OPTIMAL INVESTMENT IN A DEFAULTABLE BOND

Peter Lakner* and Weijian Liang†
New York University

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Abstract

The present paper analyzes the optimal investment strategy in a defaultable (corporate) bond and a money market account in a continuous time model. Due to jumps in the bond price our market model is incomplete. The treatment of information on the firm's asset value is based on an approach unifying the structural model and the reduced-form model. Specifically, the asset value will be assumed to be observable only at finitely many time points before the maturity of the bond. The optimal investment process will be worked out first for a short time-horizon with a general risk-averse utility function, then a multi-period optimal strategy with logarithmic and power utility will be presented using backward induction. The optimal investment strategy is analyzed numerically for the logarithmic utility.

Key Words: corporate bond, default risk, utility maximization, optimal investment

*Dept of IOMS, Stern School of Business, NYU, 44 W 4th St, New York, NY 10012. Email: plakner@stern.nyu.edu
†Email: wliang@stern.nyu.edu

1 Introduction

This paper addresses a utility maximization problem of an investor who can invest in a defaultable (corporate) bond or shortsell such bond continuously over a finite time-horizon. In order to pose this problem we need to adopt a model for the evolvement of the price of the bond, the default time, and the recovery value in case of a default. There is an abundance of literature available on this subject. The first work dates back to Black and Scholes (1973) and Merton (1974). These authors proposed to price a discount corporate bond as a contingent-claim on the firm's asset value, which is often assumed to be a (jump) diffusion process. A default is triggered when the firm's asset value process first hits a pre-defined threshold. This approach, called "structural", has been extended and enriched by other researchers. For example, Black and Cox (1976) considered the default barriers as a safety covenant, Geske (1977) developed a model for pricing coupon bonds. In addition, Longstaff and Schwartz (1995) provided closed form expressions for the price of coupon bonds with a stochastic interest rate and complex capital structures; Collin-Dufresne and Goldstein (2001) proposed a model that provided stationary leverage ratios and generalized the Longstaff and Schwartz (1995) model to a multi-factor case. However, the structural approach is criticized for its dependence on the complete observability of the firm's asset value process, which is arguable from an investor's point of view. In addition it yields a predictable default time which is counterintuitive. In practice default always has a bit of a surprise effect participants in the financial market.

Another approach, called "reduced form" was developed in the 1990's. In these models the default time has an intensity and thus is not predictable. Typically the intensity is modeled as a function of the state variables, and a default occurs when the cumulative intensity reaches a unit exponential random variable that is independent of all state variables. The reader can find several of papers on the reduced form models, including Jarrow and Turnbull (1995), Jarrow et al. (1997), Madan and Unal (1998), Lando (1998), Duffie and Singleton (1999). The reduced-form approach yields a totally inaccessible default time, and integrates very well the techniques developed for the default-free term-structure models (Musiela and Rutkowski (2002), or Sundaresan (2001)). It also yields relatively simple pricing formulas. However, the introduction of an exogenous exponential random variable to determine the default lacks economic justification.

Despite the apparent difference between the structural and reduced-form models, researchers noticed the connection between them and tried to unify the two models by means of information reduction. An incomplete list of references consists of Duffie and Lando (2001), Collin-Dufresne, Goldstein and Helwege (2003), Cetin, Jarrow, Protter and Yildirim (2004), Guo, Jarrow and Zeng (2005a, 2005b). Jarrow and Protter (2004) comment that the difference between the two approaches is whether the information regarding the firm asset value can be observed by the market or not, and that a structural model with a predictable default time can be transformed into a reduced-form model with a default intensity by information reduction. Guo et al. (2005a) actually calculate the default intensity in various reduced information models.

The purpose of this paper is to investigate the optimal investment in a defaultable bond. While there is a sizable literature on the valuation of defaultable bonds, little work has been done on optimal investment in such bonds. In our model an investor's portfolio consists of a bank account and a discount corporate bond. The investor observes the firm's asset value process only at discrete time points, i.e., we adopt the reduced information model. As pointed out in Guo, Jarrow and Zeng (2005a) this seems to be a realistic assumption, since investors usually observe the asset value when the firm provides its quarterly reports. The default will have an intensity process in this case, as pointed out by the same authors. The asset value process is assumed to follow a geometric Brownian motion, and the bond contract provides a constant default boundary and a constant recovery rate. Default occurs if the asset value drops to the level of the default boundary. The bond market is assumed to be frictionless with no transaction costs, and investors can trade continuously in time. The interest rate is assumed to be fixed. Meindl and Primbs (2006) consider this problem through the reduced-form approach, using receding horizon control and a simple binomial optimization technique.

In this paper we derive an optimal investment strategy to maximize the investor's expected utility of wealth at a terminal time. The optimal investment process turns out to be continuous when there is no corporate news coming in, but it has jumps at times of corporate news release; by corporate news release we mean the announcement of the firm asset value. Due to the the jumps of the bond price process at times of corporate news announcement, our market model is *incomplete*; please see an example for a non-hedgable contingent claim at the beginning of the Appendix.

The remainder of this paper is organized in the following way. In Section 2 we present the model of the firm's asset value process and derive the bond price. In Section 3 we define the optimization problem of an investor. In section 4 we solve the problem assuming that the terminal time for the optimization problem is short enough so that after the initial news release at time zero there is no additional news release before the terminal time. We solve this optimization problem for general utility functions. In Section 5 we study the cases of logarithmic, power, and negative exponential utility functions. In section 6 we generalize our solution to a multiple period setting allowing an arbitrary number of news releases between the initial and the terminal times. In Section 7 we discuss the numerical properties of the logarithmic utility case. We use here backwards induction, and present closed-form solutions for the cases of logarithmic and power utility functions. Finally, we summarize and give concluding remarks in Section 8.

2 Model for the defaultable bond

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\{\mathcal{F}_t, t \in [0,T]\}$ be an augmented filtration, where T is the maturity time of the defaultable bond issued by a firm. All processes appearing in this paper will be adapted to the above filtration (though some will be adapted to a smaller filtration as well). Let $\{X_t,\ t \leq T\}$ be the asset value process of the firm, and $\{p_t, \mathcal{F}_t,\ t \leq T\}$ be the price process of a zero coupon bond issued by the firm. Suppose that the face value of the bond (the amount paid at maturity if default does not occur) is one unit, and $F < 1$ is the predetermined recovery of treasury paid at time T if default occurs. We assume that the wealth of the firm can be observed only at times $0 = t_0 < t_1 < ... < t_n = T$ and at the default time τ if such default happens before (or at) the maturity of the bond. Suppose that $X_0 = x$ is a known constant. We model the asset value of the firm with a geometric Brownian motion

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad t \leq T \tag{2.1}$$

where $\{W_t, \mathcal{F}_t;\ t \leq T\}$ is a standard Brownian motion, μ and $\sigma > 0$ are known constants. Let $l > 0$ be a constant representing the liabilities of the firm. The default time τ is defined as $\inf\{t \geq 0 : X_t \leq l\}$, i.e., the bond defaults when the asset value reaches the liabilities. The risk-free interest rate r assumed to be a fixed constant.

The discounted asset value $X_t^* = e^{-rt} X_t$ satisfies

$$dX_t^* = (\mu - r) X_t^* dt + \sigma X_t^* dW_t.$$

We define

$$d\widetilde{W}_t = dW_t + \frac{\mu - r}{\sigma} dt, \quad t \le T \tag{2.2}$$

and note that by Girsanov's theorem (Karatzas & Shreve, (1998), Theorem 3.5.1) $\{\widetilde{W}_t,\ t \le T\}$ is a Brownian motion under the probability measure Q that is given by

$$\frac{dQ}{dP} = Z_T,$$

where

$$Z_t = \exp\left\{ -\frac{\mu - r}{\sigma} W_t - \frac{(\mu - r)^2}{2\sigma^2} t \right\}, \quad t \le T. \tag{2.3}$$

We cast the equations for the asset value and the discounted asset value in the form

$$dX_t = rX_t dt + \sigma X_t d\widetilde{W}_t$$

$$dX_t^* = \sigma X_t^* d\widetilde{W}_t$$

and express X_t as

$$X_t = x \exp\{\sigma \widetilde{W}_t + (r - \frac{\sigma^2}{2}) t\}. \tag{2.4}$$

The information available to the market is represented by the filtration $\{\mathcal{G}_t,\ t \le T\}$ where the sigma algebra $\mathcal{G}_t$ at time $t \in [t_i, t_{i+1})$ is given by

$$\mathcal{G}_t = \sigma\left\{ X_0, \ldots, X(t_i), H_u, u \le t \right\},$$

with H_t being the indicator function

$$H_t = 1_{\{t \ge \tau\}}, \quad t \le T.$$

The above filtration represents the reduced information that has been mentioned in the introduction. Had we assumed that investors have access to the larger $\{\mathcal{F}_t;\ t \le T\}$ filtration, we would have a structural model. However, in our model investors observe the asset value only at the times of corporate news release $t_0, t_1, \ldots, t_n$, and observe default whenever it occurs. As we already pointed out in the introduction, this is an incomplete market model. Now we are in the position to make this statement a bit more precise. It is incomplete with respect to the filtration $\{\mathcal{G}_t;\ t \le T\}$ that represents the information available to the public. It is clear from the derivation at the beginning of the Appendix that there are even bounded $\mathcal{G}_t$ measurable contingent claims that can not be hedged. There is a unique martingale measure Q but this is based on the larger filtration $\{\mathcal{F}_t;\ t \le T\}$ that is available only to the insiders of the firm. Using this larger information set would yield arbitrage opportunities since the default time is predictable with respect to $\{\mathcal{F}_t;\ t \le T\}$. In other words, the discounted bond price is not a Q-martingale with respect to the larger filtration $\{\mathcal{F}_t;\ t \le T\}$ because it has a negative jump at a predictable time. However, such trading is illegal and we assume in our model that it is not happening. The arbitrage opportunity disappears if we switch to the filtration $\{\mathcal{G}_t;\ t \le T\}$ that is available to the investors since the discounted bond price becomes a martingale. Market participants observing $\{\mathcal{F}_t;\ t \le T\}$ are not allowed to trade in the bond.

In order to compute the arbitrage-free price for the bond we need the distribution of τ under Q. We express τ as

$$\begin{aligned}\tau &= \inf\{t \ge 0 : X_t \le l\} = \inf\left\{t \ge 0 : \sigma \widetilde{W}_t + \left(r - \frac{\sigma^2}{2}\right) t \le \log \frac{l}{x}\right\} \\ &= \inf\left\{t \ge 0 : \widetilde{W}_t + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right) t \le \frac{1}{\sigma} \log \frac{l}{x}\right\},\end{aligned}$$

and from Karatzas & Shreve (1998), Section 3.5c follows that

$$Q(\tau > t) = \psi(t, \tilde{r}, b(x))$$

where

$$\tilde{r} = \frac{r}{\sigma} - \frac{\sigma}{2}, \tag{2.5}$$

$$b(x) = \frac{1}{\sigma} \log \frac{l}{x} < 0, \tag{2.6}$$

and the function $\psi(\cdot, \cdot, \cdot)$ is defined as

$$\psi(t, \lambda, \kappa) = 1 - \int_0^t \frac{|\kappa|}{\sqrt{2\pi u^3}} \exp\left\{-\frac{(\kappa - \lambda u)^2}{2u}\right\} du. \tag{2.7}$$

Clearly under the probability measure P we have

$$P(\tau > t) = \psi(t, \tilde{\mu}, b(x))$$

with

$$\tilde{\mu} = \frac{\mu}{\sigma} - \frac{\sigma}{2}. \tag{2.8}$$

Next we compute the price of the defaultable bond. We shall apply the general arbitrage-free pricing formula

$$e^{-rt} p_t = e^{-rT} \mathbb{E}_Q \left[1_{\{\tau > T\}} + F \cdot 1_{\{\tau \le T\}} \mid \mathcal{G}_t\right].$$

Clearly we have

$$e^{-rt}p_t 1_{\{\tau \le t\}} = e^{-rT} F 1_{\{\tau \le t\}}$$

so it suffices to compute the bond price on $\{t < \tau\}$. Suppose that $t \in [t_i, t_{i+1})$ for some $i = 0, \ldots, n-1$ and let

$$\tau_i = \inf \{t \ge t_i;\ X_t \le l\}. \tag{2.9}$$

On the event $\{t < \tau\}$ the discounted bond $e^{-rt}p_t$ is given by

$$\begin{aligned}
& e^{-rT}\mathbb{E}_Q \left[1_{\{\tau > T\}} + F 1_{\{\tau \le T\}} \mid X_0, \ldots, X(t_i), t < \tau\right] \\
= \;& e^{-rT}\mathbb{E}_Q \left[1_{\{\tau > T\}} + F 1_{\{\tau \le T\}} \mid X_0, \ldots, X(t_i), t < \tau, t_i < \tau\right] \\
= \;& e^{-rT} \frac{Q\left[\tau_i > T \mid X_0, \ldots, X(t_i), t_i < \tau\right] + FQ\left[t < \tau_i \le T \mid X_0, \ldots, X(t_i), t_i < \tau\right]}{Q\left[\tau > t \mid \tau > t_i, X_0, \ldots, X(t_i)\right]} \\
= \;& e^{-rT} \left[(1-F)\frac{\psi\left(T - t_i, \tilde{r}, b\left(X\left(t_i\right)\right)\right)}{\psi\left(t - t_i, \tilde{r}, b\left(X\left(t_i\right)\right)\right)} + F\right].
\end{aligned}$$

For brevity we introduce

$$\gamma(u_1, u_2, z) = (1-F)\frac{\psi\left(u_2, \tilde{r}, b\left(z\right)\right)}{\psi\left(u_1, \tilde{r}, b\left(z\right)\right)} + F, \tag{2.10}$$

and summarize our result: for $t \in [t_i, t_{i+1})$

$$e^{-rt}p_t = e^{-rT}\gamma\left(t - t_i, T - t_i, X\left(t_i\right)\right) 1_{\{\tau > t\}} + e^{-rT} F 1_{\{\tau \le t\}}. \tag{2.11}$$

This formula holds for $t = t_n = T$ as well. There is a jump of the price at τ and $p_\tau < p_{\tau-}$. The bond price has additional jumps at times $t_1, \ldots, t_{n-1}$. However, it is left-continuous at the maturity time T.

3 The optimization problem

Let $s \le T$ be the terminal time of the investor and π_t be the number of bonds held by her or him at time t. All available funds not invested in the bond will be put in a bank account continuously earning interests at rate r. The number of bonds π_t may be negative corresponding to shorting the bond. Borrowing at the rate r is also permissible, but we shall require that the overall wealth of the investor is almost surely non-negative at any time $t \in [0, s]$. Here are the rigorous details of our requirements concerning investment processes.

Definition 3.1. A process $\{\pi_t;\ t \in [0,s]\}$ is called an investment process if it is predictable with respect to the filtration $\{\mathcal{G}_t;\ t \in [0,T]\}$ and $\pi_t = \pi_\tau$ on $\{t \geq \tau\}$.

Predictability of the investment process implies that it is also adapted to $\{\mathcal{G}_t;\ t \in [0,s]\}$, so the investor can base her or his decisions only on the available information. On the other hand predictability prevents arbitrage at times $t = \tau$ or $t = t_i$ when the bond price has jumps. The assumption that π_t is flat on $[\tau, T]$ (whenever $\tau < T$) is only a formality. In that case, formally assuming that the investor holds the defaulted bond is identical to assuming that all the wealth is put in a bank account earning the continuously compounded risk-free rate.

Let V_t be the wealth of the investor at time t. We postulate that the investor is self-financed, so

$$dV_t = \pi_t dp_t + (V_t - \pi_t p_t) r dt.$$

This holds on both $\{\tau > t\}$ and $\{\tau \leq t\}$. Obviously, it simplifies to $dV_t = rV_t dt$ if $\tau \leq t$. We can write the discounted wealth $V_t^* = e^{-rt} V_t$ (by Ito's formula) as

$$dV_t^* = e^{-rt} \pi_t (dp_t - r p_t dt) = \pi_t dp_t^*, \tag{3.1}$$

where $p_t^* = e^{-rt} p_t$ is the discounted bond price. In particular, on $\{\tau \leq t\}$ we have $V_t = V_\tau e^{r(t-\tau)}$.

Definition 3.2. A function as $U : [0,\infty) \to \mathbb{R} \bigcup \{-\infty\}$ will be called a utility function if it is concave and strictly increasing on $[0,\infty)$, twice continuously differentiable on $(0,\infty)$, the derivative function $U'(\cdot)$ is strictly decreasing on $(0,\infty)$, and the following additional requirements hold:

(i) $\lim_{z\to\infty} U'(z) = 0$

(ii) the pseudo-inverse of $U'(\cdot)$ defined as $I(z) = \inf\{u \geq 0 :\ U'(u) \leq z\}$ satisfies $I(z) \leq (z^{-m} + 1) K_1$ for some $m \in \mathbb{N}$, $K_1 > 0$ and all $z \geq 0$.

(iii) If $U(0) = -\infty$ then $-U(I(x)) \leq K_2 (x^m + 1)$ for some $m \in \mathbb{N}$, $K_2 > 0$, and all $x > 0$.

(iv) If $U'(0) = \lim_{z\downarrow 0} U'(z) < \infty$ then $\lim_{\vartheta\downarrow 0} U''(\vartheta)$ exists and is finite.

The function $I(\cdot)$ becomes the inverse function of $U'(\cdot)$ if $U'(0) = \infty$. Otherwise it is strictly decreasing on $(0, U'(0))$ and zero on $[U'(0), \infty)$.

The notation V^π will be used for the wealth process corresponding to the investment process π whenever we want to emphasize this correspondence. We shall drop the superscript π whenever it is possible without causing confusion.

Definition 3.3. An investment process will be called admissible if the corresponding wealth process satisfies $V_t^\pi \geq 0$, almost surely, for all $t \in [0, s]$.

We shall study the following optimization problem

$$\max \{\mathbb{E}[U(V_s^\pi)];\ \pi \in \mathcal{A}\}$$

where $\mathcal{A}$ is the class of admissible investment processes satisfying $\mathbb{E}[U(V_s)^-] < \infty$. The last requirement is included in order to guarantee that $\mathbb{E}[U(V_s^\pi)]$ exists.

From (3.1) follows that for every investment process the corresponding discounted wealth process $\{V_t^*, \mathcal{G}_t; t \leq s\}$ is a Q-local martingale (Protter (2004), Chapter III, Theorem 29). If the investment process is admissible, then it is a non-negative local martingale hence also a supermartingale under Q with respect to the filtration $\{\mathcal{G}_t;\ t \leq s\}$. Let $V_0 = v$ be the fixed initial wealth at time zero. The supermartingale property implies that for every admissible investment process the corresponding terminal wealth satisfies the *budget constraint*

$$\mathbb{E}_Q(V_s^*) \leq v \tag{3.2}$$

4 Solution to the optimization problem for a short time-horizon

In this section we shall assume that $s < t_1$ (the general case will be discussed in section 6). The random variable

$$\zeta_s = \mathbb{E}[Z_s \mid \mathcal{G}_s]$$

will play an important role in our analysis. Note that if Y is a $\mathcal{G}_s$-measurable r.v., then $\mathbb{E}_Q(Y) = \mathbb{E}(\zeta_s Y)$.

Theorem 4.1. *Suppose that there exists an admissible investment process $\{\hat{\pi}_t, t \leq s\}$ such that the corresponding wealth process $\{\widehat{V}_t, t \leq s\}$ satisfies $\widehat{V}_s = I(y\zeta_s)$ for some constant*

$y = y(s,x,v) > 0$ *and*

$$\mathbb{E}_Q(e^{-rs}\widehat{V}_s) = v. \tag{4.1}$$

Then the investment process $\hat{\pi}$ is optimal.

Proof. First we show that $E\left[\left(U\left(\hat{V}_s\right)\right)^-\right] < \infty$. This is obvious if $U(0) > -\infty$ so we assume that $U(0) = -\infty$, in which case by item (iii) of Definition 3.2

$$E\left[\left(U\left(\hat{V}_s\right)\right)^-\right] = -E\left[U\left(I\left(y\zeta_s\right)\right)1_{\{U(I(y\zeta_s))\le 0\}}\right] \le E\left[K_2\left(y^m\zeta_s^m + 1\right)\right].$$

This last expression is finite since by Jensen's inequality

$$E\left[\zeta_s^m\right] = E\left[\left(E\left[Z_s \mid \mathcal{G}_s\right]\right)^m\right] \le E\left[Z_s^m\right] = \exp\left\{m(m-1)\frac{(\mu-r)^2}{2\sigma^2}s\right\} < \infty. \tag{4.2}$$

Next we are going to show that $\hat{\pi}$ is indeed optimal. By the concavity of $U(\cdot)$

$$U(I(c)) \ge U(a) + [I(c) - a]c \quad \text{for all } a \ge 0,\ c > 0$$

thus

$$U\left(I(y\zeta_s)\right) \ge U(V_s) + \left[I(y\zeta_s) - V_s\right]y\zeta_s,$$

where V_s is the terminal wealth corresponding to an arbitrary admissible investment process. After taking expectations we get

$$\mathbb{E}\left[U\left(\widehat{V}_s\right)\right] \ge \mathbb{E}\left[U\left(V_s\right)\right] + \mathbb{E}\left[y\zeta_s(\widehat{V}_s - V_s)\right].$$

Both $\widehat{V}_s$ and V_s are $\mathcal{G}_s$-measurable, hence by (4.1) and (3.2) we have

$$\mathbb{E}[U(\widehat{V}_s)] - \mathbb{E}[U(V_s)] \ge y\mathbb{E}[\zeta_s(\widehat{V}_s - V_s)] = y\mathbb{E}_Q(\widehat{V}_s - V_s) = ye^{rs}[v - \mathbb{E}_Q(V_s^*)] \ge 0. \ \square$$

In the rest of the section we shall identify the investment process $\hat{\pi}$ characterized in the above theorem.

Lemma 4.2. *There exists a unique constant $y = y(s,x,v) > 0$ such that $\mathbb{E}_Q\left[e^{-rs}I(y\zeta_s)\right] = v$.*

Proof. The function $\xi(y) = e^{-rs}I(y\zeta_s)$ is continuous and decreasing on $(0,\infty)$, satisfying $\lim_{y\to 0}\xi(y) = \infty$ and $\lim_{y\to\infty}\xi(y) = 0$. The Monotone Convergence Theorem implies that $\lim_{y\to 0}\mathbb{E}_Q[e^{-rs}I(y\zeta_s)] = \infty$. Next we show that $\lim_{y\to\infty}\mathbb{E}_Q[e^{-rs}I(y\zeta_s)] = 0$ and the function

$y \mapsto \mathbb{E}_Q[e^{-rs}I(y\zeta_s)]$ is continuous. Both properties will follow from the Dominated Convergence Theorem once we established $\mathbb{E}_Q[e^{-rs}I(y\zeta_s)] < \infty$ for every $y \geq 0$. However, by item (ii) of Definition 3.2

$$\mathbb{E}_Q[e^{-rs}I(y\zeta_s)] \leq e^{-rs}\mathbb{E}_Q[(y^{-m}\zeta_s^{-m}+1)K_1] = e^{-rs}K_1\left(y^{-m}\mathbb{E}\left[\zeta_s^{1-m}\right]+1\right),$$

and (4.2) demonstrates that this expression is finite (we have to replace m in (4.2) with $(1-m)$. □

Next we are going to represent the optimal terminal wealth in the form

$$I(y\zeta_s) = \alpha \cdot 1_{\{s<\tau\}} + \beta(\tau)\cdot 1_{\{s\geq\tau\}} \tag{4.3}$$

for some deterministic function $\beta(t) = \beta(t,s,x,v)$ and non-negative constant $\alpha = \alpha(s,x,v)$.In order to simplify the notation we shall write b instead of $b(x)$.

Lemma 4.3. *In the previous expression for $I(y\zeta_s)$, we have*

$$\beta(t) = I\left(ye^{Kt}\left(\frac{l}{x}\right)^L\right) \tag{4.4}$$

where $L = (r-\mu)/\sigma^2$, $K = (\mu + r - \sigma^2)(\mu - r)/(2\sigma^2)$, and

$$\alpha = I\left(y\frac{\psi(s,\tilde{r},b)}{\psi(s,\tilde{\mu},b)}\right). \tag{4.5}$$

The constants $\tilde{\mu}$, $\tilde{r}$, $b = b(x)$ and ψ have been defined in (2.8), (2.5), (2.6), and (2.7). The constant $y = y(s,x,v)$ is the one implicitly determined by Lemma (4.2).

Proof. From (2.2), (2.3), and (2.4) follows that

$$Z_s = e^{Ks}\left(\frac{X_s}{x}\right)^L. \tag{4.6}$$

X_t is a Markov process, hence for all $t \leq s$

$$\begin{aligned}
\mathbb{E}[Z_s \mid \tau = t] &= \mathbb{E}\left[e^{Ks}\left(\frac{X_s}{x}\right)^L \,\middle|\, X_u > l,\ \forall\, u < t,\ X_t = l\right] \\
&= \mathbb{E}\left[e^{Ks}\left(\frac{X_s}{x}\right)^L \,\middle|\, X_t = l\right] \\
&= \mathbb{E}\left[e^{Ks}\left(\frac{X_s}{x}\right)^L \,\middle|\, \mathcal{F}_t^X\right]\Big|_{X_t=l} \\
&= e^{Kt}\left(\frac{l}{x}\right)^L,
\end{aligned}$$

where the last identity follows from the fact that the process $\left\{e^{Kt}\left(\frac{X_t}{x}\right)^L,\ t \le T\right\}$ is a martingale (see (4.6)).

If $\tau > s$ then

$$\mathbb{E}[Z_s \mid \tau > s] = \frac{\mathbb{E}[Z_s \cdot 1_{\{\tau>s\}}]}{P(\tau > s)} = \frac{Q(\tau > s)}{P(\tau > s)} = \frac{\psi(s, \tilde{r}, b)}{\psi(s, \tilde{\mu}, b)}.$$

It follows that

$$\zeta_s = e^{K\tau}\left(\frac{l}{x}\right)^L \cdot 1_{\{\tau \le s\}} + \frac{\psi(s, \tilde{r}, b)}{\psi(s, \tilde{\mu}, b)} \cdot 1_{\{\tau>s\}},$$

which completes the proof. □

Lemma 4.4. *The constant $y = y(s, x, v)$ is determined by*

$$e^{-rs}\left[\alpha\psi(s, \tilde{r}, b) - \int_0^s \beta(t)\psi'(t, \tilde{r}, b)dt\right] = v,$$

where ψ' is the derivative of ψ with respect to the variable t.

Proof. We have

$$\begin{aligned} v &= \mathbb{E}_Q[e^{-rs}I(y\zeta_s)] \\ &= e^{-rs}\mathbb{E}_Q[\alpha \cdot 1_{\{\tau>s\}} + \beta(\tau) \cdot 1_{\{\tau \le s\}}] \\ &= e^{-rs}[\alpha Q(\tau > s) + \int_0^s \beta(t)Q(\tau \in dt)] \\ &= e^{-rs}[\alpha\psi(s, \tilde{r}, b) - \int_0^s \beta(t)\psi'(t, \tilde{r}, b)dt], \end{aligned}$$

hence the lemma follows. □

We are going to search for the optimal investment process in the form

$$\hat{\pi}_t = h(t)1_{\{t \le \tau\}} + h(\tau)1_{\{\tau<t\}} \tag{4.7}$$

for some continuous function $h(t) = h(t, s, T, x, v)$. In the following theorem we shall identify this function. Note that if $U'(0) = \infty$ then $\beta(\cdot)$ is continuously differentiable on $(0, \infty)$ since $U(\cdot)$ was required to be twice continuously differentiable on $(0, \infty)$. However, if $U'(0) < \infty$ then there may exists a point u_0 such that $\beta(\cdot)$ is continuously differentiable on $(0, u_0) \cup (u_0, \infty)$, but has a breakpoint in u_0. Let $\beta'(\cdot)$ be the derivative of $\beta(\cdot)$ with the understanding that if $\beta(\cdot)$ has a breakpoint in u_0 then $\beta'(u_0)$ is arbitrary.

Theorem 4.5. *The optimal investment process is determined by (4.7) where $h(t)$ is continuous on $[0,s]$ and is given by $h(t) = h(0) + \int_0^t h'(u)du$,*

$$h(0) = \frac{e^{rT}}{1-F} \cdot \frac{v - e^{-rs}\beta(0)}{\psi(T,\tilde{r},b)} = \frac{v - e^{-rs}\beta(0)}{p_0 - e^{-rT}F}, \tag{4.8}$$

$$h'(t) = -e^{r(T-s)}\frac{\beta'(t)}{\gamma(t) - F}. \tag{4.9}$$

Formula (4.9) can be cast in the form

$$h'(t) = -e^{-r(s-t)}\frac{\beta'(t)}{p_t(\tau > t) - e^{-r(T-t)}F} \tag{4.10}$$

where the notation $p_t(\tau > t)$ represents the time-t value of the bond on the event$\{\tau > t\}$.

Proof. It is clear that the above defined function $h(\cdot)$ is either included in $C^1((0,\infty))$ or in $C^1((0,u_0) \cup (u_0,\infty))$, similarly to $\beta(\cdot)$. We are going to show that even in the second case $h(\cdot)$ is continuous in u_0. Suppose that indeed we are in the second case, so $U'(0) < \infty$. We need to show that the expression defining $h'(t)$ on the right-hand side of (4.9) has finite left and right limits in $t = u_0$ which amounts to showing that both $\lim_{t\uparrow u_0}\beta'(t)$ and $\lim_{\downarrow u_0}\beta'(t)$ are finite. By (4.4) breakpoint u_0 is determined by

$$y\exp\{Ku_0\}\left(\frac{l}{x}\right)^L = U'(0).$$

If the above equation leads to a nonpositive value for u_0 then we are in the case of $h(\cdot)$ being in $C^1((0,\infty))$, so we assume that this equation yields a positive u_0. In the following calculation we shall assume that $K > 0$; the complementary case can be covered in a very similar fashion. For the left limit

$$\lim_{t\uparrow u_0}\beta'(t) = \lim_{z\uparrow U'(0)} KI'(z)z = \lim_{z\uparrow U'(0)}\frac{Kz}{U''(I(z))} = \lim_{\vartheta\downarrow 0}\frac{KU'(\vartheta)}{U''(\vartheta)},$$

and this is finite by the last requirement of Definition 3.2. For the right limit

$$\lim_{t\downarrow u_0}\beta'(t) = \lim_{z\downarrow U'(0)} KI'(z)z = 0,$$

so we established the continuity of $h(\cdot)$ on $[0,s)$.

In the remaining part of the proof we are going to show that $\hat{\pi}$ satisfies

$$e^{-rs}\widehat{V}_s = v + \int_0^s \hat{\pi}_t dp^*_t. \tag{4.11}$$

Formula (2.11) for the case of $t \in [0, t_1)$ gives

$$e^{-rt}p_t = e^{-rT}\gamma(t)\, 1_{\{\tau>t\}} + e^{-rT}F 1_{\{\tau\le t\}}, \tag{4.12}$$

where we use $\gamma(t) = \gamma(t, T, x)$ for the sake of brevity.

We separate two cases:
Case 1: $\tau > s$. In this case (4.11) is equivalent to

$$e^{-rs}\alpha = v + e^{-rT}\int_0^s h(t)\gamma'(t)dt \tag{4.13}$$

Case 2: $\tau = u \le s$. Now (4.11) is equivalent to

$$e^{-rs}\beta(u) = v + e^{-rT}\int_0^u h(t)\gamma'(t)dt + h(u)[e^{-rT}F - e^{-rT}\gamma(u)] \tag{4.14}$$

We shall work on case 2 first. We already established that $h(\cdot)$ is continuous, so (4.14) holds if it is true for $u = 0$, and the derivatives of both sides in every point $u \neq u_0$ are identical. Naturally if both sides are differentiable on $(0, \infty)$ then we disregard the $u \neq u_0$ condition in the previous sentence. Taking derivatives of (4.14) with respect to u we get

$$e^{-rs}\beta'(u) = h'(u)[e^{-rT}F - e^{-rT}\gamma(u)] = -h'(u)e^{-rT}(1-F)\frac{\psi(T,\tilde{r},b)}{\psi(u,\tilde{r},b)},$$

which follows from (4.9). In addition, (4.14) with $u = 0$ follows from (4.8) and the definition of $\gamma(\cdot)$.

Next we consider case 1. Let $\gamma_1(t) = e^{-rT}\gamma(t) - e^{-rT}F$. By integration by parts, we have

$$e^{-rT}\int_0^s h(t)\gamma'(t)dt = h(s)\gamma_1(s) - h(0)\gamma_1(0) - \int_0^s h'(t)\gamma_1(t)dt.$$

In order to show that (4.13) holds for $h'(t)$ specified in (4.9), we substitute $\gamma_1(t)$, (4.8) and (4.9) into the right-hand side of the above equation after which it can be simplified to

$$e^{-rT}\int_0^s h(t)\gamma'(t)dt = \frac{1}{\psi(s)}\left[v + e^{-rs}\int_0^s \psi'(u)\beta(u)du\right] - v.$$

We apply Lemma 4.4 here, and it is straightforward to see (using another partial integration) that the right-hand side of the above expression is exactly $e^{-rs}\alpha - v$, which implies (4.13). Note that the partial integrations above are correct even if $h(\cdot)$ and $\beta(\cdot)$ have a breakpoint, because both functions are continuous. □

Below we are going to state some simple features of our solution (4.8) and (4.9). Notice that $h(\cdot)$ is linear in $1/(1-F)$. Another interesting feature of the optimal investment process is that the terminal wealth $\hat{V}_s = I(y\zeta_s)$ does not depend on the maturity of the bond T. It is worth noting that $h(\cdot)$ depends on x and l only through the proportion x/l. In addition both $|h(0)|$ and $|h'(t)|$ are increasing with respect to the maturity time T (this is happening because $\psi(T, \tilde{r}, b)$ is decreasing in T).

Next follows a simple intuitive derivation of (4.8). Let us suppose that default happens at a time very close to time zero, say before time ϵ where ϵ is a small positive number. Since we are only getting an intuitive insight into formula (4.8), we shall not make this notion more precise. Simple accounting shows that if $\epsilon \to 0$ then the wealth at time s will converge to $(v - h(0)p_0)e^{rs} + Fh_0 e^{-r(T-s)}$. However, from (4.3) follows that if $\tau \to 0$ then the optimal terminal wealth converges to $\beta(0)$. Hence we must have

$$\beta(0) = (v - h_0 p_0)e^{rs} + Fh_0 e^{-r(T-s)}$$

which gives (4.8) after rearrangement of the terms.

5 Examples

In this section we shall investigate some special utility functions.

A. Power and logarithmic utilities.

Suppose the utility function is

$$U_\delta(c) = \begin{cases} \frac{1}{\delta} c^\delta, & \text{if } \delta < 1 \text{ and } \delta \neq 0; \\ \log c, & \text{if } \delta = 0. \end{cases} \tag{5.1}$$

In this case $U'_\delta(c) = c^{\delta-1}$ and $I(u) = u^{\frac{1}{\delta-1}}$ for all $\delta < 1$.

By straightforward substitution

$$
\begin{aligned}
\alpha &= y^{1/(\delta-1)} \cdot \left[\frac{\psi(s,\tilde{r},b)}{\psi(s,\tilde{\mu},b)}\right]^{1/(\delta-1)} \\
\beta(t) &= y^{1/(\delta-1)} \cdot \exp\left\{\frac{K}{\delta-1}t\right\} \cdot \left(\frac{l}{x}\right)^{L/(\delta-1)} \\
h(0) &= \frac{e^{rT}}{1-F} \cdot \frac{v - e^{-rs} y^{1/(\delta-1)} (\frac{l}{x})^{L/(\delta-1)}}{\psi(T,\tilde{r},b)} \\
h'(t) &= y^{1/(\delta-1)} \cdot \frac{e^{r(T-s)}}{1-F} \cdot \frac{\psi(t,\tilde{r},b)}{\psi(T,\tilde{r},b)} \cdot \frac{K}{1-\delta} \cdot \exp\left\{\frac{K}{\delta-1}t\right\} \cdot \left(\frac{l}{x}\right)^{L/(\delta-1)}
\end{aligned}
$$

The optimal terminal wealth is determined by the above formulas for α, $\beta(t)$, and (4.3). By Lemma 4.4 we have

$$
y^{1/(\delta-1)} = e^{rs} v \left[\left(\psi\left(s,\tilde{r},b\right)\right)^{\frac{\delta}{\delta-1}} \left(\psi\left(s,\tilde{\mu},b\right)\right)^{-\frac{1}{1-\delta}} - \int_0^s \exp\left\{\frac{K}{\delta-1}t\right\} \left(\frac{l}{x}\right)^{\frac{L}{\delta-1}} \psi'\left(t,\tilde{r},b\right) dt \right]^{-1}.
$$

For the utility function given in (5.1) we use the notation $h_\delta(\cdot)$ instead of $h(\cdot)$, and with an eye on future developments we also write $h_\delta(t,s,T,v,x)$. We summarize the above results in the following theorem.

Theorem 5.1. *The optimal investment process for the above utility function is determined by*

$$
h_\delta(0) = h_\delta(0,s,T,v,x) = \frac{v}{1-F} A(s,T,x) \tag{5.2}
$$

$$
h'_\delta(t) = \frac{\partial}{\partial t} h_\delta(t,s,T,v,x) = \frac{v}{1-F} B(t,s,T,x) \tag{5.3}
$$

where $A(s,T,x)$ and $B(t,s,T,x)$ do not depend on v or F. These functions are given by the following formulae:

$$
A(s,T,x) = C_1(T,x) \left\{ 1 - \phi(s,x) \left(\frac{l}{x}\right)^{\frac{L}{\delta-1}} \right\}
$$

$$
B(t,s,T,x) = C_1(T,x)\phi(s)\psi(t,\tilde{r},b) \frac{K}{1-\delta} C_2(t,x)
$$

$$
C_1(T,x) = \frac{e^{rT}}{\psi(T,\tilde{r},b(x))}
$$

$$
C_2(t,x) = \exp\left\{\frac{K}{\delta-1}t\right\} \left(\frac{l}{x}\right)^{\frac{L}{\delta-1}}
$$

and

$$\phi(s,x) = \left\{ \left(\psi\left(s,\tilde{r},b\right)\right)^{\frac{\delta}{\delta-1}} \left(\psi\left(s,\tilde{\mu},b\right)\right)^{-\frac{1}{\delta-1}} - \int_0^s C_2(t,x)\psi'(t,\tilde{r},b)dt \right\}^{-1}.$$

An important feature of this optimal investment process is that it is linear in the initial wealth v.

In the case of log utility ($\delta = 0$)*,* $h_0(0)$ *and* $h_0'(t)$ *can be greatly simplified because* $\mathbb{E}_Q[e^{-rs}I(y\zeta_s)] = v$ *implies* $e^{-rs}/y = v$*. Thus,*

$$h_0(0) = h_0(0,T,v,x) = \frac{v}{1-F} \cdot \frac{e^{rT}}{\psi(T,\tilde{r},b(x))} \left[1 - \left(\frac{l}{x}\right)^{-L}\right] \tag{5.4}$$

$$h_0'(t) = \frac{\partial}{\partial t} h_0(t,T,v,x) = \frac{v}{1-F} \cdot \frac{e^{rT}}{\psi(T,\tilde{r},b(x))} \psi(t,\tilde{r},b(x)) K e^{-Kt} \left(\frac{l}{x}\right)^{-L} \tag{5.5}$$

It is worth noting that in this case the optimal investment process does not depend on s.

Next we calculate the *value function* of our optimization problem for the power and logarithmic utility function, i.e., the maximal expected utility from terminal wealth

$$W_\delta(v,x,s) = E\left[U_\delta\left(\hat{V}_s\right)\right].$$

Theorem 5.2. *For all* $\delta < 1$, $\delta \neq 0$ *the value function is*

$$W_\delta(v,x,s) = D_\delta(x,s) \cdot \frac{1}{\delta} v^\delta, \tag{5.6}$$

and for $\delta = 0$ *it is*

$$W_0(v,x,s) = D_0(x,s) + \log v. \tag{5.7}$$

where

(5.8)

$$D_\delta(x,s) = e^{\delta rs} \left(\phi(s,x)\right)^\delta \left[\left(\psi\left(s,\tilde{r},b\right)\right)^{\frac{\delta}{\delta-1}} \left(\psi\left(s,\tilde{\mu},b\right)\right)^{-\frac{1}{\delta-1}} - \int_0^s \left(C_2(t,x)\right)^\delta \psi'(t,\tilde{\mu},b)dt \right]$$

and

$$D_0(x,s) = rs - \psi(s,\tilde{\mu},b) \log \frac{\psi(s,\tilde{r},b)}{\psi(s,\tilde{\mu},b)} - (1-\psi(s,\tilde{\mu},b)) \log \left(\frac{l}{x}\right)^L + \int_0^s Kt\psi'(t,\tilde{\mu},b)dt.$$

Proof. This is an immediate consequence of (4.3), (4.4), and (4.5). □

B. Negative Exponential Utility.

Not every utility function implies a linear relationship between the optimal investment process π and the initial wealth v. Negative exponential utility, $U_\theta(c) = -e^{-\theta c}$ with $\theta > 0,\ c > 0$, is one of such counterexamples.[1]

6 Solution for the case of long time-horizon

In this section we allow the terminal time s to be an arbitrary time point between zero and T.

A. Logarithmic utility

Here we consider the case of $U(v) = \log v$. We shall proceed by backwards induction, and in order to do so we have to consider initial times other than zero. Suppose that the present time is $t_i < s$ for some index i, the asset value of the corporation has just been announced to be $X(t_i) = x$, and the current wealth of the investor is $V(t_i) = v$. Also assume that default did not happen yet, i.e., $\tau > t_i$. Let $P^{t_i,x}$ be the conditional probability measure given that $X(t_i) = x$ and $\tau > t_i$, and let $\mathbb{E}^{t_i,x}$ be the corresponding expectation. We shall now adapt the notations in the previous sections to this situation. The asset value of the corporation X_t now satisfies diffusion (2.1) with initial data (t_i, x), and $\tau = \tau_i$ since we assumed $\tau > t_i$ (τ_i was defined in (2.9)). The wealth $V_u^\pi(t_i, v)$ at time u corresponding to an investment

[1] For negative exponential utility function, we have $U_\theta'(c) = \theta e^{-\theta c}$ and $I(u) = (1/\theta)\,[\log(\theta/u)]^+$. We define the intervals $N_1 = \left(0,\ \theta\,(l/x)^{-L}\right)$, $N_2 = \left(0,\ \theta e^{-Kt}\,(l/x)^{-L}\right)$ and $N_3 = (0,\ \theta\psi(s,\tilde{\mu},b)/\psi(s,\tilde{r},b))$. It is straightforward to show that

$$
\begin{aligned}
h(0) &= \frac{e^{rT}}{(1-F)\psi(T,\tilde{r},b)}\left[v - \frac{e^{-rs}}{\theta}\cdot 1_{N_1}(y)\cdot\log\left(y^{-1}\theta\left(\frac{l}{x}\right)^{-L}\right)\right]\\
h'(t) &= \frac{e^{rT}}{(1-F)\psi(T,\tilde{r},b)}\cdot\psi(t,\tilde{r},b)\cdot\frac{e^{-rs}}{\theta}K\cdot 1_{N_2}(y)
\end{aligned}
$$

According to Lemma 4.4 we have an implicit representation for y in the form of

$$
v = \frac{e^{-rs}}{\theta}\left\{1_{N_3}(y)\cdot\psi(s,\tilde{r},b)\log\left(\frac{\theta\psi(s,\tilde{\mu},b)}{y\psi(s,\tilde{r},b)}\right) - \int_0^s 1_{N_2}(y)\cdot\psi'(t,\tilde{r},b)\log\left(\frac{\theta e^{-Kt}}{y}\left(\frac{l}{x}\right)^{-L}\right)dt\right\}.
$$

Unfortunately, the dependence of the right-hand side on y is complicated hence there is no simple way to extract y in order to write down a closed form expression. Lemma 4.2 guarantees that a unique $y > 0$ satisfying the above identity exists. However, the optimal portfolio process π cannot be linear in the initial wealth v.

process π and to wealth v at time t_i is determined by the equation

$$e^{-ru}V_u^{\pi}(t_i, v) = e^{-rt_i}v + \int_{(t_i,u]} \pi_t dp_t^* \ .$$

Let $\mathcal{A}^{(i)}(v, x)$ be the class of investment processes such that $V_u^{\pi}(t_i, v) \geq 0$, a.s., for all $u \in [t_i, s]$ and $\mathbb{E}^{t_i,x}\left[(\log V_s^{\pi}(t_i, v))^-\right] < \infty$. The value function at time t_i becomes

$$W_0^{(i)}(v, x) = \sup\left\{\mathbb{E}^{t_i,x}\left[\log V_s^{\pi}(t_i, v)\right];\ \pi \in \mathcal{A}^{(i)}(v, x)\right\}$$

Suppose now that the present time is "just before" t_i for some $t_i \leq s$ (in the case of $s = T$ we consider only $t_i < s$), so our information is represented by $\mathcal{G}(t_i-)$. In this case the information concerning the asset value of the corporation can be summarized as $X(t_{i-1}) = x$, $\tau > t_i$. To reduce the burden of notation, let $P^{t_i-,x}$ be the conditional probability measure given this information, $\mathbb{E}^{t_i-,x}$ be the corresponding expectation, and suppose that $V(t_i-) = v$. It is important to note that x in this case corresponds to the asset value at t_{i-1} due to the discrete nature of corporate news. The wealth $V_u^{\pi}(t_i-, v)$ at time u corresponding to an investment process π and to wealth v at time t_i- is determined by

$$e^{-ru}V_u^{\pi}(t_i-, v) = e^{-rt_i}v + \int_{[t_i,u]} \pi_t dp_t^* \ .$$

Notice that in the above calculation the discounted gain, represented by the integral on the right-hand side, is calculated on the closed interval $[t_i, u]$, so gains or losses derived from the jump of the default bond at time t_i is included in the wealth at time u.

Let $\mathcal{A}^{(i-)}(v, x)$ be the class of investment processes such that $V_u^{\pi}(t_i-, v) \geq 0$, a.s., for all $u \in [t_i, s]$ and $\mathbb{E}^{t_i-,x}\left[(\log V_s^{\pi}(t_i-, v))^-\right] < \infty$. The value function at time t_i- now becomes

$$W_0^{(i-)}(v, x) = \sup\left\{\mathbb{E}^{t_i-,x}\left[\log V_s^{\pi}(t_i-, v)\right];\ \pi \in \mathcal{A}^{(i-)}(v, x)\right\}.$$

Let j be the index for which $t_j < s \leq t_{j+1}$.

Theorem 6.1. *For all $0 \leq i \leq j$ there exist functions $c_i(x)$ and $\tilde{c}_i(x)$ such that*

$$W_0^{(i)}(v, x) = \log v + c_i(x) \tag{6.1}$$

$$W_0^{(i-)}(v, x) = \log v + \tilde{c}_i(x) \tag{6.2}$$

where the functions $c_i(x)$ *and* $\tilde{c}_i(x)$ *do not depend on* v*, but both may depend on* $s - t_i$*,* $t_{i+1} - t_i$ *,* $t_{i+2} - t_i$*,* $\ldots$*,* $t_j - t_i$*,* $T - t_i$*, and* $\tilde{c}(x)$ *may in addition depend on* $t_i - t_{i-1}$ *as well.* $W_0^{(i-)}(v, x)$ *has the form given in (6.2) even if* $s = t_{j+1} < T$ *and* $i = j + 1$*.*

Proof. In this proof we shall assume that $s \neq t_{j+1}$ (the complementary case can be covered with a minimal modification of this proof and will be omitted). We are going to proceed by backward induction. In step 1 we show that (6.1) holds for $i = j$, in step 2 we show that (6.2) holds assuming that (6.1) holds, and finally in step 3 we shall show that (6.1) holds with i replaced by $i - 1$ provided that (6.2) holds.

Step 1 follows from (5.7) since

$$W_0^{(j)}(v, x) = W_0(v, x, s - t_j),$$

which by (5.7) indeed gives the form required by (6.1).

We continue with step 2, i.e., assume (6.1) and show (6.2). By the dynamic programming principle

$$W_0^{(i-)}(v, x) = \sup \left\{ \mathbb{E}^{t_i-,x} \left[W_0^{(i)} \left(V_{t_i}^{\pi} \left(t_{i-}, v \right), X \left(t_i \right) \right) \right] \right\}$$

where the supremum is constrained to investment processes such that $V_{t_i}^{\pi}(t_i-, v) \geq 0$. Notice that this is a non-stochastic one-dimensional maximization problem, since we are maximizing with respect to $\pi(t_i)$ which, by the predictability of π can not depend on $X(t_i)$, so it is non-random under $P^{t_i-,x}$. By our assumption (6.1) the above identity can be written as

$$W_0^{(i-)}(v, x) = \sup \left\{ \mathbb{E}^{t_i-,x} \left\{ \log V_{t_i}^{\pi} \left(t_i-, v \right) \right] \right\} + \mathbb{E}^{t_i-,x} \left[c_i \left(X \left(t_i \right) \right) \right].$$

Let $\pi(t_i) = \lambda$. By (2.11) the above can be written as

$$W_0^{(i-)}(v, x) = \sup_{\lambda} \left\{ \mathbb{E}^{t_i-,x} \left[\log \left\{ v + \lambda e^{-r(T-t_i)} \left[\gamma \left(0, T - t_i, X \left(t_i \right) \right) - \gamma \left(t_i - t_{i-1}, T - t_{i-1}, x \right) \right] \right\} \right] \right\}$$
$$+ \mathbb{E}^{t_i-,x} \left[c_i \left(X \left(t_i \right) \right) \right]$$

with the constraint that

$$v + \lambda e^{-r(T-t_i)} \left[\gamma \left(0, T - t_i, X \left(t_i \right) \right) - \gamma \left(t_i - t_{i-1}, T - t_{i-1}, x \right) \right] \geq 0, \quad P^{t-,x} - \text{a.s.}$$

By $\tau > t_i$ the support of $X(t_i)$ is (l, ∞), so by Lemma A.2 in the Appendix our constrains can be written as

$$v + \lambda e^{-r(T-t_i)} \left[1 - \gamma \left(t_i - t_{i-1}, T - t_{i-1}, x \right) \right] \geq 0 \quad (6.3)$$

$$v + \lambda e^{-r(T-t_i)} \left[F - \gamma \left(t_i - t_{i-1}, T - t_{i-1}, x \right) \right] \geq 0 \quad (6.4)$$

(we must consider both constraints because λ may be negative when the investor shorts the bond). These become

$$\lambda \geq \frac{-v}{1-\gamma\left(t_i - t_{i-1}, T - t_{i-1}, x\right)} e^{r(T-t_i)} \tag{6.5}$$

$$\lambda \leq \frac{v}{\gamma\left(t_i - t_{i-1}, T - t_{i-1}, x\right) - F} e^{r(T-t_i)} \tag{6.6}$$

We conclude that $\pi(t_i) = \lambda$ is the solution of the (deterministic) maximization problem

$$\sup_{\lambda} \left\{ \mathbb{E}^{t_i-,x} \left[\log\left\{ v + \lambda e^{-r(T-t_i)} \left[\gamma\left(0, T-t_i, X\left(t_i\right)\right) - \gamma\left(t_i - t_{i-1}, T - t_{i-1}, x\right) \right] \right\} \right] \right\}, \tag{6.7}$$

subject to constraints (6.5) and (6.6). The expression that we are maximizing is a strictly concave function of λ, hence it achieves its maximum over the compact interval determined by the constraints at a unique point. Due to the linearity of the constraints with respect to v and to the algebraic properties of the logarithm function the point where the maximum is achieved is a linear function of v. Let this maximum point be $v\hat{\lambda}(x) = v\hat{\lambda}(x, t_i - t_{i-1}, \ldots, T - t_{i-1})$. It follows that

$$W_0^{(i-)}(v,x) = \log v + \tag{6.8}$$

$$\mathbb{E}^{t_i-,x} \left[\log\left\{ 1 + \hat{\lambda}(x) e^{-r(T-t_i)} \left[\gamma\left(0, T-t_i, X\left(t_i\right)\right) - \gamma\left(t_i - t_{i-1}, T - t_{i-1}, x\right) \right] \right\} \right] + \mathbb{E}^{t_i-,x} \left[c_i\left(X\left(t_i\right)\right) \right]$$

which is indeed the form required by (6.2).

In step 3 we are showing that $W_0^{(i-1)}(v,x)$ satisfies (6.1) with i replaced by $i-1$ provided that (6.2) holds. By the dynamic programming principle we have

$$W_0^{(i-1)}(v,x) = \sup \left\{ E^{t_{i-1},x} \left[W_0^{(i-)}\left(V_{t_i-}^{\pi}\left(t_{i-1}, v\right), x \right) \right] \right\}$$

where the supremum is taken for admissible π's. However, by (6.2) and (5.7) this can be cast into

$$\begin{aligned} W_0^{(i-1)}(v,x) &= \sup \left\{ E^{t_{i-1},x} \log V_{t_i-}^{\pi}(t_{i-1}, v) \right\} + \tilde{c}_i(x) \\ &= \log v + D_0(x, t_i - t_{i-1}) + \tilde{c}_i(x) \end{aligned}$$

which is indeed the form required by (6.1). □

For the purposes of writing down the optimal investment process recall the function $h_0(t,T,v,x)$ from (5.4) and (5.5). Also recall that the function $\hat{\lambda}(x)$ is defined so that (6.7) achieves its unique maximum at $\lambda = v\hat{\lambda}(x)$, subject to constraints (6.5) and (6.6). In other words, $\hat{\lambda}(x)$ is the point where (6.7) achieves the supremum under constrains (6.5) and (6.6) with $v = 1$.

THEOREM 6.2. *At a time $t \leq s$ such that $t \in (t_i, t_{i+1})$ for some $i \leq j$ the optimal investment is*

$$\hat{\pi}_t 1_{\{\tau>t\}} = h_0(t - t_i, T - t_i, V(t_i), X(t_i)) 1_{\{\tau>t\}}.$$

This formula is correct for $t = 0$ as well with the selection of $t_i = 0$. Additionally, this gives the value of the optimal investment process at time $t = T$ if $s = T$ in which case $i = n - 1$. For all $i \leq j$ the optimal investment process at time t_i is

$$\hat{\pi}(t_i) 1_{\{\tau>t_i\}} = V(t_i-) \hat{\lambda} \left(X \left(t_{i-1} \right) \right) 1_{\{\tau>t_i\}}. \tag{6.9}$$

This is correct for $i = j + 1$ as well if $s = t_{j+1} < T$.

Proof. This follows from the theorem 6.1 and its proof. In the case of $s = T$ we have to consider the fact that the bond price is left-continuous at the maturity time T. □

REMARK 6.3. *An interesting feature of the logarithmic utility function is that the optimal investment process does not depend on the terminal time s.*

For the actual computation of $\hat{\lambda}(x)$ we need the conditional distribution of $X(t_i)$ given that $X(t_{i-1}) = x$ and $\tau > t_i$. This is given in Lemma A.3 in the Appendix.

COROLLARY 6.4. *Assume that the utility function is the logarithmic utility. If $\mu = r$ then for each $i \leq j$ (as well as for $i = j + 1$ if $s = t_{j+1} < T$) the optimal investment value $\hat{\pi}(t_i) 1_{\{\tau>t_i\}}$ is zero. In other words, there will be zero investment in the $\mu = r$ case at the time of the corporate news.*

The proof is deferred to the Appendix.

B. Power Utility

In this subsection, the utility function is assumed to be the power function with parameter $\delta < 1, \delta \neq 0$ given in (5.1). Since the results are essentially analogous to those in section A, we shall only state the main results and outline the proofs without providing details.

Define the value function for the power utility similarly to section A, i.e.,

$$W_\delta^{(i)}(v, x) = \sup \left\{ \mathbb{E}^{t_i, x} \left[\frac{1}{\delta} \left(V_s^\pi(t_i, v) \right)^\delta \right] ; \ \pi \in \mathcal{A}^{(i)}(v, x) \right\},$$

and

$$W_\delta^{(i-)}(v,x) = \sup\left\{\mathbb{E}^{t_i-,x}\left[\frac{1}{\delta}\left(V_s^\pi(t_i-,v)\right)^\delta\right];\ \pi\in\mathcal{A}^{(i-)}(v,x)\right\}.$$

Recall that j is the index for which $s\in(t_j,t_{j+1}]$.

Theorem 6.4. *For all $i\le j$ there exist functions $c_{\delta,i}(x)$ and $\tilde{c}_{\delta,i}(x)$ such that*

$$W_\delta^{(i)}(v,x) = \frac{1}{\delta}v^\delta\cdot c_{\delta,i}(x) \tag{6.10}$$

$$W_\delta^{(i-)}(v,x) = \frac{1}{\delta}v^\delta\cdot \tilde{c}_{\delta,i}(x) \tag{6.11}$$

$W_\delta^{(i-)}(v,x)$ has the form given in (6.11) even if $s=t_{j+1}<T$ and $i=j+1$.

Proof Outline. Step 1. We assume that $s\ne t_{j+1}$. From (5.6) follows that

$$W_\delta^{(j)}(v,x) = W_\delta(v,x,s-t_j) = D_\delta(x,s-t_j)\cdot\frac{1}{\delta}v^\delta.$$

Therefore,

$$c_{\delta,j}(x) = D_\delta(x,s-t_j). \tag{6.12}$$

Step 2: Assume that (6.10) holds. By the dynamic programming principle

$$W_\delta^{(i-)}(v,x) = \sup\left\{\mathbb{E}^{t_i-,x}\left[W_\delta^{(i)}\left(V_{t_i}^\pi\left(t_{i-},v\right),X\left(t_i\right)\right)\right]\right\}$$

(6.13)

$$= \sup\left\{\mathbb{E}^{t_i-,x}\left[\frac{1}{\delta}\Big(v+\lambda e^{-r(T-t_i)}\left[\gamma(0,T-t_i,X(t_i))-\gamma(t_i-t_{i-1},T-t_{i-1},x)\right]\Big)^\delta c_{\delta,i}(X(t_i))\right]\right\}$$

where $\lambda=\pi(t_i)$. Here the supremum is taken over all λ's satisfying the admissibility constrains of (6.5) and (6.6).

The expression after the sup on the right-hand side of (6.13) is a strictly concave (deterministic) function of λ, therefore the supremum over a compact interval is achieved at a unique point. In addition, the point where the supremum is achieved is a linear function of v. Denote by $\hat{\lambda}(x)$ the function such that the supremum is achieved at $\hat{\lambda}(x)v$. Now we have

$$W_\delta^{(i-)}(v,x)$$

$$= \frac{1}{\delta}v^\delta\mathbb{E}^{t_i-,x}\left[\Big(1+\hat{\lambda}(x)e^{-r(T-t_i)}\left[\gamma(0,T-t_i,X(t_i))-\gamma(t_i-t_{i-1},T-t_{i-1},x)\right]\Big)^\delta c_{\delta,i}(X(t_i))\right]$$

which is indeed of the form required by (6.11), with

(6.14)

$$\tilde{c}_{\delta,i}(x) = \mathbb{E}^{t_i-,x}\left[\left(1+\hat{\lambda}(x)e^{-r(T-t_i)}\left[\gamma(0,T-t_i,X(t_i)) - \gamma(t_i-t_{i-1},T-t_{i-1},x)\right]\right)^{\delta} c_{\delta,i}(X(t_i))\right].$$

Step 3: Assume that (6.11) holds. Using again the dynamic programming principle we can show that

$$W_{\delta}^{(i-1)}(v,x) = \frac{1}{\delta}v^{\delta}\cdot D_{\delta}(x,t_i-t_{i-1})\cdot\tilde{c}_{\delta,i}(x)$$

where $D_\delta(\cdot,\cdot)$ is given in (5.8). It is clear that

$$c_{\delta,i-1}(x) = D_{\delta}(x,t_i-t_{i-1})\cdot\tilde{c}_{\delta,i}(x). \tag{6.15}$$

□

In order to compute the optimal investment process, we have to calculate the functions $\tilde{c}_{\delta,i}$ for all $i=1,2,\ldots,n-1$. This makes the power utility much more computation extensive than the logarithmic utility. The computation can be done by backward induction using (6.12), (6.14), and (6.15).

The optimal investment process for $t\in(t_i,t_{i+1})$ such that $t\le s$ is given by

$$\hat{\pi}_t 1_{\{\tau>t\}} = h_\delta(t-t_i, s-t_i, T-t_i, V(t_i), X(t_i))1_{\{\tau>t\}},$$

where the h_δ function is determined by (5.2) and (5.3). This formula is correct for $t=0$ and $t=T$ as well (the latter is relevant only if $s=T$). In the case of $t=0$ we select $t_i=0$, and in the case of $t=T$ we select $i=n-1$ (for the $t=T$ case keep in mind that the bond price is left-continuous at T). For all $1\le i\le j$ the optimal investment process at time t_i is given by (6.9),where now $\hat{\lambda}(x)$ is the unique point where the supremum on the right-hand side of (6.13) is achieved, under the constraints (6.5) and (6.6), and with $v=1$. This is correct for $i=j+1$ as well if $s=t_{j+1}<T$. In order to compute $\hat{\lambda}(x)$ we need the conditional distribution of $X(t_i)$ given that $X(t_{i-1})=x$ and $\tau>t_i$, which is calculated in Lemma A.3 of the Appendix.

7 Numerical analysis for the logarithmic utility function

The logarithmic utility function is an important case worth further discussion, because maximizing the expected logarithm of terminal wealth of an investor amounts to maximizing the

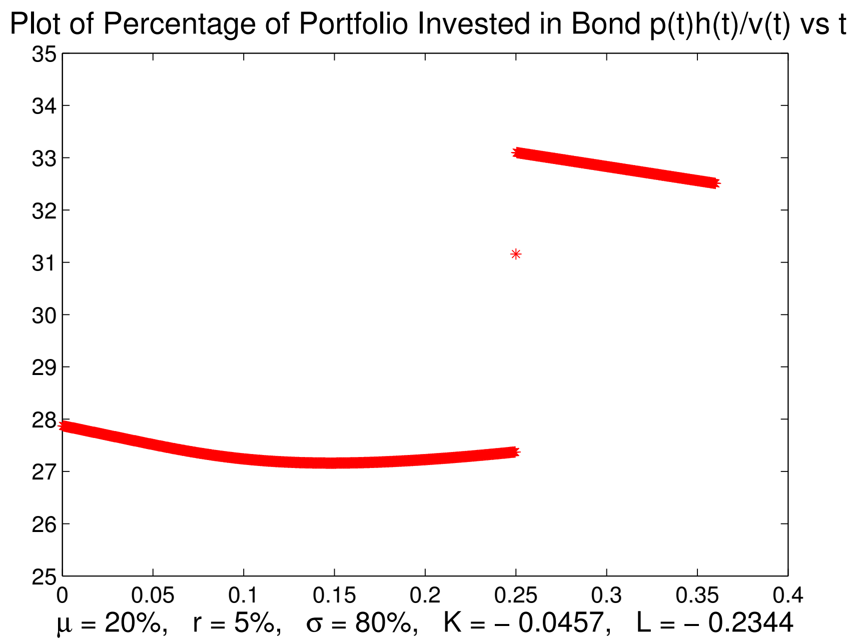

Figure 1: Case 1: $\mu > r$ and $\sigma^2 > \mu + r$, $l/x = 0.5$, K and L are close to 0.

expected return rate of her or his portfolio. In addition, the optimal investment process, given in (5.4) and (5.5) has a particularly simple form; it is linear in the initial wealth v, and does not depend on the terminal time s.

In Figure 1 through Figure 4 we show the percentage of the portfolio invested in the bond (i.e., $100 \times \pi(t)p(t)/v(t)$) as a function of time, presuming that default does not occur by the time .36. With the notation of (4.7) these plots show the graph of the function $t \mapsto 100 \times h_0(t)p(t)/v(t)$. For brevity we shall refer to the *p*ercent of the portfolio invested *i*n the *b*ond as PIB. We illustrated the following four cases: case 1, $\mu > r$ and $\sigma^2 > \mu + r$; case 2, $\mu > r$ and $\sigma^2 < \mu + r$; case 3, $\mu < r$ and $\sigma^2 > \mu + r$; case 4, $\mu < r$ and $\sigma^2 < \mu + r$. We selected these four cases because in the first case $h_0(0) > 0$ and $K < 0$, in the second case $h_0(0) > 0$ and $K > 0$, in the third case $h_0(0) < 0$ and $K > 0$, and in the fourth case $h_0(0) < 0$ and $K < 0$. We note that $h_0(\cdot)$ is increasing if $K > 0$ and decreasing if $K < 0$. For the purpose of illustration, we set $v = \$10,000$, $T = 5$, $F = 0.4$, $x = \$1,000,000$, $l/x = 0.5$. Various values of μ, r and σ are listed below the corresponding graphs. We plotted the graph of $100 \times h_0(\cdot)p(t)/v(t)$ on the time horizon $[0, .36]$. We assume quarterly announcements of the firm's asset values, so during the time period $[0, .36]$ there is one announcement at time 0.25. In these plots, a negative PIB value means a short position in the bond.

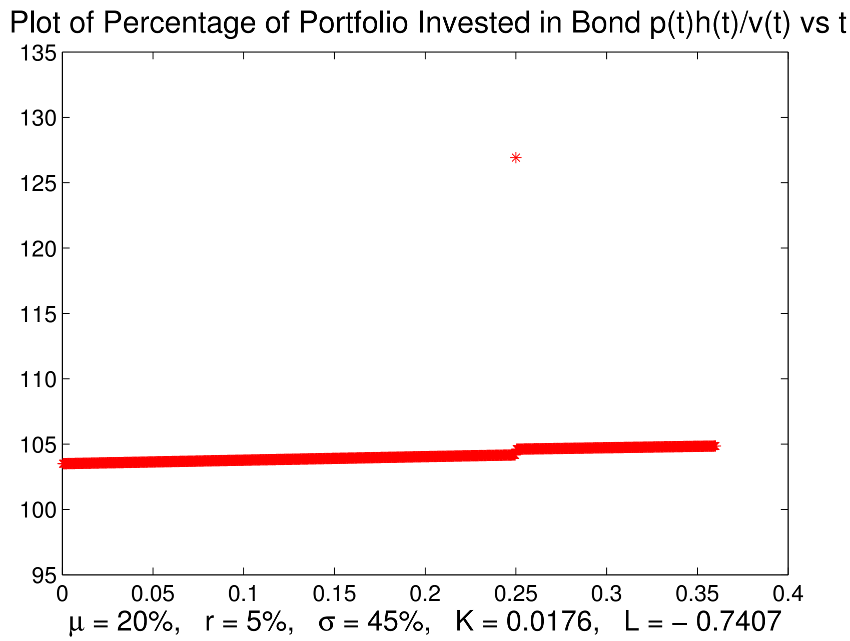

Figure 2: Case 2: $\mu > r$ and $\sigma^2 < \mu + r$, $l/x = 0.5$, K and L are close to 0.

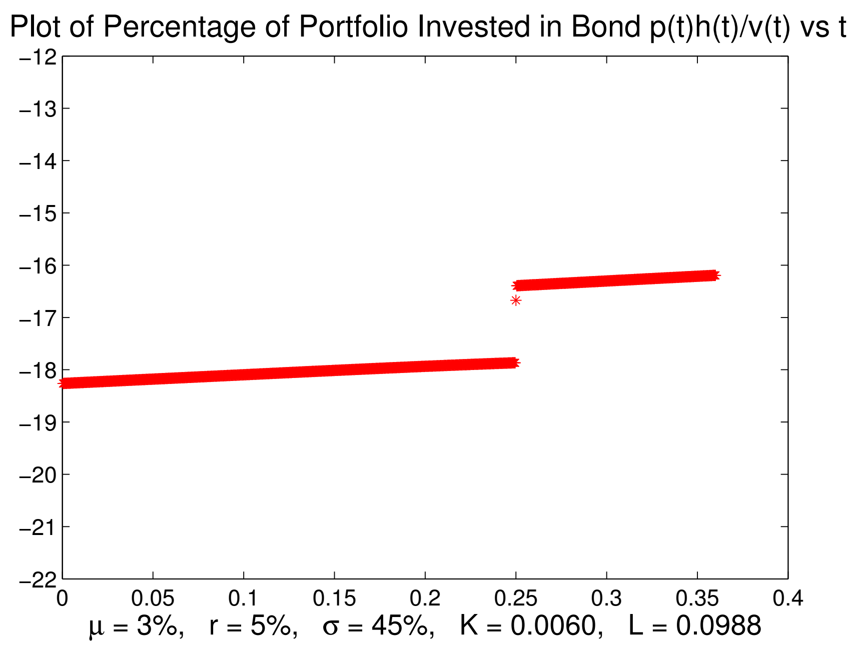

Figure 3: Case 3: $\mu < r$ and $\sigma^2 > \mu + r$, $l/x = 0.5$, K and L are close to 0.

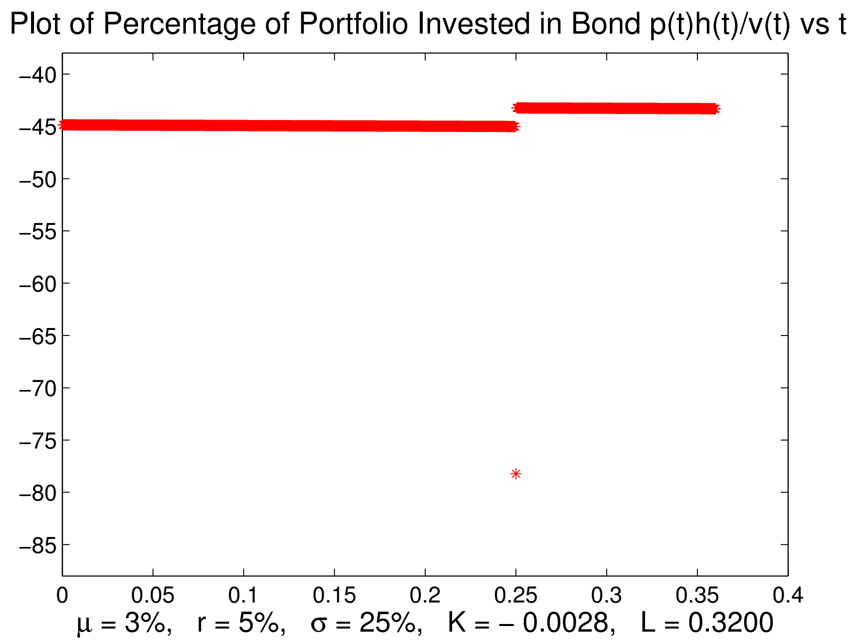

Figure 4: Case 4: $\mu < r$ and $\sigma^2 < \mu + r$, $l/x = 0.5$, K and L are close to 0.

From (2.11) follows that the return rate of the bond before default is always higher than r (because the discounted bond price is increasing w.r.t. time t). There are two cases, the investor is holding either long or short positions in the bond. In the former case he/she is earning a higher than r interest rate in the hope that default will not happen. In the second case he/she is losing money up to default, but hoping that default will happen which will then bring in a rather large profit. Thus intuitively the investor should hold a long position in the bond if the default probability is "small", and a short position if the default probability is "large". This intuition is nicely supported by the fact that $h_0(0)$ and also the PIB value is positive, negative, or zero according to $\mu > r$, $\mu < r$, or $\mu = r$ (see (5.4) and the definition of L in Lemma 4.3). Clearly the probability of default happening by time s, that is, $P(\tau \leq s)$ is a decreasing function of μ (keeping all other parameters the same).

Looking at Figures 1 - 4 it is apparent that the investor has different behaviors *between* the quarterly financial announcements and exactly *at* (or one may say infinitesimally before) the announcements. Following is the explanation of this phenomenon. The investment position in the bond between the announcements at time s is basically determined by the likelihood of default by the time of the next announcement and the size of the jump of the bond price at the time of a possible default. The conditional probability of not having default by the time of the next announcement given that there was no default up to now is an increasing function of the present time, i.e., $P[\tau > t_1 \mid \tau > s]$ is increasing on $s \in [0, t_1]$. On the other hand, the absolute value of the discounted jump size is an increasing function of τ (see (2.11)). These two factors compete in determining the PIB value at a time between two announcements. Since the conditional default probability is decreasing, one would be tempted to increase the PIB value. On the other hand, if there is a default then the absolute value of the jump size is getting larger with time, which would motivate the investor to decrease the PIB. However, at a time infinitesimally prior to the time of an announcement the default has no influence on the position of the investor, since there is zero probability that default would happen exactly at the time of the announcement of the asset value. At the time of the announcement the investor knows that there will be a jump in the bond price, makes the best statistical estimate of that jump and changes her/his position so that she or he gains as much utility from the jump as possible, while keeping in mind the admissibility constraints (6.5) and (6.6).

In examples 1-4 the PIB value increases or decreases extremely slowly between two consecutive asset value announcements. It can be seen in Table 1 that in case 2 the change of the PIB value is 0.67% in a quarter of a year (increases from 103.5% to 104.17%), and in the other three cases the change is even tinier. From a practical standpoint one can say that the PIB value is constant between announcements in cases 1-4. The PIB value can change more considerably (see Figure 7) if the debt ratio l/x is very large (in the cases shown in Figure 7 we have $l/x = 0.95$).

We interpret case 1 and 3 here (Figure 1 and Figure 3 respectively) more in detail. In case 1, the investor decides to buy the bond at time zero, because the expected return rate of the asset value is greater than the interest rate. The investor is continually making profit on the time interval $[0, .25)$ because the bond increases at a higher rate than r, if default does not happen by the time of the next announcement at $t = .25$. (By making profit we mean that the discounted value of the portfolio is increasing.) The portfolio increases from 10,000 at time zero to 10,305 at $t = .25$. If all the initial capital had been invested in the bank only, the value at $r = 5\%$ would be 10,126 only. There would be a sizable loss if default happened. However, the probability of default before time $t = .25$ is only about .09 (see Figure 5). At time $t = 0.25-$, the investor increases her or his position significantly, the PIB value grows from 27.37% to 31.16% (see Figure 1 or Table 1). Strictly on the intuitive level one could explain this with the fact that since μ is large, we expect the soon to be announced asset value to be rather large. Then it follows that the bond price is more likely to have a positive than a negative jump. In order to profit from this jump the investor takes more long positions in the bond. We hypothetically set the value of the Brownian motion driving the firm asset value one standard deviation above its mean at $t = 0.25$, i.e., $w_{0.25} = 0.5$. Then the firm asset value arrives at $x = \$1,248,806$, much higher than that at time zero. As shown in Table 1, the bond price increases from \$0.182 at $t = 0.25-$ to \$0.2428 at $t = 0.25$, and the investor gains from the jump; the portfolio value increases from \$10,305 to \$11,177. After the jump the PIB value goes up to 33.1%.

In case 3 the investor shorts the bond at time zero since the mean return rate of the asset value is below the interest rate ($\mu < r$). Since the bond grows at a rate higher than r, shorting the bond amounts to a (small) loss, in case there is no default. However, if there is a default then the investor makes a profit. In our case the value of the portfolio at $t = 0.25-$ is \$10,124 provided that default did not happen, slightly below the \$10,126 figure that would be the value if all the capital had been invested in the bank only. Should a default happen in case 3 say at time $t = 0.20$, the bond price would drop and the portfolio value would jump from $\$10,100$ to $\$11,900$, earning $\$1,800$. The probability of default by time $t = 0.25$ is very small, about 0.0025 (see Figure 6). However, the cost of shorting the bond in the case of no default is very tiny as seen above. At time $t = 0.25-$ the PIB value changes from -17.87% to -16.67%. This can be explained by the fact that μ is small, so the soon to be announced asset value is expected to be rather small, hence the investor expects a downwards jump in the bond price. In our illustration we suppose that the firm asset value drops only slightly at $t = 0.25$. As shown in Table 1, this causes the bond price to drop a bit, from \$.5159 to \$.4831. The investor's portfolio value grows from $\$10,124$ to $\$10,240$. After the announcement the PIB level goes up to -16.39%.

Table 1. Bond Prices, Portfolio Values and Percentage of Portfolio in Bond (PIB) values.

	$v(0.25-)$	$p(0.25-)$	$v(0.25)$	$p(0.25)$	PIB at 0	PIB at 0.25−	PIB at 0.25	PIB at 0.25+
case 1	\$10305	0.1820	\$11177	0.2428	27.87%	27.37%	31.16%	33.10%
case 2	\$10136	0.5159	\$11263	0.5601	103.50%	104.17%	126.92%	104.61%
case 3	\$10124	0.5159	\$10240	0.4831	−18.26%	−17.87%	−16.67%	−16.39%
case 4	\$10126	0.7067	\$10156	0.7040	−44.86%	−45.01%	−78.23%	−43.24%

The next question we consider here is whether an investor would switch from holding long positions to holding short, or the other way around between announcements. We shall show that such switches are rare due to practical constraints. It can be seen from (5.4) and (5.5) that such switch occurs only if $\sigma^2 > \mu + r$. We first consider the case of $\mu > r$, which implies

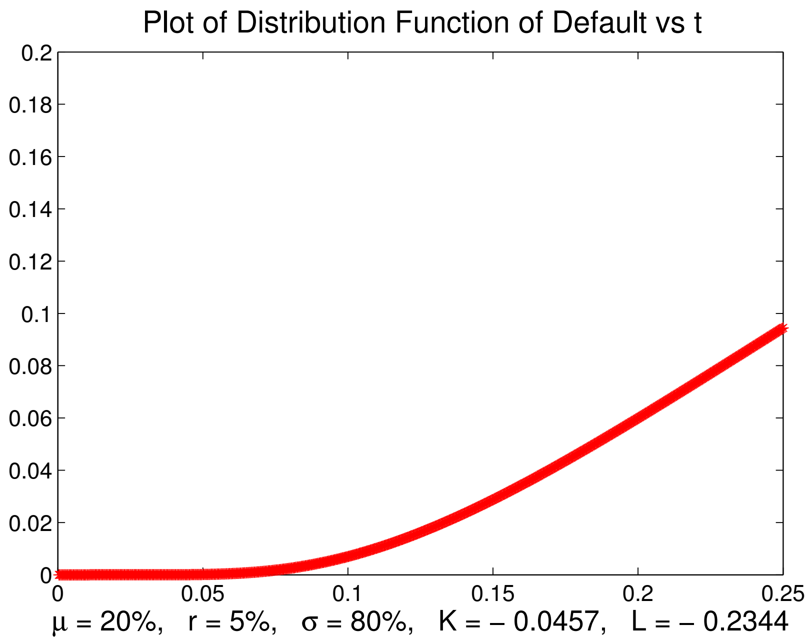

Figure 5: Case 1: $\mu > r$ and $\sigma^2 > \mu + r$, $l/x = 0.5$, K and L are close to 0.

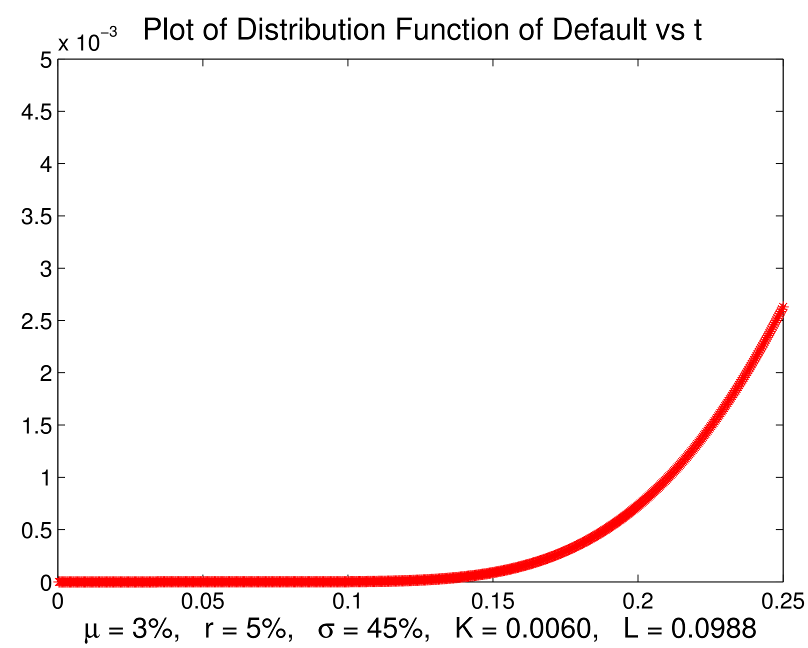

Figure 6: Case 3: $\mu < r$ and $\sigma^2 > \mu + r$, $l/x = 0.5$, K and L are close to 0.

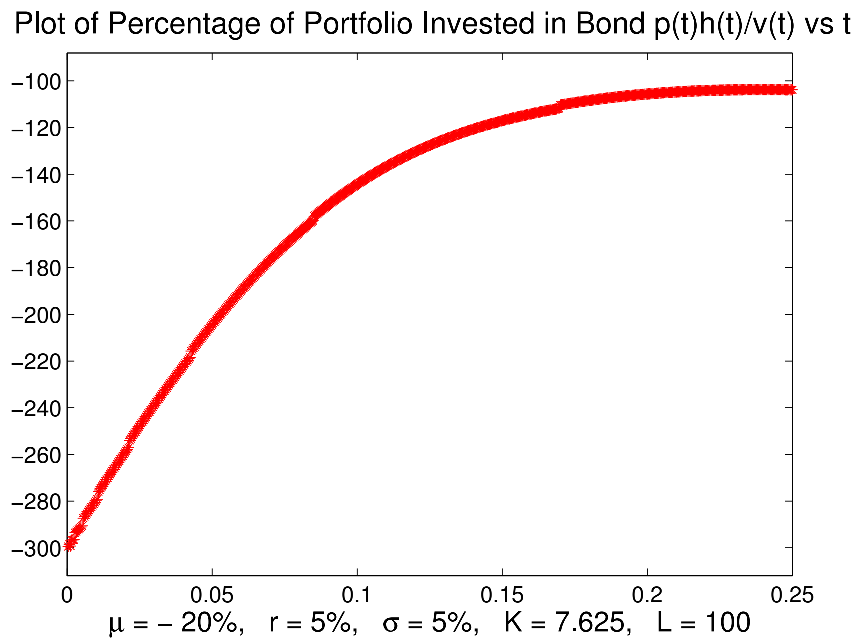

Figure 7: $\mu < r$ and $l/x = 0.95$, K and L are far from 0.

the investor starts with a long position ($h_0(0) > 0$) and reduces her or his position over time ($K < 0$ and $h_0'(t) < 0$). Clearly, the necessary condition of observing $h_0(t) < 0$ for some $t \leq .25$ is

$$1 - \left(\frac{l}{x}\right)^{-L} + K\left(\frac{l}{x}\right)^{-L}\int_0^t e^{-Ku}\psi(u, \tilde{r}, b)du < 0.$$

Since $0 \leq \psi(t, \tilde{r}, b) \leq 1$ and $t \leq .25$, the above condition implies

$$1 - \left(\frac{l}{x}\right)^{-L} e^{-.25\times K} < 0,$$

which gives

$$\frac{l}{x} > \exp\left\{-\frac{1}{8}[\sigma^2 - (\mu + r)]\right\}.$$

In the case of $\mu < r$, the exactly same necessary condition is concluded.

The right-hand side of the above expression decreases in σ, so the higher the asset volatility is, the more likely we observe a position switch. However, even for a very large volatility the debt ratio l/x must be very large in order to observe a position switch within a quarter of a year. For example, suppose that $\mu + r \geq 0$ and the volatility is as large as $\sigma = .5$. Our necessary condition for a position switch becomes $l/x > \exp\{-\frac{1}{8} \times .5^2\} = .97$. However, in this case the bond (presumably having a very low rating at this point) is a nuance away from default, and very likely will default before such position switch would take place.

Next we provide a simple intuitive derivation of (4.8). Let us suppose that default happens at a time very close to time zero, say before time ϵ where ϵ is a small positive number. Since we are only getting an intuitive insight into formula (4.8), we shall not make this notion more precise. Simple accounting shows that if $\epsilon \to 0$ then the wealth at time s will converge to $(v - h_0 p_0)e^{rs} + Fh_0 e^{-r(T-s)}$. However, from (4.3) follows that if $\tau \to 0$ then the optimal terminal wealth converges to $\beta(0)$. Hence we must have

$$\beta(0) = (v - h_0 p_0)e^{rs} + Fh_0 e^{-r(T-s)}$$

which gives (4.8) after rearrangement of the terms.

8 Conclusion

The major contribution of this paper is deriving the optimal investment in a defaultable zero coupon bond assuming a discrete information flow concerning the firm's asset value at specified time points. The methodology we present here can be applied to general utility functions. For the logarithmic and power utility functions we presented closed-form expressions for the investment strategy, which can be easily implemented. As an important example, we numerically analyzed the strategy for an investor who wants to maximize her or his expected return rate. We observed that under various reasonable parameter settings the investor changes her or his bond position very slowly between successive news releases, and most of the changes in the optimal portfolio happen "just after" each news release. Most of the gains (and possible losses) are realized at the time of the news releases and at the time of default in case it happens.

Appendix

We start this section with an example showing that our market model is not complete. This amounts to showing that there exist G_t-measurable random variables, i.e., contingent claims with expiration t that can not be replicated in this model. Let us select $t = t_1$ and let the payoff at time t_1 be $Y = 1_{\{\tau > t_1\}} f\left(X\left(t_1\right)\right)$ where the function $f(\cdot)$ will be determined below. Suppose that Y can be replicated by a trading strategy π of the form $\pi_u = \varphi(u) 1_{\{\tau \geq u\}} + \varphi(\tau) 1_{\{\tau < u\}}$, $u \leq t_1$ (similarly to (4.7)). By (3.1) and (4.11) we have

$$\exp\{-rt_1\} f\left(X(t_1)\right) = v + e^{-rT} \int_0^{t_1} \varphi(t) \gamma'\left(t, T, X_0\right) dt +$$
$$\varphi(t_1) e^{-rT} \left[\gamma\left(0, T - t_1, X\left(t_1\right)\right) - \gamma\left(t_1, T, X_0\right)\right].$$

This identity clearly imposes a special form on the function $f(\cdot)$, i.e., we must have $f(x) = c_1 + c_2 \gamma(0, T - t_1, x)$ for some constants c_1, c_2 for every $x > l$. Selecting any different $f(\cdot)$ will yield an unhedgable contingent claim.

Lemma A.1. *For all $\kappa < 0$, we have the following alternative formula for ψ:*

$$\psi(t, \lambda, \kappa) = \Phi\left(-\frac{\kappa}{\sqrt{t}} + \lambda\sqrt{t}\right) - e^{2\lambda\kappa} \Phi\left(\frac{\kappa}{\sqrt{t}} + \lambda\sqrt{t}\right) \tag{A.1}$$

where Φ is the standard normal distribution function.

Proof. The right-hand side converges to 1 as $t \to 0$, and it is straightforward to show that its derivative with respect to t agrees with that of the right-hand side of (2.7). ☐

Lemma A.2. *For all $u > 0$ we have $\sup_{z>l} \gamma(0, u, z) = 1$ and $\inf_{z>l} \gamma(0, u, z) = F$.*

Proof. From the definition of $\gamma(0, u, z)$ in (2.7) immediately follows that $F \leq \gamma(0, u, z) \leq 1$ and $\gamma(0, u, l) = F$. Next we show that $\lim_{z \to \infty} \gamma(0, u, z) = 1$. By (2.7) it suffices to show $\psi(t, \tilde{r}, b(z)) \to 1$, as $z \to \infty$. However,

$$\psi(t, \tilde{r}, b(z)) = Q[\tau > t \mid X_0 = z] = Q\big(\inf_{0 \leq u \leq t} \widetilde{W}_u + \tilde{r} u > b(z)\big)$$

which indeed converges to 1 since $\lim_{z \to \infty} b(z) = -\infty$. ☐

Lemma A.3. *For $i = 1, 2, \ldots, n-1$ the conditional distribution of $X(t_i)$ given that $X(t_{i-1}) = x$ and $\tau > t_i$ is given by*

$$P\left[X(t_i) > z \mid X(t_{i-1}) = x,\ \tau > t_i\right] = \begin{cases} \varphi(z, t_i - t_{i-1}, x), & \text{if } z > l; \\ 1, & \text{if } z \le l. \end{cases}$$

where

$$\varphi(z,t,x) = (\psi(t,\tilde{\mu},b))^{-1}\left[\Phi\left(\frac{1}{\sigma\sqrt{t}}\log\frac{x}{z} + \tilde{\mu}\sqrt{t}\right) - e^{2b\tilde{\mu}}\Phi\left(\frac{1}{\sigma\sqrt{t}}\log\frac{l^2}{xz} + \tilde{\mu}\sqrt{t}\right)\right].$$

Here $b = b(x)$ given in (2.6) and Φ is the standard normal distribution function.

Proof. The case of $z \le l$ is straightforward so we only consider the case of $z > l$. By the time homogeneity of X_t, we have

$$P\left[X(t_i) > z \mid X(t_{i-1}) = x,\ \tau > t_i\right] = P\left[X(t_i - t_{i-1}) > z \mid X_0 = x,\ \tau > t_i - t_{i-1}\right].$$

For brevity, we use $t = t_i - t_{i-1}$, so the above probability is equal to

$$\frac{P[X_t > z,\ \tau > t \mid X_0 = x]}{P[\tau > t \mid X_0 = x]}. \tag{A.2}$$

The denominator is $\psi(t, \tilde{\mu}, b(x))$ with $b(x)$ given in (2.6); we need to compute the numerator. Let $Y_t = W_t + \tilde{\mu}t$, then $\log X_t = \log x + \sigma Y_t$ and $\tau = \inf\{u \ge 0 : Y_u \le b\}$. Let Q_t be a probability measure, equivalent to P, given by

$$\frac{dQ_t}{dP} = \exp\left\{-\tilde{\mu}W_t - \frac{\tilde{\mu}^2}{2}t\right\} = \exp\left\{-\tilde{\mu}Y_t + \frac{\tilde{\mu}^2}{2}t\right\},$$

and E_t be the corresponding expectation. From Girsanov's theorem (Karatzas & Shreve, 1998) follows that $\{Y_u; u \le t\}$ is a Brownian Motion under Q_t. The numerator of (A.2) can be written as

$$P\left[Y_t > \frac{1}{\sigma}\log\frac{z}{x},\ \tau > t\right] = \mathbb{E}_t\left[1_{\left\{\tau > t,\ Y_t > \frac{1}{\sigma}\log\frac{z}{x}\right\}}\exp\left\{\tilde{\mu}Y_t - \frac{\tilde{\mu}^2}{2}t\right\}\right].$$

Let $b' = \frac{1}{\sigma}\log\frac{z}{x}$. We assumed $z > l$ so we have $b' > b$. The above expression is equal to

$$\mathbb{E}_t\left[1_{\{Y_t > b'\}}\exp\left\{\tilde{\mu}Y_t - \frac{\tilde{\mu}^2}{2}t\right\}\right] - \mathbb{E}_t\left[1_{\{\tau \le t,\ Y_t > b'\}}\exp\left\{\tilde{\mu}Y_t - \frac{\tilde{\mu}^2}{2}t\right\}\right]$$

By straightforward calculation the first expectation in this expression is

$$\Phi\left(\frac{1}{\sigma\sqrt{t}}\log\frac{x}{z} + \tilde{\mu}\sqrt{t}\right).$$

In order to evaluate the second expectation, we apply the reflection principle. We reflect Y

on the level $b < 0$ after τ, and denote the reflected process by $\widetilde{Y}$ which is also a Brownian motion under Q_t. Then $Y_t = 2b - \widetilde{Y}_t$ on the event $\{\tau > t\}$, hence the second expectation becomes

$$\mathbb{E}_t \left[1_{\{\tau \le t,\ \widetilde{Y}_t < 2b - b'\}} \exp\left\{ \tilde{\mu}(2b - \widetilde{Y}_t) - \frac{\tilde{\mu}^2}{2} t \right\} \right]$$

Since $2b - b' < b$, this is

$$\mathbb{E}_t \left[1_{\{\widetilde{Y}_t < 2b - b'\}} \exp\left\{ \tilde{\mu}(2b - \widetilde{Y}_t) - \frac{\tilde{\mu}^2}{2} t \right\} \right],$$

which can be easily seen to be equal to

$$e^{2b\tilde{\mu}} \Phi \left(\frac{1}{\sigma\sqrt{t}} \log \frac{l^2}{xz} + \tilde{\mu}\sqrt{t} \right).$$

□

Remark. *Expression (A.1) for ψ verifies that the conditional distribution function computed in the above lemma is continuous at $z = l$ as expected.*

Proof of corollary 6.4:

Differentiating the quantity after the supremum sign in (6.7) with respect to λ and then substituting $\lambda = 0$ gives

$$E^{t_i -, x} \left[\frac{1}{v} e^{-r(T - t_i)} \left\{ \gamma\left(0, T - t_i, X\left(t_i\right)\right) - \gamma\left(t_i - t_{i-1}, T - t_{i-1}, x\right) \right\} \right]$$

which can be written as

$$\frac{1}{v} E\left[p(t_i) - p(t_i -) \mid \tau > t_i,\ X(t_{i-1}) = x \right].$$

Notice that $\mu = r$ implies $P \equiv Q$ and that $\{p_t^*, \mathcal{G}_t;\ t \le T\}$ is a Q-martingale, hence also a P-martingale. But this implies that the above quantity is zero. Now from the concavity of the function we are maximizing follows that the maximum is achieved at $\lambda = 0$ (which obviously satisfies the constraints (6.5) and (6.6)).

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