\ln X_i^{1/n^{1/2}} + \langle\tilde{\mu}^{l,n}(0),\xi\rangle - \tilde{S}^n(0)p^{l,n}(0).\end{aligned}$$

Next, since $\langle\tilde{\mu}^{l,n}(t),\xi\rangle \ge \tilde{S}^n(t)p^{l,n}(t)$, it also follows that we may write

$$\begin{aligned}\langle\tilde{\mu}^{l,n}(t),\xi\rangle - \tilde{S}^n(t)\sup_{0\le s\le t} p^{l,n}(s) &\ge \tilde{S}^n(t)\left(p^{l,n}(t) - \sup_{0\le s\le t} p^{l,n}(s)\right) \quad (192)\\ &= \tilde{S}^n(t)\left((p^{l,n}(t)-p^{l,n}(0)) - \sup_{0\le s\le t}(p^{l,n}(s)-p^{l,n}(0))\right).\end{aligned}$$

Thus, using the basic identity $(x_1+\dots+x_m)^2 \le m(x_1^2+\dots+x_m^2), m\ge 1$, we have that (191) and (192) imply that

$$\begin{aligned}&(1/3)E\left[\sup_{0\le t\le\nu_\varepsilon^n}\left(\langle\tilde{\mu}^{l,n}(t),\xi\rangle - \tilde{S}^n(t)\sup_{0\le s\le t} p^{l,n}(s)\right)^2\right]\\ \le\ & E\left[\sup_{0\le t\le\nu_\varepsilon^n}(\tilde{S}^n(t)\max_{i=1,\dots,N_L^n(t)}\ln X_i^{1/n^{1/2}})^2\right] \quad (193)\\ &+E[(\langle\tilde{\mu}^{l,n}(0),\xi\rangle - \tilde{S}^n(0)p^{l,n}(0))^2] \quad (194)\end{aligned}$$

$$+E\left[\sup_{0\le t\le \nu_\varepsilon^n}\left(\tilde{S}^n(t)\left((p^{l,n}(t)-p^{l,n}(0))-\sup_{0\le s\le t}(p^{l,n}(s)-p^{l,n}(0))\right)\right)^2\right]. \tag{195}$$

We now show that each of (193) through (195) converge to 0 as n tends to ∞, which will complete our treatment of (189). Regrading (193), note that since $P(X_i\le C)=1$, we have that

$$E\left[\sup_{0\le t\le \nu_\varepsilon^n}(\tilde{S}^n(t)\max_{i=1,\dots,N_L^n(t)}\ln X_i^{1/n^{1/2}})^2\right] \;\le\; \frac{\ln C}{n}E\left[\sup_{0\le t\le T}(\tilde{S}^n(t))^2\right] \;\to\; 0 \;\text{ as }\; n\to\infty.$$

Next, for (194), recall that by assumption we have that

$$E[(\langle\tilde{\mu}^{l,n}(0),\xi\rangle-\tilde{S}^n(0)p^{l,n}(0))^2] \;\to\; 0 \;\text{ as }\; n\to\infty.$$

Finally, regarding (195), note that by the Cauchy Schwarz inequality, Proposition 9.13 and Lemma 9.7 of Section 9.2.1, we have that

$$\begin{aligned}
&E\left[\sup_{0\le t\le \nu_\varepsilon^n}\left(\tilde{S}^n(t)\left((p^{l,n}(t)-p^{l,n}(0))-\sup_{0\le s\le t}(p^{l,n}(s)-p^{l,n}(0))\right)\right)^2\right] \qquad (196)\\
\le\;& E\left[\sup_{0\le t\le T}(\tilde{S}^n(t))^4\right]E\left[\sup_{0\le t\le \nu_\varepsilon^n}\left((p^{l,n}(t)-p^{l,n}(0))-\sup_{0\le s\le t}(p^{l,n}(s)-p^{l,n}(0))\right)^4\right]\\
\to\;& 0 \;\text{ as }\; n\to\infty.
\end{aligned}$$

This completes our treatment of (189).

Finally, note that (190) follows by (196) upon noting that

$$\begin{aligned}
&E\left[\sup_{0\le t<\nu_\varepsilon^n}\left(\tilde{S}^n(t)(\sup_{0\le s\le t}p^{l,n}(s)-p^{l,n}(t))\right)^2\right]\\
=\;& E\left[\sup_{0\le t\le \nu_\varepsilon^n}\left(\tilde{S}^n(t)\left((p^{l,n}(t)-p^{l,n}(0))-\sup_{0\le s\le t}(p^{l,n}(s)-p^{l,n}(0))\right)\right)^2\right].
\end{aligned}$$

This completes the proof. □

9.3 Proof of Proposition 9.14

In this section, we provide the statement and proof of Proposition 9.14 concerning tightness of the sequence of diffusion scaled, measure-valued order book processes in the high frequency regime. In particular, we prove the following.

Proposition 9.14. *If $E[\ln X]>0$ and $E[(\Delta\tilde{\mu}^{l,n}(0))^2]\to 0$ and $(p^n(0),\tilde{S}^n(0))\Rightarrow(p(0),\tilde{S}(0))$ as $n\to\infty$, and if Items 1-4 of Definition 3.1 hold, then the sequence $\{\tilde{\mu}^n, n\ge 1\}$ is tight in $D(\mathbb{R}_+,\mathcal{M}_F(\mathbb{R}))$.*

Proposition 9.14 is used in Section 6 in the proof of Theorem 6.1 which is our main result concerning the weak convergence of the sequence of diffusion scaled, measure-valued order book processes in the high-frequency regime. However, before providing the proof of Proposition 9.14, we first need the following supporting result. Let $\mathrm{L}_c(\mathbb{R})$ be the set of Lipschitz continuous functions with compact support on $\mathbb{R}$. We then have the following.

Proposition 9.15. *If $E[\ln X] > 0$ and $E[(\Delta\tilde{\mu}^{l,n}(0))^2] \to 0$ and $(p^n(0), \tilde{S}^n(0)) \Rightarrow (p(0), \tilde{S}(0))$ as $n \to \infty$, and if Items 1-4 of Definition 3.1 hold, then, for each $m \geq 1$ and $\varphi_1, ..., \varphi_m \in \mathrm{L}_c(\mathbb{R})$, we have the joint convergence*

$$(\langle\tilde{\mu}^n, \varphi_1\rangle, ..., \langle\tilde{\mu}^n, \varphi_m\rangle) \;\Rightarrow\; (\tilde{S}\varphi_1(p), ..., \tilde{S}\varphi_m(p)) \;\; in \;\; \mathbb{D}^m([0,\infty), \mathbb{R}) \;\; as \;\; n \to \infty. \tag{197}$$

Proof. First note that for each $n \geq 1$ and $\varphi \in \mathrm{L}_c(\mathbb{R})$, we may write $\langle\tilde{\mu}^n, \varphi\rangle = \langle\tilde{\mu}^{l,n}, \hat{\varphi}\rangle$, where $\hat{\varphi} \in \mathrm{L}_c(\mathbb{R})$ is given by $\hat{\varphi}(x) = \varphi(e^x)$ for $x \in \mathbb{R}$. Hence, in order to show (197), it suffices to show that for each $m \geq 1$ and $\varphi_1, ..., \varphi_m \in \mathrm{L}_c(\mathbb{R})$,

$$(\langle\tilde{\mu}^{l,n}, \varphi_1\rangle, ..., \langle\tilde{\mu}^{l,n}, \varphi_m\rangle) \;\Rightarrow\; (\tilde{S}\varphi_1(p^l), ..., \tilde{S}\varphi_m(p^l)) \;\; \text{in} \;\; \mathbb{D}^m([0,\infty), \mathbb{R}) \;\; \text{as} \;\; n \to \infty. \tag{198}$$

Recall first from (132) the definition

$$\tilde{A}^n(t) \;=\; \int_0^t \frac{E[\ln X]}{\tilde{S}^n(s)} d\bar{N}_L^n(s), \quad t \geq 0,$$

and let $\tilde{A}^n = \{\tilde{A}^n(t), t \geq 0\}$. Next, let $m \geq 1$ and let $\varphi_1, ..., \varphi_m \in \mathrm{BL}(\mathbb{R})$. It then follows that for each $t \geq 0$, we may write

$$(\langle\tilde{\mu}^{l,n}(t), \varphi_1\rangle, ..., \langle\tilde{\mu}^{l,n}(t), \varphi_m\rangle) \;=\; (\tilde{S}^n(t)\varphi_1(\tilde{A}^n(t)), ..., \tilde{S}^n(t)\varphi_m(\tilde{A}^n(t))) + \tilde{\varepsilon}^n(t), \tag{199}$$

where

$$\tilde{\varepsilon}^n(t) \;=\; (\langle\tilde{\mu}^{l,n}(t), \varphi_1\rangle - \tilde{S}^n(t)\varphi_1(\tilde{A}^n(t)), ..., \langle\tilde{\mu}^{l,n}(t), \varphi_m\rangle - \tilde{S}^n(t)\varphi_m(\tilde{A}^n(t))). \tag{200}$$

Moreover, using similar arguments as in the proof of Proposition 9.4 of Section 9.1 above, it is straightforward to show that we have the convergence

$$(\tilde{S}^n\varphi_1(\tilde{A}^n), ..., \tilde{S}^n\varphi_m(\tilde{A}^n)) \;\Rightarrow\; (\tilde{S}\varphi_1(p^l), ..., \tilde{S}\varphi_m(p^l)) \;\; \text{in} \;\; \mathbb{D}^m([0,\infty), \mathbb{R}) \;\; \text{as} \;\; n \to \infty.$$

Hence, letting $\tilde{\varepsilon}^n = \{\tilde{\varepsilon}^n(t), t \geq 0\}$, in order to show (198) and complete the proof, it suffices by (199) to show that $\tilde{\varepsilon}^n \Rightarrow 0$ as $n \to \infty$.

First note that for each $i = 1, ..., m$, we may write

$$\langle\tilde{\mu}^{l,n}, \varphi_i\rangle - \tilde{S}^n\varphi_i(\tilde{A}^n) \;=\; (\langle\tilde{\mu}^{l,n}, \varphi_i\rangle - \tilde{S}^n\varphi_i(p^{l,n})) + (\tilde{S}^n(\varphi_i(p^{l,n}) - \varphi_i(\tilde{A}^n))). \tag{201}$$

We now show that both

$$(\langle\tilde{\mu}^{l,n}, \varphi_i\rangle - \tilde{S}^n\varphi_i(p^{l,n})) \;\Rightarrow\; 0 \;\; \text{as} \;\; n \to \infty \tag{202}$$

and

$$(\tilde{S}^n(\varphi_i(p^{l,n}) - \varphi_i(\tilde{A}^n))) \;\Rightarrow\; 0 \;\; \text{as} \;\; n \to \infty. \tag{203}$$

By (201), this will then show that $\langle\tilde{\mu}^{l,n}, \varphi_i\rangle - \tilde{S}^n\varphi_i(\tilde{A}^n) \Rightarrow 0$ as $n \to \infty$, which, by (200), will imply that $\tilde{\varepsilon}^n \Rightarrow 0$ as $n \to \infty$ and will compete the proof.

We begin with (202). First note that for each $t \geq 0$, the measure $\tilde{\mu}^{l,n}(t)$ is a linear combination of Dirac measures and so we may write

$$\tilde{\mu}^{l,n}(t) \;=\; \frac{1}{\sqrt{n}} \sum_{k=1}^{S^n(t)} \delta_{x_k^n(t)}.$$

Note also that in the above it is possible that $x_j^n(t) = x_k^n(t)$ for $j \neq k$. It now follows from the above that for each $t \geq 0$ we have that

$$\begin{aligned}\langle \tilde{\mu}^{l,n}(t), \varphi_i \rangle - \tilde{S}^n(t)\varphi_i(p^{l,n}(t)) &= \frac{1}{\sqrt{n}} \sum_{k=1}^{S^n(t)} \varphi_i\left(x_k^n(t)\right) - \tilde{S}^n(t)\varphi_i(p^{l,n}(t)) \\ &= \frac{1}{\sqrt{n}} \sum_{k=1}^{S^n(t)} (\varphi_i(x_k^n(t)) - \varphi_i(p^{l,n}(t))).\end{aligned}$$

Now let $T \geq 0$ and $\epsilon > 0$. We then have from the above that we may write

$$\begin{aligned}&\sup_{0\leq t\leq T} \left| \langle \tilde{\mu}^{l,n}(t), \varphi_i \rangle - \tilde{S}^n(t)\varphi_i(p^{l,n}(t)) \right| \\ &= \sup_{0\leq t\leq T} \left| \frac{1}{\sqrt{n}} \sum_{k=1}^{S^n(t)} \left(\varphi_i\left(x_k^n(t)\right) - \varphi_i(p^{l,n}(t))\right) \right| 1\{T \leq \nu_\epsilon^n\} \qquad (204) \\ &\quad + \sup_{0\leq t\leq T} \left| \frac{1}{\sqrt{n}} \sum_{k=1}^{S^n(t)} \left(\varphi_i\left(x_k^n(t)\right) - \tilde{S}^n(t)\varphi_i(p^{l,n}(t))\right) \right| 1\{T > \nu_\epsilon^n\}, \qquad (205)\end{aligned}$$

where we recall the definition of ν_ε^n from (79) of Section 9.1 above. Hence, in order to show (202), it now suffices to show that both (204) and (205) weakly converge to 0 as n tends to ∞. We first show that (204) converges to 0 in $L^2(P)$ as n tends to ∞, which implies weak convergence to 0. Let K be the Lipschitz constant associated with φ_i. We then have that

$$\begin{aligned}&E\left[\sup_{0\leq t\leq T} \left| \frac{1}{\sqrt{n}} \sum_{k=1}^{S^n(t)} \left(\varphi_i\left(x_k^n(t)\right) - \varphi_i(p^{l,n}(t))\right) \right|^2 1\{T \leq \nu_\epsilon^n\}\right] \\ &\leq K^2 E\left[\sup_{0\leq t\leq T} \left| \frac{1}{\sqrt{n}} \sum_{k=1}^{S^n(t)} \left(x_k^n(t) - p^{l,n}(t)\right) \right|^2 1\{T \leq \nu_\epsilon^n\}\right] \\ &= K^2 E\left[\sup_{0\leq t\leq T} \left(\Delta\tilde{\mu}^{l,n}(t)\right)^2 1\{T \leq \nu_\epsilon^n\}\right] \\ &\to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

where the final convergence above follows by Lemma 9.2 of Section 9.1. Thus, (204) weakly converges to 0 as n tends to ∞. Next, we show that (205) converges to zero in probability as n tends ∞, which implies weak convergence to 0. First note that by Proposition 9.6 of Section 9.1, we have that for every $\delta > 0$, there exists an $\epsilon > 0$ and an $N = N(\delta, \epsilon)$ such that for $n \geq N$, we have $P(\nu_\epsilon^n < T) < \delta$. This then implies for every $\gamma > 0$,

$$P\left(\sup_{0\leq t\leq T} \left| \frac{1}{\sqrt{n}} \sum_{k=1}^{S^n(t)} \left(\varphi_i\left(x_k^n(t)\right) - \tilde{S}^n(t)\varphi_i(p^{l,n}(t))\right) \right| 1\{T > \nu_\epsilon^n\} > \gamma\right) \leq P(\nu_\epsilon^n < T) < \delta.$$

Hence, (205) converges to zero in probability as n tends ∞, which completes the proof of (202).

We next proceed to (203). First note that by (140) of Theorem 9.1, we have that

$$p^{l,n} - \tilde{A}^n \quad \Rightarrow \quad 0 \ \text{ as } \ n \to \infty. \tag{206}$$

Hence, it follows since the function $\varphi_i \in \mathrm{L}_c(\mathbb{R})$ that $\varphi_i(p^{l,n}) - \varphi_i(\tilde{A}^n) \Rightarrow 0$ as $n \to \infty$. Thus, since $\tilde{S}^n \Rightarrow \tilde{S}$ as $n \to \infty$, it is immediate that (203) holds. This completes the proof. □

We are now in a position to provide the proof of Proposition 9.14. We have the following.

Proof of Proposition 9.14. It suffices to check that $\{\tilde{\mu}^n, n \geq 1\}$ satisfies the following two conditions of Jakubowski [15]

1. For each $T > 0$ and $\eta > 0$, there exists a compact set $\mathcal{K}_{T,\eta} \subset \mathcal{M}_F(\mathbb{R})$ such that

$$\liminf_{n\to\infty} P(\tilde{\mu}^n(t) \in \mathcal{K}_{T,\eta} \ \text{ for all } \ t \in [0,T]) > 1 - \eta.$$

2. There exists a family $\mathbb{F}$ of real, continuous functions F in $\mathcal{M}_F(\mathbb{R})$ that separates points on $\mathcal{M}_F(\mathbb{R})$ and is closed under addition such that the sequence $\{F(\tilde{\mu}^n), n \geq 1\}$ is tight in $D(\mathbb{R}_+, \mathbb{R})$ for each $F \in \mathbb{F}$.

We begin with Condition 1. Let $T > 0$ and $\eta > 0$. Next, recall from Theorem 15.7.5 of Kallenberg [16] that a set $\mathcal{M} \subset \mathcal{M}_F(\mathbb{R})$ is relatively compact if and only if $\sup_{\xi\in\mathcal{M}} \xi(\mathbb{R}) < \infty$ and $\sup_{\xi\in\mathcal{M}} \xi((-\infty, -n) \cup (n, \infty)) \to 0$ as $n \to \infty$. Now let $K^1_{T,\eta/2}$ be such that

$$\sup_{n\geq 1} P\left(\sup_{0\leq t\leq T} \tilde{S}^n(t) > K^1_{T,\eta/2}\right) \quad \leq \quad \eta/2. \tag{207}$$

Note that such a $K^1_{T,\eta/2}$ exists by Lemma 9.7 of Section 9.2.1. Next, let $K^2_{T,\eta/2}$ be such that

$$\sup_{n\geq 1} P\left(\sup_{0\leq t\leq T} (p^n(t) + C) > K^2_{T,\eta/2}\right) \quad \leq \quad \eta/2. \tag{208}$$

Note that such a $K^2_{T,\eta/2}$ exists by Theorem 5.1 of Section 5. We now define the set $\mathcal{M}_{T,\eta} \subset \mathcal{M}_F(\mathbb{R})$ by setting

$$\mathcal{M}_{T,\eta} \quad = \quad \{\xi \in \mathcal{M}_F(\mathbb{R}) : \xi(\mathbb{R}) \leq K^1_{T,\eta/2} \ \text{ and } \ \xi((-\infty, 0) \cup (K_{T,\eta/2}, \infty)) = 0\}.$$

It is then clear by Theorem 15.7.5 of Kallenberg [16] that $\mathcal{M}_{T,\eta}$ is relatively compact and, moreover, by (207), (208), system equation (6) and the assumption that $E[(\Delta\tilde{\mu}^{l,n}(0))^2] \to 0$ as $n \to \infty$, we have that

$$\liminf_{n\to\infty} P(\tilde{\mu}^n(t) \in \mathcal{M}_{T,\eta} \ \text{ for all } \ t \in [0,T]) > 1 - \eta.$$

Thus, the sequence $\{\tilde{\mu}^n, n \geq 1\}$ satisfies Condition 1 above.

Next, we note that Condition 2 follows from Proposition 9.15 above since the family of continuous functions $\mathbb{F} = \{\langle \cdot\ , \varphi\rangle, \varphi \in \mathrm{BL}(\mathbb{R})\}$ separates points on $\mathcal{M}_F(\mathbb{R})$ and is closed under addition. This completes the proof. □

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