Portfolio Optimization with Downside Constraints*

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Abstract

We consider the portfolio optimization problem for an investor whose consumption rate process and terminal wealth are subject to downside constraints. In the standard financial market model that consists of d risky assets and one riskless asset, we assume that the riskless asset earns a constant instantaneous rate of interest, $r > 0$, and that the risky assets are geometric Brownian motions. The optimal portfolio policy for a wide scale of utility functions is derived explicitly. The gradient operator and the Clark-Ocone formula in Malliavin calculus are used in the derivation of this policy. We show how Malliavin calculus approach can help us get around certain difficulties that arise in using the classical "delta hedging" approach.

KEY WORDS: optimal portfolio selection, utility maximization, downside constraint, Malliavin calculus, gradient operator, Clark-Ocone formula

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1 Introduction

We consider maximizing the expected utility from both consumption and terminal wealth for an investor whose consumption rate process must not fall below a given level R and whose terminal wealth must not fall below a given level K. This problem is closely related to two individual optimization problems: one is maximizing expected utility from consumption when consumption rate process must not fall below the constant R; the other is maximizing expectted utility from investment when the terminal wealth must not fall below the constant K. The optimal consumption rate and the optimal terminal wealth in the two individual expected utility maximization problems are given. The main purpose of this paper is to derive the optimal portfolio process for an expected utility maximizing investor who generates utility both from "living well" (i.e., from consumption) and from "becoming rich" (i.e., from terminal wealth) and whose consumption rate and terminal wealth are subject to deterministic downside constraints. We are going to use Malliavin calculus, in particular the gradient operator and the Clark-Ocone formula. This technique for computing hedging portfolios has been used before by Ocone & Karatzas (1991), Lakner (1998), and Bermin (1999) (2000) (2002). The "usual approach" to deriving hedging portfolios is the so called "delta hedging", which works in the following way. In a Markovian setting one can usually write the optimal wealth process in the form of $g(t, R_t)$ for some function $g(t, x_1, \ldots, x_d)$ where R_t is the d-dimensional return process for the stocks. A simple application of Ito's rule shows that if $g \in C^{1,2}([0, T] \times \Re^d)$, the amount of money invested in the i^{th} risky security at time t should be $\partial g(t, R_t)/\partial x_i$. However, the problem with this approach, as pointed out by Bermin (1999), is that one can not always guarantee the necessary differentiability condition for g. In our case g has an integral form such that the integrand is not even once differentiable in the variable x. Moreover, for some utility functions, g is not Lipschitz continuous. In order to calculate the Malliavin derivatives of g for a wide scale of utility functions, we shall formulate an auxiliary result (Proposition 5.2) stating that if a functional of a Wiener process F is in the class $\mathbb{D}_{1,1}$ (for the definition of $\mathbb{D}_{1,1}$ and additional references please see Section 5.1), then for a "piecewise continuously differentiable" function ϕ, the function $\phi(F)$ is also in the class $\mathbb{D}_{1,1}$. Using this proposition and the Clark-Ocone formula, we will derive the explicit expression of the optimal portfolio process for an investor subject to downside constraints.

The paper is organized in the following way. Section 2 sets up the model for the financial market and the investor, respectively; the latter has at his disposal the choice of a *portfolio* (investment strategy) and a *consumption strategy*, which determine the evolution of his wealth. Section 3 is concerned with an investor's optimization problem in which utility is derived only from consumption and the consumption rate is subject to a downside constraint. We provide quite explicit expressions for the optimal consumption and wealth processes. The "dual" situation, with utility derived only from terminal wealth which is subject to an insurance constraint, is discussed in Section 4; again, explicit expressions are obtained for the above-mentioned quantities. We combine the two problems in Section 5, where we take up the more realistic case of utility coming from both consumption and terminal wealth that are subject to downside constraints. Explicit expressions are provided for the optimal consumption and wealth processes. Malliavin calculus approach is introduced and used to derive the explicit expression for the optimal portfolio strategy. Section 6 concludes the paper.

2 The economy

The model under consideration here is that of a complete financial market as in Merton (1971), Karatzas (1989) and others, wherein there are one riskless asset and d (correlated) risky assets generated by d independent Brownian motions. As our intent in this paper is to obtain the optimal portfolio process $\hat{\pi}$ in a very explicit feedback form on the current level of wealth, we shall assume that the riskless asset earns a constant instantaneous rate of interest, $r > 0$, and that the risky assets are geometric Brownian motion. More specifically, the respective prices $S_0(\cdot)$ and $S_1(\cdot), \ldots, S_d(\cdot)$ of these financial instruments evolve according to the equations

$$\begin{aligned} \mathrm{d}S_0(t) &= rS_0(t)\mathrm{d}t, \qquad S_0(0) = 1 && (2.1) \\ \mathrm{d}S_i(t) &= S_i(t)\left[b_i\mathrm{d}t + \sum_{j=1}^{d} \sigma_{ij}\mathrm{d}W^j(t)\right] \qquad S_i(0) = s_i > 0; \quad i = 1, \ldots, d. && (2.2) \end{aligned}$$

We fix a finite time-horizon [0,T], on which we are going to treat all our problems. In the above equations, $W(\cdot) = (W^1(\cdot), \ldots, W^d(\cdot))'$ is a standard d-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with an augmented filtration $\boldsymbol{F}$ $= \mathcal{F}(t)_{0\leq t\leq T}$ generated by the Brownian motion $W(\cdot)$. The coefficients r (interest rate), $b = (b_1, \ldots, b_d)^*$ (vector of stock return rates) and $\sigma = (\sigma_{ij})_{1\leq i,j\leq d}$ (matrix of stock-volatilities) are all assumed to be constant. Furthermore, the matrix σ is assumed to be invertible.

The investor in our model is endowed with initial wealth $x > 0$. We shall denote by $X(t)$ the wealth of this agent at time t, by $\pi_i(t)$ the amount that he invests in the ith stock at that time $(1 \leq i \leq d)$, and by $c(t)$ the rate at which he withdraws funds for consumption.

We call $\pi(t) = (\pi_1(t), \cdots, \pi_d(t))^*, 0 \le t \le T$ a *portfolio process* if it is measurable, adapted and satisfies

$$\int_0^T \|\pi(t)\|^2 \, \mathrm{d}t < \infty, \qquad a.s.$$

We define $c(t), 0 \le t \le T$ as a *consumption rate process* if it is nonnegative, progressive measurable and satisfies

$$\int_0^T c(t) \, \mathrm{d}t < \infty, \qquad a.s.$$

The investor's wealth process satisfies the equation

$$\begin{aligned} \mathrm{d}X(t) &= -c(t)\mathrm{d}t + \left[X(t) - \sum_{i=1}^d \pi_i(t)\right] r \, \mathrm{d}t + \sum_{i=1}^d \pi_i(t) \left[b_i \, \mathrm{d}t + \sum_{j=1}^d \sigma_{ij} \, \mathrm{d}W^j(t)\right] \\ &= (rX(t) - c(t)) \, \mathrm{d}t + \pi^*(t)(b - r\mathbf{1}) \, \mathrm{d}t + \pi^*(t)\sigma \, \mathrm{d}W(t); \quad X(0) = x. \end{aligned} \tag{2.3}$$

For the initial wealth $x \ge 0$, we shall restrict the investor's portfolio and consumption rate processes to the ones that ensure the solution process X of (2.3) is bounded from below; we call such pair (π, c) of portfolio and consumption rate processes *admissible*. We define the vector

$$\theta = \sigma^{-1}(b - r\mathbf{1}), \tag{2.4}$$

where $\mathbf{1}$ is the d-dimensional vector with all entries equal to 1. We also introduce the processes (d and 1-dimensional, respectively)

$$\tilde{W}(t) = W(t) + \theta t; \quad 0 \le t \le T \tag{2.5}$$

$$Z(t) = \exp\{-\theta^* W(t) - \frac{1}{2}\|\theta\|^2 t\} = \exp\{-\theta^* \tilde{W}(t) + \frac{1}{2}\|\theta\|^2 t\}; \quad 0 \le t \le T, \tag{2.6}$$

and the auxiliary probability measure $\tilde{P}$ defined on $(\Omega, \mathbf{F})$

$$\tilde{P}(A) = E[Z(T) \cdot 1_A]. \tag{2.7}$$

According to the Girsanov theorem the process $\tilde{W}(t)$ is a $\tilde{P}$-Brownian motion on $[0,T]$. From (2.3), we can derive

$$\mathrm{d}X(t) = (rX(t) - c(t))\,\mathrm{d}t + \pi^*(t)\sigma\,\mathrm{d}\tilde{W}(t) \tag{2.8}$$

Let us introduce the notation

$$\beta(t) \triangleq \frac{1}{S_0(t)} = \exp\{-rt\}. \tag{2.9}$$

The solution of (2.8) with initial wealth $X(0) = x \geq 0$ is easily seen to be given by

$$\beta(t)X(t) = x - \int_0^t \beta(s)c(s)\,\mathrm{d}s + \int_0^t \beta(s)\pi^*(s)\sigma\,\mathrm{d}\tilde{W}(s), \quad 0 \leq t \leq T. \tag{2.10}$$

We can deduce that the process $M(t) \triangleq \beta(t)X(t) + \int_0^t \beta(s)c(s)\,\mathrm{d}s, 0 \leq t \leq T$, consisting of current discounted wealth plus total discounted consumption-to-date, is a continuous local martingale under $\tilde{P}$. Let us now introduce the process

$$\zeta(t) = \beta(t)Z(t). \tag{2.11}$$

With the help of the "Bayes rule", we can deduce that the process $N(t) \triangleq \zeta(t)X(t) + \int_0^t \zeta(s)c(s)\,\mathrm{d}s, 0 \leq t \leq T$ is a continuous local martingale under P. This process is also bounded from below. An application of Fatou's lemma shows that N is a supermartingale under P. Consequently, with $\mathcal{S}_{u,v}$ denoting the class of $\{\mathcal{F}_t\}$-stopping times with values in the interval $[u,v]$, we have by the optional sampling theorem the equivalent inequality

$$E\left[\zeta(\tau)X(\tau) + \int_0^\tau \zeta(s)c(s)\,\mathrm{d}s\right] \leq x. \tag{2.12}$$

for every $\tau \in \mathcal{S}_{0,T}$. This inequality, called the budget constraint, implies that the expected total value of terminal wealth and consumption-to-date, both deflated down to $t = 0$, does not exceed the initial capital.

The investor's preferences are assumed to be given by a continuous, strictly increasing, strictly concave and continuously differentiable utility function U whose derivative satisfies $\lim_{x\to\infty} U'(x) = 0$. Next, we are going to consider maximization of utility from consumption when the consumption rate process is subject to a downside constraint. We fix a level $R > 0$ and require the consumption rate process $c(t), \forall t \in [0, T]$ is almost surely bounded below by R. One can think of R as the investor's minimum consumption needs.

3 Maximization of utility from consumption subject to a downside constraint

The investor, endowed with initial wealth $x_1 > 0$, choose at every time his stock portfolio $\pi(t)$ and his consumption rate $c(t)$, which has to be greater or equal to the minimum living expenditure, in order to obtain a maximum expected utility from consumption. Let us consider a utility function U_1. We can formulate the constrained optimization problem as

$$\begin{aligned} \max_{(\pi(t),c(t))} \quad & E\int_0^T U_1(t, c(t))\,\mathrm{d}t \\ \text{s.t.} \quad & c(t) \geq R, \quad 0 \leq t \leq T \end{aligned} \tag{3.1}$$

We note that for each $t \in [0, T]$, $U_1(t, \cdot)$ is also a utility function. We denote by U_1' the differentiation with respect to the second argument. Let $L_1(t) \triangleq \lim_{c\to R+} U_1'(t, c)$, and assume $L_1(t) < \infty,\ 0 \leq t \leq T$. Define $I_1 : [0, T] \times (0, \infty) \mapsto [0, T] \times [R, \infty)$ as the pseudo-inverse of U_1', i.e. for each $t \in [0, T]$, $I_1(t, z) = \min\{c \geq R;\ U_1'(t, c) \leq z\}$. The following proposition characterizes the investor's optimal consumption rate process.

Proposition 3.1. *For any $x_1 \geq \frac{R}{r}(1 - \beta(T))$, the investor's optimal consumption process is $c_1(t) = I_1(t, \lambda_1(x_1)\zeta(t)), 0 \leq t \leq T$, where $\lambda_1(x_1)$ is chosen to satisfy $E[\int_0^T \zeta(t)c_1(t)\,\mathrm{d}t] = x_1$.*

The optimal wealth process X_1 is given by

$$\begin{aligned}\beta(t)X_1(t) &= \tilde{E}\left[\int_t^T \beta(s)c_1(s)\,\mathrm{d}s|\mathcal{F}(t)\right] \\ &= x_1 - \int_0^t \beta(s)c_1(s)\,\mathrm{d}s + \int_0^t \beta(s)\pi_1^*(s)\sigma\,\mathrm{d}\tilde{W}(s).\end{aligned} \tag{3.2}$$

In particular, X_1 is positive on $[0,T)$ and vanishes at $t=T$, almost surely.

Proof. In addition to the downside constraint $c(t) \geq R$, the optimization problem in (3.1) is also subject to the so called budget constraint: $\tilde{E}\left[\int_0^T \beta(t)c(t)\,\mathrm{d}t\right] = E\left[\int_0^T \zeta(t)c(t)\,\mathrm{d}t\right] \leq x_1$. Let λ_c and λ_1 denote the Lagrange multipliers associated with the downside constraint and the budget constraint, respectively. The first order condition to this problem is

$$U_1'(t, c_1(t)) = \lambda_1\zeta(t) - \lambda_c. \tag{3.3}$$

From the complementary slackness conditions, $\lambda_c(c_1(t) - R) = 0$, $\lambda_c \geq 0$, and $c_1(t) \geq R$, we obtain that

$$\lambda_c = [\lambda_1\zeta(t) - U_1'(t,R)]^+, \tag{3.4}$$

substitute this back into (3.3), we obtain that

$$c_1(t) = I_1(t, \lambda_1\zeta(t)). \tag{3.5}$$

□

The case of $x_1 = R/r(1-\beta(T))$ is rather trivial (in that case $c_1 \equiv R$ and $\pi_1 \equiv 0$), thus for the rest of the paper we assume that $x_1 > R/r(1-\beta(T))$.

4 Maximization of utility from investment subject to an insurance constraint

Let us consider now the complementary problem to that of Section 3, namely the maximization of the expected utility from terminal wealth which must not fall below a given level K. An investor working under such constraint will be called an *insurer*.

Definition 4.1. We call a portfolio process *insured* if the corresponding wealth process $X(T)$ is bounded below on $[0, T]$, and

$$X(T) \geq K, \quad a.s. \tag{4.1}$$

4.1 The optimization problem

The optimization problem of a portfolio insurer is to maximize $E[U_2(X(T))]$ over all insured portfolio processes for a given initial wealth x_2.

The optimal terminal wealth $X_2(T)$ for the above problem is well-known (see Grossman and Vila (1989), Grossman and Zhou (1996), and Teplá (2001)). In order to formulate it we need some additional notations and facts. Let $L_2 \stackrel{\triangle}{=} \lim_{x_2 \to K+} U_2'(x_2)$, and assume $L_2 < \infty$. Define $I_2 : (0, \infty) \mapsto [K, \infty)$ as the pseudo-inverse of U_2', i.e., $I_2(z) = \min\{x_2 \geq K : U_2'(x_2) \leq z\}$ and notice that even if U_2 is twice continuously differentiable on (K, ∞), $I_2(z)$ may not be differentiable in $z = L_2$. Some other properties of I_2 are as follows: I_2 is strictly decreasing on $(0, L_2)$; $\lim_{z \to 0} I_2(z) = \infty$; and $I_2(z) = K$ if $z \geq L_2$. We denote by (π_2, c_2) the optimal strategy for a portfolio insurer. Given that the consumption rate is not an argument of utility function U_2, we have $c_2 \equiv 0$ and the corresponding wealth process X_2 is given by

$$\beta(t)X_2(t) = x_2 + \int_0^t \beta(s)\pi_2^*(s)\sigma \, \mathrm{d}\tilde{W}(s), \quad 0 \leq t \leq T. \tag{4.2}$$

Hence the discounted wealth process $(\beta(t)X_2(t), 0 \leq t \leq T)$ is a continuous $\tilde{P}$-local martingale, bounded below by a constant for every insured portfolio process. Now Fatou's lemma implies that this process is a $\tilde{P}-$supermartingale, and $\tilde{E}[\beta(T)X_2(T)] \leq x_2$. This budget-constraint implies that the class of insured portfolio processes is empty unless we have $\beta(T)K \leq x_2$. The case of $\beta(T)K = x_2$ is rather trivial (in that case the only insured portfolio process is $\pi_2 \equiv 0$), thus for the rest of the paper we assume that $\beta(T)K < x_2$.

For an insured portfolio process π we define the Markov time

$$\tau(\pi) = \tau = \inf\{t \leq T : X(t) = K\beta(T)/\beta(t) = Ke^{-r(T-t)}\} \tag{4.3}$$

and let $\tau = \infty$ if the set in the left-hand side of (4.3) is empty. It is worth pointing out that

$$X(s) = Ke^{-r(T-s)}, \quad s \in [\tau, T] \text{ holds a.s. on } \{\tau < \infty\} \tag{4.4}$$

and

$$\pi(s) = 0, \quad s \in [\tau, T] \text{ holds a.s. on } \{\tau < \infty\}. \tag{4.5}$$

Indeed, (4.4) follows from Karatzas & Shreve (1991), Problem 1.3.29 and from the fact that the process $(e^{-rt}X(t) - Ke^{-rT}, t \leq T)$ is a nonnegative supermartingale. Formula (4.5) then follows from the stochastic integral representation (4.2). These two equations signify that once the wealth process hits the curve of $t \mapsto Ke^{-r(T-t)}$, it will follow this curve and no investment in the risky securities will take place.

Now we are ready to state the result characterizing the optimal terminal wealth for a portfolio insurer. The reader is referred to Grossman & Vila (1989), Grossman & Zhou (1996), and Teplá (2001) for a detailed proof of this proposition.

Proposition 4.1. *Suppose that for every constant* $\lambda_2 > 0$*, we have* $\tilde{E}[I_2(\lambda_2\zeta(T))] < \infty$*. Then the optimal terminal wealth for a portfolio insurer is*

$$X_2(T) = I_2(\lambda_2\zeta(T)), \tag{4.6}$$

where the constant $\lambda_2 > 0$ *is uniquely determined by*

$$\tilde{E}[X_2(T)e^{-rT}] = x_2. \tag{4.7}$$

Additionally, the discounted optimal wealth process is a $\tilde{P}$*-martingale, i.e.,*

$$X_2(t) = e^{-r(T-t)}\tilde{E}[I_2(\lambda_2\zeta(T))|\mathcal{F}(t)]. \tag{4.8}$$

Remark 4.1. In fact, the wealth process of an optimally behaving insurer will not hit the boundary $Ke^{-r(T-t)}$ before the terminal time T. In other words, we observe that $\tau(\pi_2) \geq T$, almost surely.[1]

5 Maximization of utility from both consumption and terminal wealth subject to downside constraints

Let us consider now an investor who derives utility both from "living well" (i.e., from consumption) and from "becoming rich" (i.e., from terminal wealth) when the downside constraints are imposed on both consumption rate process and terminal wealth. His *expected total utility* is then

$$J(x;\pi,c) \triangleq E\int_0^T U_1(t,c(t))\,\mathrm{d}t + EU_2(T,X(T)), \tag{5.1}$$

[1]We skip the proof of this assertion since it is less crucial to the understanding of the rest of the paper. A detailed proof is available upon request.

and the mathematical formulation of this investor's optimization problem is:

$$\begin{aligned} V(x;R,K) \triangleq \max_{(\pi(t),c(t))} \quad & J(x;\pi,c) \\ \text{s.t.} \quad & c(t) \geq R, \quad 0 \leq t \leq T \\ & X(T) \geq K. \end{aligned} \tag{5.2}$$

Here, $V(x;R,K)$ is the *value function* of this problem.

In contrast to the problems of sections 3 and 4, this one requires to balance competing objectives. One can show that the situation calls for the kind of compromise analogous to the unconstrained maximization of utility from both consumption and terminal wealth. More specifically, the optimal strategy is: at time $t=0$, the investor divides his endowment x into two nonnegative parts x_1 and x_2, with $x_1 + x_2 = x$. For x_1, he solves the problem of section 3 (with utility U_1 from consumption and the downside constraint on the consumption rate process); for x_2, he solves the problem of section 4 (with utility U_2 from terminal wealth and the downside constraint on the terminal wealth). Let us denote by $V_1(x_1,R)$ the value function of the constrained optimization problem in section 3 and $V_2(x_2,K)$ the value function of the constrained optimization problem in section 4. The superposition of his actions for these two problems will lead to the optimal policy for the problem of (5.2), provided x_1 and x_2 are chosen for which the"marginal expected utilities" $V_1'(x_1,R)$ and $V_2'(x_2,K)$ from the two individual constrained optimization problems are identical.

We start with an admissible pair (π,c) and define

$$x_1 \triangleq \tilde{E}\int_0^T \beta(t)c(t)\,\mathrm{d}t, \qquad x_2 \triangleq x - x_1. \tag{5.3}$$

Proposition 3.1 gives us a pair (π_1,c_1) which is optimal for $V_1(x_1,R)$, with corresponding wealth process X_1 satisfying $X_1(T) = 0$, almost surely. On the other hand, Proposition 4.1 provides a pair $(\pi_2, 0)$ which is optimal for $V_2(x_2, K)$, with corresponding wealth process X_2. If we define now

$$\tilde{\pi} \triangleq \pi_1 + \pi_2, \quad \tilde{c} \triangleq c_1, \quad \text{and} \quad \tilde{X} \triangleq X_1 + X_2 \tag{5.4}$$

and add (3.2) and (4.6), we obtain

$$\begin{aligned} \beta(t)\tilde{X}(t) &= \tilde{E}\left[\int_t^T \beta(s)\tilde{c}(s)\,\mathrm{d}s + \beta(T)\tilde{X}(T)|\mathcal{F}(t)\right] \\ &= x + \int_0^t \beta(s)\tilde{c}(s)\,\mathrm{d}s + \int_0^t \beta(s)(\tilde{\pi}(s))^*\sigma\,\mathrm{d}\tilde{W}(s), \quad 0 \le t \le T. \end{aligned} \tag{5.5}$$

In other words, $\tilde{X}$ is the wealth process corresponding to the pair $(\tilde{\pi}, \tilde{c})$.

We know from Proposition 3.1 and Proposition 4.1 that $E\int_0^T U_1(t, c_1(t))\,\mathrm{d}t \ge E\int_0^T U_1(t, c(t))\,\mathrm{d}t$ and $EU_2(T, X_2(T)) \ge EU_2(T, X(T))$ hold. Adding them up memberwise, we obtain

$$J(x; \pi, c) \le V_1(x_1, R) + V_2(x_2, K), \tag{5.6}$$

hence

$$V(x; R, K) \le V_*(x; R, K) \triangleq \max_{\substack{x_1 \ge R/r(1-\beta(T)) \\ x_2 \ge \beta(T)K \\ x_1 + x_2 = x}} [V_1(x_1, R) + V_2(x_2, K)]. \tag{5.7}$$

Therefore, if we find x_1, x_2 for which this maximum is achieved, then the total expected utility corresponding to the pair $(\tilde{\pi}, \tilde{c})$ of (5.4) will be exactly equal to $V_*(x; R, K)$; this will in turn imply $V(x; R, K) = V_*(x; R, K)$. Thus the pair $(\tilde{\pi}, \tilde{c})$ of (5.4) will be shown to be *optimal* for the problem of (5.2).

The optimal solution (x_1, x_2) to the maximization problem (5.7) is described by the equation

$$V_1'(x_1, R) = V_2'(x_2, K). \tag{5.8}$$

In order to see the values of x_1, x_2 that satisfy (5.8) lie in the interior of the constraints, we first introduce the functions

$$\mathcal{X}_1(\lambda_1) \triangleq E\left[\int_0^T \zeta(t)c_1 \,\mathrm{d}t\right], \tag{5.9}$$

$$\mathcal{X}_2(\lambda_2) \triangleq E[\zeta(T)X_2(T)]. \tag{5.10}$$

For $I_1(t,\cdot)$ on $(0, L_1(t))$, $0 \leq t \leq T$, and I_2 on $(0, L_2)$, we have $\mathcal{X}_1(\lambda_1) = E\left[\int_0^T \zeta(t)I_1(t, \lambda_1\zeta(t))\,\mathrm{d}t\right]$, and $\mathcal{X}_2(\lambda_2) = E[\zeta(T)I_2(\lambda_2\zeta(T))]$. Using the convex duals of V_1 and V_2, one can show that $V_1'(x_1, R) = \lambda_1(x_1)$, and $V_2'(x_2, K) = \lambda_2(x_2)$ (see Karatzas & Shreve (1998), chapter 3). It then follows from $V_1'(x_1, R) = V_2'(x_2, K)$ that $\lambda_1(x_1) = \lambda_2(x_2) = \lambda \Leftrightarrow x_1 = \mathcal{X}_1(\lambda), x_2 = \mathcal{X}_2(\lambda)$. The constant λ is determined uniquely as follows: we introduce the function

$$\begin{aligned}\mathcal{X}(\lambda) &\triangleq \mathcal{X}_1(\lambda) + \mathcal{X}_2(\lambda)\\ &= E\left[\int_0^T \zeta(t)I_1(t, \lambda\zeta(t))\,\mathrm{d}t + I_2(T, \lambda\zeta(T))\right].\end{aligned} \tag{5.11}$$

Let $\mathcal{Y} = \mathcal{X}^{-1}$ be the inverse of $\mathcal{X}$; then $\lambda = \mathcal{Y}(x)$, and the "optimal partition" of the initial wealth is given by $x_1 = \mathcal{X}_1(\lambda(x))$, $x_2 = \mathcal{X}_2(\lambda(x))$. Since $\mathcal{X}_1(\lambda(x)) > R/r(1 - \beta(T))$ and $\mathcal{X}_2(\lambda(x)) > \beta(T)K$, we conclude that the pair (x_1, x_2) selected to satisfy (5.8) lies in the interior of the constraints.

We have established the following result.

Proposition 5.1. *For a fixed initial capital* $x \geq \frac{R}{r}(1 - \beta(T)) + \beta(T)K$*, the optimal consumption rate process and the optimal level of terminal wealth of* (5.2) *are given by*

$$\hat{c}(t) = I_1(t, \lambda_1(x_1)\zeta(t)), \quad 0 \leq t \leq T, \quad \textit{and} \quad \hat{X}(T) = I_2(\lambda_2(x_2)\zeta(T)), \tag{5.12}$$

respectively; the corresponding wealth process $\hat{X}$ *is given by*

$$\hat{X}(t) = \beta^{-1}(t)\tilde{E}\left[\int_t^T \beta(s)I_1(s, \lambda_1(x_1)\zeta(s))\,\mathrm{d}s + \beta(T)I_2(\lambda_2(x_2)\zeta(T))|\mathcal{F}(t)\right] \tag{5.13}$$

almost surely, for every $0 \le t \le T$.

5.1 Derivation of the optimal portfolio process using the Clark-Ocone formula

For a background material on the gradient operator and the Clark-Ocone formula we refer the reader to Nualart (1995), Ocone & Karatzas (1991), or Karatzas, Ocone, & Li (1991). For easy later reference and usage, we recall the definition of the gradient operator D and the class $\mathbb{D}_{1,1}$, as applied to the probability space $(\Omega, \mathcal{F}, \tilde{P})$ and the Brownian motion $\{\tilde{W}(t); t \le T\}$.

Let $\mathcal{P}$ denote the family of all random variables $F : \Omega \to \Re$ of the form

$$F(\omega) = \varphi(\theta_1, \dots, \theta_n)$$

where $\varphi(x_1, \dots, x_n) = \sum_\alpha a_\alpha x^\alpha$ is a polynomial in n variables $x_1, \dots, x_n$ and $\theta_i = \int_0^T f_i(t) \mathrm{d}\tilde{W}(t)$ for some $f_i \in L^2([0, T])$ (deterministic). Such random variables are called *Wiener polynomials*. Note that $\mathcal{P}$ is dense in $L^2(\Omega)$. Consider the space of continuous, real functions ω on $[0, T]$ such that $\omega(0) = 0$, denoted as $C_0([0, T])$. This space is called *the Wiener space*, because we can regard each path $t \to \tilde{W}(t, \omega)$ of the Wiener process starting at 0 as an element ω of $C_0([0, 1])$. Thus we may identify $\tilde{W}(t, \omega)$ with the value $\omega(t)$ at time t of an element $\omega \in C_0([0, T])$: $\tilde{W}(t, \omega) = \omega(t)$. With this identification the Wiener process simply becomes the space $\Omega = C_0([0, T])$ and the probability law $\tilde{P}$ of the Wiener process becomes the measure μ defined on the cylinder sets of Ω by

$$\mu(\{\omega; \omega(t_1) \in F_1, \dots, \omega(t_k) \in F_k\}) = P[\tilde{W}(t_1) \in F_1, \dots, \tilde{W}(t_k) \in F_k]$$
$$= \int_{F_1 \times \cdots \times F_k} \rho(t_1, x, x_1)\rho(t_1 - t_0, x, x_2) \cdots \rho(t_k - t_{k-1}, x_{k-1}, x_k) \mathrm{d}x_1, \cdots, \mathrm{d}x_k$$

where $F_i \subset \Re$; $0 \leq t_1 < t_2 < \cdots < t_k$ and

$$\rho(t,x,y) = (2\pi t)^{-1/2} \exp\big(-\frac{1}{2t}|x-y|^2\big); \quad t > 0; x, y \in \Re.$$

The measure μ is called *the Wiener measure* on Ω. In other words, we identify our probability space $(\Omega, \mathcal{F}, \tilde{P})$ with $(C_0([0,T]), \mathcal{B}(C_0([0,T])), \mu)$ such that $\tilde{W}(t,\omega) = \omega(t)$ for all $t \in [0,T]$. Here $\mathcal{B}(C_0([0,T]))$ denotes the corresponding Borel σ-algebra. Next, we define the Cameron-Martin space $\mathcal{H}$ according to

$$\mathcal{H} = \left\{ \gamma : [0,T] \to \Re : \gamma(t) = \int_0^t \dot{\gamma}(s)\mathrm{d}s, |\gamma|^2_{\mathcal{H}} = \int_0^T \dot{\gamma}^2(s)\mathrm{d}s < \infty \right\}.$$

With this setup we can define the directional derivative of a random variable $F \in \mathcal{P}$ in all the directions $\gamma \in \mathcal{H}$ by

$$D_\gamma F(\omega) = \frac{d}{d\xi}[F(\omega + \xi\gamma)]_{\xi=0}. \tag{5.14}$$

Notice from the above equation that the map $\gamma \to D_\gamma F(\omega)$ is continuous for all $\omega \in \Omega$ and linear, consequently there exists a stochastic variable $\bigtriangledown F(\omega)$ with values in the Cameron-Martin space $\mathcal{H}$ such that $D_\gamma F(\omega) = (\bigtriangledown F(\omega), \gamma)_{\mathcal{H}} := \int_0^T \frac{d(\bigtriangledown F)}{dt}(t)\dot{\gamma}(t)\mathrm{d}t$. Moreover, since $\bigtriangledown F(\omega)$ is an $\mathcal{H}$-valued stochastic variable, the map $t \to \bigtriangledown F(t,\omega)$ is absolutely continuous with respect to the Lebesgue measure on $[0,T]$. Now we let the Malliavin derivative $D_t F(\omega)$ denote the Radon-Nikodym derivative of $\bigtriangledown F(\omega)$ with respect to the Lebesgue measure such that

$$D_\gamma F(\omega) = \int_0^T D_t F(\omega)\dot{\gamma}(t)\mathrm{d}t. \tag{5.15}$$

We note that the Malliavin derivative is well defined almost everywhere $dt \times d\tilde{P}$.

We denote by $\|\cdot\|_{L^2}$ the $(L^2[0,T])^n$-norm, i.e., for $\psi = (\psi_1, \ldots, \psi_n) \in (L^2[0,T])^n$

$$\|\psi\|^2_{L^2} = \sum_{i=1}^n \int_0^T \psi_i^2(t)\,\mathrm{d}t,$$

and introduce another norm $\|\cdot\|_{1,1}$ on the set $\mathcal{P}$ according to

$$\|F\|_{1,1} = E\left[|F| + \|DF\|_{L^2}\right]. \tag{5.16}$$

Now, as the Malliavin derivative is a closable operator (see Nualart (1995)), we define by $\mathbb{D}_{1,1}$ the Banach space which is the closure of $\mathcal{P}$ under the norm $\|\cdot\|_{1,1}$.

What makes the Malliavin calculus interesting in mathematical finance is the Clark-Ocone formula (Ocone & Karatzas (1991)). An extension of the Clark-Ocone formula for $\mathbb{D}_{1,1}$, which is given in Karatzas, Ocone, & Li (1991), turns out to be more useful in solving our problem. Recall the extended version of the Clark-Ocone formula: for every $F \in \mathbb{D}_{1,1}$ we have the stochastic integral representation

$$F = \tilde{E}[F] + \int_0^T \tilde{E}\left[(D_t F)^* | \mathcal{F}(t)\right] \mathrm{d}\tilde{W}(t) \tag{5.17}$$

and also

$$\tilde{E}[F|\mathcal{F}(t)] = \tilde{E}[F] + \int_0^t \tilde{E}\left[(D_s F)^* | \mathcal{F}(s)\right] \mathrm{d}\tilde{W}(s); \quad t \leq T. \tag{5.18}$$

We want to apply this formula to the functional $\hat{X}(T)$ of (5.12). The problem is that existing results such as Lemma A1 in Ocone & Karatzas (1991), or Proposition 1.2.3 in Nualart (1995), would require $I_2(z)$ being either continuously differentiable or Lipschitz continuous. However, $I_2(z)$ is not differentiable in $z = L_2$, and assuming Lipschitz continuity would exclude the most frequently used utility functions, such as the logarithm and power utilities. Thus we proceed with a proposition which is applicable to the present situation. First, we propose the following definition, for every fixed $-\infty \leq a < b \leq \infty$:

Definition 5.1. A function $\phi : (a, b) \to \Re$ is called *piecewise continuously differentiable* if the following conditions are satisfied:

(i) ϕ is continuous on (a, b);

(ii) There exist finitely many points $a = c_0 < c_1 < \ldots < c_{m+1} = b (m \geq 0)$ such that ϕ is continuously differentiable on (c_i, c_{i+1}) for every $i = 0, \ldots, m$;

(iii) The function ϕ' is bounded on every compact subinterval of (a, b), where

$$\phi'(x) \triangleq \begin{cases} \text{the derivative of } \phi \text{ in } x, & \text{if } x \in (a, b) \backslash \{c_1, \ldots, c_m\} \\ 0, & \text{if } x \in \{c_1, \ldots, c_m\} \end{cases}$$

(iv) The limits

$$\lim_{x \to a+} \phi(x) \quad \lim_{x \to b-} \phi(x)$$
$$\lim_{x \to a+} \phi'(x) \quad \lim_{x \to b-} \phi'(x)$$

exist;

(v)

$$\text{If} \quad |\lim_{x \to a+} \phi'(x)| = \infty, \quad \text{then } |\lim_{x \to a+} \phi(x)| = \infty$$

and

$$\text{if} \quad |\lim_{x \to b-} \phi'(x)| = \infty, \quad \text{then } |\lim_{x \to b-} \phi(x)| = \infty.$$

We denote the class of piecewise continuously differentiable functions by $PC^1(a, b)$. For a function $\phi \in PC^1(a, b)$ the points $c_1, \ldots, c_m$ will be called the breakpoints of ϕ. We allow $m = 0$ in which case ϕ is continuously differentiable on (a, b). For any $\phi \in PC^1(a, b)$ we denote by ϕ' the function defined in item (iii) of the above definition.

Now we are ready to state our proposition.

Proposition 5.2. *Suppose that we have constants a, b, a function ϕ, and a random variable F such that $-\infty \le a < b \le \infty$, $\phi \in PC^1(a,b)$ with breakpoints $c_1, \ldots, c_m$, $F \in \mathbb{D}_{1,1}$, and $\tilde{P}(F \in (a,b)) = 1$. If*

$$\tilde{E}[|\phi(F)| + \|\phi'(F)DF\|_{L^2}] < \infty \tag{5.19}$$

and

$$\tilde{P}(F \in \{c_1, \ldots, c_m\}) = 0 \tag{5.20}$$

then $\phi(F) \in \mathbb{D}_{1,1}$ and

$$D_t\phi(F) = \phi'(F)D_tF. \tag{5.21}$$

Proof. See Appendix. □

The optimal portfolio process for the optimization problem (5.2) is given by the following theorem.

Theorem 5.1. *Suppose that the relations*

$$\lim_{z \to 0+} I_2'(z) = \infty, \tag{5.22}$$

$$|I_2(z)| + |I_2'(z)| \le K_1(1 + z^{-\alpha}); \qquad z \in (0, \infty) \tag{5.23}$$

hold for some positive constants K_1 and α. Then the hedging portfolio with the downside constraints is

$$\begin{aligned} \hat{\pi}(t) = {} & -\lambda_2 \exp\left\{-\theta^*\tilde{W}(t) + \frac{T}{2}\|\theta\|^2 - r\tau\right\}(\sigma^*)^{-1}\theta \\ & \times \int_{\Re^d} I_2'\left(\lambda_2 \exp\left\{-\theta^*\eta - \theta^*\tilde{W}(t) + \frac{T}{2}\|\theta\|^2\right\}\right) \exp\{-\theta^*\eta\}\varphi_\tau(\eta)\, d\eta. \end{aligned} \tag{5.24}$$

where $\tau \triangleq T - t$ and φ_τ is the d-dimensional normal density function with variance-covariance matrix τI_d (here I_d is the $d \times d$ identity matrix).

Proof. Notice that $\hat{X}(T)$ of (5.12) is consistent with $X_2(T)$ of (4.6). The condition of Proposition 4.1, i.e., that $\tilde{E}[I_2(\lambda_2\zeta(T))]$ is finite for every $\lambda_2 > 0$ follows from (5.23) and the fact that all powers of $\zeta(T)$ have finite $\tilde{P}-$ expectation. We need to show that $I_2(\lambda_2\zeta(T)) \in \mathbb{D}_{1,1}$ and compute $D_t I_2(\lambda_2\zeta(T))$, because having done that, the Clark-Ocone formula and (5.5), (5.12), and (5.13) would imply

$$\hat{\pi}(t) = e^{-r(T-t)}(\sigma^*)^{-1}\tilde{E}[D_t I_2(\lambda_2\zeta(T))|\mathcal{F}(t)]. \tag{5.25}$$

From Ocone & Karatzas (1991), Lemma A1 follows that $\zeta(T) \in \mathbb{D}_{1,1}$, and

$$D_t\zeta(T) = -\theta\zeta(T); \quad t \leq T. \tag{5.26}$$

The essential step of the proof is applying Proposition 5.2 with I_2 playing the role of ϕ and $\lambda_2\zeta(T)$ playing the role of F. From the properties of utility function U_2 follows that $I_2 \in PC^1(0,\infty)$ with a single breakpoint $z = L_2$. Condition (5.19) for $\phi = I_2$ and $F = \lambda_2\zeta(T)$ is an easy consequence of (5.23), (5.26), and the fact that all powers of $\zeta(T)$ have finite $\tilde{P}$-expectation. Thus we can indeed apply Proposition 5.2 which guarantees that $I_2(\lambda_2\zeta(T)) \in \mathbb{D}_{1,1}$ and

$$D_t I_2(\lambda_2\zeta(T)) = -\lambda_2 I_2'(\lambda_2\zeta(T))\theta\zeta(T); \quad t \leq T. \tag{5.27}$$

Now we combine (5.25), (5.27), (2.11), (2.9), and (2.6) to obtain

$$\hat{\pi}(t) = -e^{-r(T-t)}\lambda_2(\sigma^*)^{-1}\theta\tilde{E}\left[I_2'\left(\lambda_2\exp\left\{-\theta^*\tilde{W}(T) + \frac{T}{2}\|\theta\|^2\right\}\right)\exp\left\{-\theta^*\tilde{W}(T) + \frac{T}{2}\|\theta\|^2\right\}|\mathcal{F}(t)\right], \tag{5.28}$$

which implies (5.24), using some well-known properties of the Brownian motion.

□

In the following two examples, we discuss the special cases of the logarithmic and the power utility functions. We shall also compare the optimal portfolio processes with downside constraints to the optimal portfolio processes without constraints. The latter is well known from the portfolio optimization literature (see, for example, Ocone & Karatzas (1991)). We shall use the notations $\bar{\pi}(t)$ and $\bar{X}(t)$ for the optimal portfolio process and wealth process without constraints, respectively.

Example 5.1. We select $U_1(c) = \log c \quad c \in [R, \infty)$ and $U_2(x) = \log x \quad x \in [K, \infty)$, in which case $L_2 \triangleq \lim_{x\to K+} U_2'(x) = 1/K$ and

$$I_2(x) = \frac{1}{x} 1_{\{x\le L_2\}} + K 1_{\{x>L_2\}} \tag{5.29}$$

$$I_2'(x) = -\frac{1}{x^2} 1_{\{x\le L_2\}}. \tag{5.30}$$

To simplify the calculation, let's assume that the consumption rate c is always greater than or equal to R, in which case $I_1(t, y) = 1/y$. In order to specialize our formula for the optimal portfolio process to this example, we cast (5.25) and (5.27) in the form

$$\hat{\pi}(t) = \frac{1}{\lambda_2} e^{-r(T-t)} (\sigma^*)^{-1} \theta \tilde{E}\left[\frac{1}{\zeta(T)} 1_{\{\lambda_2\zeta(T)\le L_2\}} | \mathcal{F}(t)\right]. \tag{5.31}$$

By (5.13), we can write the optimal wealth process as

$$\begin{aligned}\hat{X}(t) =& e^{rt}\tilde{E}\left[\int_t^T \beta(s)\frac{1}{\lambda_1\zeta(s)}\,\mathrm{d}s|\mathcal{F}(t)\right] + \frac{1}{\lambda_2}e^{-r(T-t)}\tilde{E}\left[\frac{1}{\zeta(T)}1_{\{\lambda_2\zeta(T)\le L_2\}}|\mathcal{F}(t)\right] \\ &+ e^{-r(T-t)}K\tilde{P}(\lambda_2\zeta(T) > L_2|\mathcal{F}(t)),\end{aligned} \tag{5.32}$$

in which the first part on the right hand side $e^{rt}\tilde{E}\left[\int_t^T \beta(s)\frac{1}{\lambda_1\zeta(s)}\,\mathrm{d}s|\mathcal{F}(t)\right]$ is equal to $\frac{T-t}{\lambda_1}e^{rt}$, thus we can conclude that

$$\hat{\pi}(t) = \left(\hat{X}(t) - \frac{T-t}{\lambda_1}e^{rt} - e^{-r(T-t)}K\tilde{P}(\lambda_2\zeta(T) > L_2|\mathcal{F}(t))\right)(\sigma^*)^{-1}\theta. \tag{5.33a}$$

In order to make this formula more explicit, we use (2.11), (2.9), and (2.6) to write the conditional probability on the right-hand side of (5.33a) for $\theta \neq 0$ as

$$\tilde{P}(\lambda_2 \zeta(T) > L_2 | \mathcal{F}(t)) = \Phi\left(\frac{1}{\|\theta\|\sqrt{T-t}}\left(\log\left(\frac{\lambda_2}{L_2}\right) + \frac{T}{2}\|\theta\|^2 - \theta^* \tilde{W}(t) - rT\right)\right), \quad (5.33b)$$

where Φ is the (one-dimensional) standard normal distribution function. Now (5.33a)-(5.33b) give an explicit representation for $\hat{\pi}(t)$ in the case of logarithmic utility functions. If $\theta = 0$, (5.24) implies $\hat{\pi} \equiv 0$ (for any utility function which satisfies the conditions of Theorem 5.1).

It's interesting to compare this result to the optimal portfolio process for maximizing utility from consumption and terminal wealth without downside constraints. The optimal portfolio process under logarithmic utilities without downside constraints has the well known feedback form

$$\bar{\pi}(t) = \bar{X}(t)(\sigma^*)^{-1}\theta. \quad (5.34)$$

Defining the process $\hat{X}_c(t) \triangleq \left(\hat{X}(t) - \frac{T-t}{\lambda_1}e^{rt} - e^{-r(T-t)}K\tilde{P}(\lambda_2\zeta(T) > L_2|\mathcal{F}(t))\right)$, we can rewrite (5.33a) as $\hat{\pi}(t) = \hat{X}_c(t)(\sigma^*)^{-1}\theta$, from which we can see that the optimal portfolio process with downside constraints is captured in an explicit feedback form on $\hat{X}_c(t)$. Here $\hat{X}_c(t)$ can be interpreted as the constraint–adjusted current level of wealth.

Example 5.2. In this example we specialize our result to the case of the power utilities $U_1(t,c) = \frac{1}{\delta}c^\delta, c \geq R, \delta \in (-\infty, 1), \delta \neq 0$, and $U_2(x) = \frac{1}{\delta}x^\delta, x \geq K, \delta \in (-\infty, 1), \delta \neq 0$. In this case,

$$I_2(x) = x^\epsilon 1_{\{x \leq L_2\}} + K 1_{\{x > L_2\}}; \quad x \in (0, \infty), \quad (5.35)$$

where $\epsilon = 1/(\delta - 1)$. Also, we have $L_2 = K^{\delta-1}$ and

$$I_2'(x) = \epsilon x^{\epsilon-1} 1_{\{x \leq L_2\}}; \quad x \in (0, \infty). \quad (5.36)$$

We can analyze this similarly to the previous example. To simplify the calculation, we again assume that the downside constraint on the consumption rate process is not binding, in which case we have $I_1(t, y) = y^{\epsilon}$. From (5.25) and (5.27), we get

$$\hat{\pi}(t) = -e^{-r(T-t)}\lambda_2^{\epsilon}\epsilon(\sigma^*)^{-1}\theta\tilde{E}\left[(\zeta(T))^{\epsilon}1_{\{\lambda_2\zeta(T)\leq L_2\}}|\mathcal{F}(t)\right], \tag{5.37}$$

and (5.13) implies

$$\begin{aligned}\hat{X}(t) =& e^{rt}\lambda_1^{\epsilon}\tilde{E}\left[\int_t^T \beta(s)(\zeta(s))^{\epsilon}\,\mathrm{d}s|\mathcal{F}(t)\right] + e^{-r(T-t)}\lambda_2^{\epsilon}\tilde{E}\left[(\zeta(T))^{\epsilon}1_{\{\lambda_2\zeta(T)\leq L_2\}}|\mathcal{F}(t)\right]\\ &+ e^{-r(T-t)}K\tilde{P}(\lambda_2\zeta(T) > L_2|\mathcal{F}(t))\end{aligned} \tag{5.38}$$

Rearranging and simplifying (5.38) using algebra and independent increments property of Brownian motion, we get

$$\hat{\pi}(t) = -\epsilon\left(\hat{X}(t) - \frac{\exp\{\nu(T-t)+rt\}\lambda_1^{\epsilon}}{\nu} - e^{-r(T-t)}K\tilde{P}(\lambda_2\zeta(T) > L_2|\mathcal{F})\right)(\sigma^*)^{-1}\theta, \tag{5.39}$$

where $\nu \triangleq -r(\epsilon+1) + \frac{1}{2}\theta^2\epsilon + \frac{1}{2}\theta^2\epsilon^2$, and $\nu \neq 0$. (5.39) together with (5.33b) give an explicit representation for $\hat{\pi}(t)$ whenever $\theta \neq 0$.

Following the argument in last example, we introduce the constrain-adjusted wealth process $\hat{X}_c(t) \triangleq \hat{X}(t) - \frac{\exp\{\nu(T-t)+rt\}\lambda_1^{\epsilon}}{\nu} - e^{-r(T-t)}K\tilde{P}(\lambda_2\zeta(T) > L_2|\mathcal{F})$, $0 \leq t \leq T$. We may rewrite the expression (5.39) for the optimal portfolio process as $\hat{\pi}(t) = -\epsilon\hat{X}_c(t)(\sigma^*)^{-1}\theta$, which has an explicit feedback form on the constraint–adjusted current level of wealth $\hat{X}_c(t)$.

6 Concluding remarks

We have developed a method for deriving explicit expression for the optimal portfolio process when the investor's consumption rate process and terminal wealth are subject to downside constraints. The gradient operator and the Clark-Ocone formula are used to obtain the optimal portfolio policies for a wide scale of utility functions. In order to calculate the required Malliavin derivatives in the Clark-Ocone formula, we extend the classic chain rule that holds for Lipschitz functions to be valid for any piecewise continuously differentiable functions. The methods developed in this paper seem preferable for investors with a liability stream. This raises an issue for further study, which is to explore the adaptability of the theory developed here in pension fund management.

7 Appendix

Proof of Proposition 5.2. In the case of $m = 0$ our proposition becomes identical to Lemma A1 in Ocone & Karatzas (1991). There $a = -\infty$ and $b = \infty$ was assumed but it can be easily generalized to include finite a and b. In the following proof we shall assume that $m = 1$ and $c_1 = c$; the proof for $m > 1$ would be technically the same with additional notations.

The proof will be carried out in two steps. In the first step we assume that ϕ and ϕ' are bounded on (a, b), i.e.,

$$K_2 \triangleq \sup_{x \in (a,b)} \{|\phi(x)| + |\phi'(x)|\} < \infty. \tag{7.1}$$

We select an increasing sequence $(a_k)_{k \geq 1} \subset (a, c)$ and a decreasing sequence $(b_k)_{k \geq 1} \subset (c, b)$ such that

$$\lim_{k \to \infty} a_k = \lim_{k \to \infty} b_k = c, \tag{7.2}$$

and for every $k \geq 1$ define the function

$$\psi_k(x) = \begin{cases} \phi'(x), & \text{if } x \in (a, a_k] \bigcup [b_k, b); \\ \frac{1}{b_k - a_k} \left((b_k - x)\phi'(a_k) + (x - a_k)\phi'(b_k) \right), & \text{if } x \in (a_k, b_k). \end{cases} \tag{7.3}$$

Note that ψ_k is continuous and bounded by K_2 on (a, b). Next we define for every $k \geq 1$

$$\phi_k(x) = \phi(c) + \int_c^x \psi_k(z) \mathrm{d}z; \quad x \in (a, b), \tag{7.4}$$

a continuously differentiable function on (a, b) satisfying the relation

$$\phi_k'(x) = \psi_k(x); \quad x \in (a, b), \tag{7.5}$$

and note that

$$\lim_{k\to\infty} \psi_k(x) = \phi'(x); \quad x \in (a, b)\backslash\{c\}. \tag{7.6}$$

The function ϕ is absolutely continuous on any compact subinterval of (a, b) thus we have

$$\phi(x) = \phi(c) + \int_c^x \phi'(z)\mathrm{d}z; \quad x \in (a, b), \tag{7.7}$$

and now (7.4) and (7.7) imply for $x \in (a, b)$

$$|\phi_k(x) - \phi(x)| = \left|\int_c^x (\psi_k(z) - \phi'(z))\,\mathrm{d}z\right| \leq \int_{a_k}^{b_k} |\psi_k(z) - \phi'(z)|\mathrm{d}z \leq 2(b_k - a_k)K_2, \tag{7.8}$$

i.e.,

$$\lim_{k\to\infty} \sup_{x\in(a,b)} |\phi_k(x) - \phi(x)| = 0. \tag{7.9}$$

Additionally, by (7.8) we have

$$|\phi_k(x)| \leq K_2 + 2(b_1 - a_1)K_2; \quad x \in (a, b), \tag{7.10}$$

thus by (7.9), (7.10), and the Dominated Convergence Theorem

$$\lim_{k\to\infty} \tilde{E}|\phi_k(F) - \phi(F)| = 0. \tag{7.11}$$

For every $k \geq 1$ the function ϕ_k is continuously differentiable on (a, b), both ϕ_k and ϕ_k' are bounded, thus Lemma A1 in Ocone & Karatzas (1991) implies that $\phi_k(F) \in \mathbb{D}_{1,1}$ and

$$D\phi_k(F) = \phi_k'(F)DF. \tag{7.12}$$

Formulas (7.5), (7.6), and condition (5.20) imply

$$\lim_{k\to\infty} D_t\phi_k(F) = \phi'(F)D_tF; \quad a.e. \quad (t,\omega) \in [0,T] \times \Omega, \tag{7.13}$$

and

$$\|D_t\phi_k(F)\| \le K_2\|D_tF\|; \quad a.e. \quad (t,\omega) \in [0,T] \times \Omega. \tag{7.14}$$

From the assumption $F \in \mathbb{D}_{1.1}$ follows that

$$\tilde{E}\|DF\|_{L^2} < \infty,$$

and now (7.13), (7.14), and the Dominated Convergence Theorem imply

$$\lim_{k\to\infty} \tilde{E}\|D\phi_k(F) - \phi'(F)DF\|_{L^2} = 0 \tag{7.15}$$

Since the gradient operator D is closed, (7.11) and (7.15) guarantee that $\phi(F) \in \mathbb{D}_{1,1}$ and (5.21) holds.

In the second step of the proof we do not assume the boundedness of ϕ and ϕ'. This part of the proof will be similar to the proof of Lemma A1 in Ocone & karatzas (1991). Let $f \in C_0^\infty$ be such that $f(z) = z$ if $|z| \le 1$ and $|f(z)| \le |z|$ for all $z \in \Re$. We define for every $k \ge 1$ the function

$$l_k(x) = kf\left(\frac{\phi(x)}{k}\right); \quad x \in (a,b). \tag{7.16}$$

For every $k \ge 1$ the function l_k is obviously bounded on (a,b). One can easily see that $l_k \in PC^1(a,b)$. Indeed,

$$l_k'(x) = f'\left(\frac{\phi(x)}{k}\right)\phi'(x),$$

and since ϕ satisfies items (iv) and (v) in Definition 5.1, it follows that $\lim_{x\to a+} l_k'(x)$ and $\lim_{x\to b-} l_k'(x)$ exist and are finite. Therefore l_k also satisfies items (iv) and (v). The first three items of Definition 5.1 are obviously satisfied by l_k. Both l_k and l'_k are bounded on (a, b), thus from the first step of this proof $l_k(F) \in \mathbb{D}_{1,1}$ and

$$Dl_k(F) = f'(\frac{\phi(F)}{k})\phi'(F)DF. \tag{7.17}$$

Additionally, we have the bound

$$|l_k(F)| \leq |\phi(F)|; \quad k \geq 1, \tag{7.18}$$

and

$$\lim_{k\to\infty} l_k(F) = \phi(F), \quad a.s., \tag{7.19}$$

thus condition (5.19) implies

$$\lim_{k\to\infty} \tilde{E}|l_k(F) - \phi(F)| = 0 \tag{7.20}$$

Furthermore, we have

$$\|D_t l_k(F)\| \leq \sup_{x\in\Re} |f'(x)| \times \|\phi'(F)D_t F\|; \quad a.e. \quad (t, \omega) \in [0, T] \times \Omega, \tag{7.21}$$

and

$$\lim_{k\to\infty} D_t l_k(F) = \phi'(F)D_t F; \quad t \in [0, T]. \tag{7.22}$$

Condition (5.19) and the Dominated Convergence Theorem imply

$$\lim_{k\to\infty} \tilde{E}\|Dl_k(F) - \phi'(F)DF\|_{L^2} = 0, \tag{7.23}$$

and now $\phi(F) \in \mathbb{D}_{1,1}$ and (5.21) are consequences of (7.20), (7.23), and the closedness of the operator D.

References

[1] Bermin, H. P. (2000): "Hedging lookback and partial lookback options using Malliavin calculus," *Applied Mathematical Finance*, 7, 75-100.

[2] Bermin, H. P. (2002): "A general approach to hedging options: applications to barrier and partial barrier options," *Mathematical Finance*, 12, 199-218.

[3] Bermin, H. P. (1999): "Hedging options: the Malliavin calculus approach versus the Δ-hedging approach," Manuscript.

[4] Cox, J. C. and C. Huang (1989): "Optimal consumption and portfolio policies when asset prices follow a diffusion process," *Journal of Economic Theory*, 49, 33-83.

[5] Grossman, S. J. and J. Vila (1989): "Portfolio insurance in complete markets: a note," *Journal of Business*, 62, 473-476.

[6] Grossman, S. J. and Z. Zhou (1996): "Equilibrium analysis of portfolio insurance," *Journal of Finance*, 51, 1379-1403.

[7] Karatzas, I. (1989): "Optimization problems in the theory of continuous trading," *SIAM J. Contr. and Opt.*, 27, 1221-1259.

[8] Karatzas, I., D. L. Ocone, and J. Li (1991): "An extension of Clark's formula," *Stochastics and Stochastic Reports*, 37, 127-131.

[9] Karatzas, I. and S. E. Shreve (1991): *Brownian Motion and Stochastic Calculus.* Springer-Verlag, New York.

[10] Karatzas, I. and S. E. Shreve (1998): *Methods of mathematical finance.* Springer-Verlag, New York.

[11] Lakner, P. (1998): "Optimal trading strategy for an investor: the case of partial information," *Stochastic Processes and their Applications*, 76, 77-97.

[12] Merton, R. C. (1971): "Optimal consumption and portfolio rules in a continuous time model," *Jour. Econ. Theory*, 3, 373-413.

[13] Nualart, D. (1995): *The Malliavin calculus and related topics.* Springer-Verlag, New York-Berlin-Heidelberg.

[14] Ocone, D. L. and Karatzas, I. (1991): "A generalized Clark representation formula with applications to optimal portfolios," *Stochastics and Stochastic Reports*, 34, 187-220.

[15] Øksendal, B. K. (1996): "An introduction to Malliavin calculus with applications to economics," Working paper, Institute of Finance and Management Science, Norwegian School of Economics and Business Administration.

[16] Øksendal, B. K. (1998): *Stochastic differential equations: an introduction with applications.* 5th ed. Springer-Verlag, Berlin Heidelberg/New York.

[17] Teplá, L. (2001): "Optimal investment with minimum performance constraints," *Journal of Economic Dynamics & Control*, 25, 1629-1645.