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Optimal control of a mean-reverting inventory

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Motivated by empirical observations, we assume that the inventory level of a company follows a mean reverting process. The objective of the management is to keep this inventory level as close as possible to a given target; there is a running cost associated with the difference between the actual inventory level and the target.

If inventory deviates too much from the target, management may perform an intervention in the form of either a purchase or a sale of an amount of the goods. There are fixed and proportional costs associated with each intervention. The objective of this paper is to find the optimal inventory levels at which interventions should be performed as well as the magnitudes of the interventions, in order to minimize the total cost. We solve this problem by applying the theory of stochastic impulse control. Our analysis yields the optimal policy which at times exhibits a behavior that is not intuitive.

Subject classifications: inventory/production: uncertainty, stochastic; probability: stochastic model applications.
Area of review: Stochastic Models.

1. Introduction

We assume that the inventory level of a company follows a mean reverting process. The objective of management is to keep this inventory level as close as possible to a given target; there is a running cost associated with the difference between the inventory level and the target. Management is allowed to perform at times interventions in the form of major purchases or sales of the goods. These interventions are subject to fixed as well as proportional costs. The objective of this paper is to find the levels of the inventory at which management should perform interventions and the magnitudes of these interventions that minimize the total cost.

This type of inventory control is presumably applicable when dealing with commodity products, such as oil, coal, water, etc. The cost of having too much inventory above a preferred target level is due to actual inventory costs, i.e., the cost of money being tied up and the cost of the actual maintenance of the inventory. The cost of having too little inventory is due to the perceived likelihood of depleting the inventory which could possibly lead to a loss of sales as well as a loss of goodwill. Clearly,the perceived likelihood of such an event occurring increases with a lower level of inventory. The mean reversion of the process can be explained by the assumption that in the medium and in the long term the supply and demand for the goods from the outside will remain more or less stable. That is, if inventory recently has gone down because of a strong demand, one could expect the demand in the near future to be weaker, allowing the inventory to revert back towards its preferred target.

Another application of a mean reverting process is the inventory of shares in a particular company held by a specialist who is responsible for trading in that company's shares (i.e., a market-maker who assures that trading in these shares always remains possible). It has been observed that such an inventory process tends to be consistently mean reverting. (The mean reversion tends to be based on the fact that the market-maker, who trades continuously, adjusts the prices of those shares in such a way that the shares move in the desired direction). However, even in this environment it could happen that the inventory moves (because of exogenous market conditions) too far from its desired level, forcing the market-maker to take drastic action (involving trades of large blocks).

In what follows, we apply the theory of stochastic impulse control to solve the problem described above. Constantinides and Richard (1978), Harrison, Sellke and Taylor (1983), Ormeci, Dai and Vande Vate (2008) and Sulem (1986) also apply stochastic impulse control methods to study inventory problems in which the manager is allowed to increase and reduce the inventory level. However, they model the dynamics of the uncontrolled inventory as a Brownian motion with drift. Hence, they assume that the uncontrolled inventory is not mean-reverting.

The special case of our model with the speed of the mean reversion being equal to zero is somewhat similar to the problems considered by the above references. The dynamics of our uncontrolled inventory without mean reversion follows a Brownian motion without drift. Including the mean reversion force is in a sense a generalization of the models considered in the literature.

The structure of the paper is as follows: in Section 2 we introduce the inventory dynamics and management objective. In Section 3 we characterize the value function, and in Section 4 we obtain the solution. In Section 5 we study the time to change the inventory level. Section 6 is devoted to the numerical solution and comparative statics analysis. Additional remarks are presented in Section 7. We close the paper with some conclusions. In addition, one Appendix contains some mathematical proofs, and a second Appendix provides a comparison to the Brownian motion with drift model.

2. The Inventory Model

We use a Brownian motion to model the uncertainty in the inventory level. Formally, we consider a probability space $(\Omega, \mathcal{F}, P)$ together with a filtration $(\mathcal{F}_t)$ generated by a one-dimensional Brownian motion W. We denote

$$X_t := \text{inventory level at time } t.$$

We assume that X is an adapted stochastic process given by

$$X_t = x + \int_0^t k(\rho - X_s)ds + \int_0^t \sigma dW_s + \sum_{i=1}^{\infty} I_{\{\tau_i < t\}}\xi_i. \tag{1}$$

Here, $k > 0$ is the speed of mean-reversion, $\rho \in (-\infty, \infty)$ is the long-term mean of the process X, $\sigma > 0$ is the volatility, τ_i is the time of the i-th intervention, and ξ_i is the intensity of the i-th intervention. We observe that, in the particular case in which there are not interventions, X is simply an Ornstein-Uhlenbeck stochastic process.

Definition 2.1 (The controls). An impulse control is a pair

$$(T, \xi) = (\tau_0, \tau_1, \tau_2, \ldots, \tau_n, \ldots; \xi_0, \xi_1, \xi_2, \ldots, \xi_n, \ldots), \tag{2}$$

where $\tau_0 = 0 < \tau_1 < \tau_2 < \cdots < \tau_n < \cdots$ is a sequence of increasing stopping times, and (ξ_n) is a sequence of random variables such that each $\xi_n : \Omega \mapsto \mathbf{R}$ is $\mathcal{F}_{\tau_n}$-measurable. We assume $\xi_0 = 0$. The management (the controller) decides to act at time τ_i by adding ξ_i to the inventory level process at time τ_i. That is, $X_{\tau_i^+} = X_{\tau_i} + \xi_i$. We note that ξ_i and X can also take negative values.

PROBLEM 2.1. The management wants to select the pair (T, ξ) that minimizes the functional J defined by

$$J(x; T, \xi) := E\left[\int_0^\infty e^{-\lambda t} f(X_t)dt + \sum_{n=1}^{\infty} e^{-\lambda \tau_n} g(\xi_n) I_{\{\tau_n < \infty\}}\right], \tag{3}$$

where

$$f(x) = (x-\rho)^2, \tag{4}$$

$$g(\xi) = \begin{cases} C + c\xi & \text{if } \xi > 0 \\ \min(C, D) & \text{if } \xi = 0 \\ D - d\xi & \text{if } \xi < 0 \end{cases} \tag{5}$$

$$\lambda > 0, \tag{6}$$

$$C, c, D, d \in (0, \infty), \tag{7}$$

$$\rho \in (-\infty, \infty). \tag{8}$$

Here, f represents the running cost incurred by deviating from the aimed inventory level ρ, C represents the fixed cost per intervention when the management pushes the inventory level upwards, D represents the fixed cost per intervention when the management pushes the inventory level downwards, c represents the proportional cost per intervention when the management pushes the inventory level upwards, d represents the proportional cost per intervention when the inventory level pushes the management downwards, and λ is the discount rate. We could have considered more general strategies in which we would allow $\tau_i \le \tau_{i+1}$ instead of just $\tau_i < \tau_{i+1}$. However, because we have a fixed cost per intervention and the management can decide the exact amount of the interventions, it is clear that in our problem it cannot be optimal to intervene more than once at the same time.

In our model, there is an optimal inventory level ρ for the company. When there is not much volatility (modeled by values of W), the manager can implicitly conduct the inventory towards the preferred level ρ at a speed k without paying any costs (or the costs are negligible). However, when there is too much volatility, the inventory level can go far away from the target ρ, and the manager will have to pay fixed and proportional costs to conduct the inventory towards ρ. This interpretation of our model is consistent with some of the applications of inventory theory in financial economics. For instance, subsection 2.2 of Manaster and Mann (1996), chapter 11 of Hasbrouck (2007), and Madhavan and Smidt (1993) study inventories of stocks which are consistent with our model.

Since we want to minimize the functional J, we should consider only those strategies for which J is well defined and finite. In particular, we need that

$$E\left[\int_0^\infty e^{-\lambda t}(X_t - \rho)^2 dt\right] < \infty, \tag{9}$$

which implies

$$E\left[\int_0^\infty e^{-\lambda t}|X_t - \rho| dt\right] < \infty.$$

Indeed, condition (9) implies

$$\begin{aligned} E\left[\int_0^\infty e^{-\lambda t}|X_t - \rho| dt\right] &= E\left[\int_0^\infty e^{-\lambda t}|X_t - \rho| I_{\{|X_t - \rho| \le 1\}} dt\right] \\ &\quad + E\left[\int_0^\infty e^{-\lambda t}|X_t - \rho|| I_{\{|X_t - \rho| > 1\}} dt\right] \\ &\le E\left[\int_0^\infty e^{-\lambda t} I_{\{|X_t - \rho| \le 1\}} dt\right] \end{aligned}$$

$$+E\left[\int_0^\infty e^{-\lambda t}(X_t-\rho)^2 I_{\{|X_t-\rho|>1\}}dt\right]$$
$$<\infty.$$

In order that

$$E\left[\sum_{n=1}^\infty e^{-\lambda\tau_n}g(\xi_n)I_{\{\tau_n<\infty\}}\right]$$

be well defined and finite, we need that

$$E\left[\sum_{n=1}^\infty e^{-\lambda\tau_n}I_{\{\tau_n<\infty\}}\right]<\infty \text{ and } E\left[\sum_{n=1}^\infty e^{-\lambda\tau_n}|\xi_n|I_{\{\tau_n<\infty\}}\right]<\infty.$$

To obtain the inequality on the left-hand-side, we need that

$$\forall T\in[0,\infty):\quad P\left\{\lim_{n\to\infty}\tau_n\le T\right\}=0. \tag{10}$$

To obtain the inequality on the right-hand-side, we need that

$$\lim_{T\longrightarrow\infty}E\left[e^{-\lambda T}X(T+)\right]=0. \tag{11}$$

Indeed,

$$E\left[\int_0^\infty e^{-\lambda t}|X_t-\rho|dt\right]<\infty \text{ and } E\left[\sum_{n=1}^\infty e^{-\lambda\tau_n}|\xi_n|I_{\{\tau_n<\infty\}}\right]<\infty$$

imply condition (11).

Definition 2.2 (Admissible controls). We shall say that an impulse control is admissible if the conditions (9)-(11) are satisfied. We shall denote by $\mathcal{A}(x)$ the class of admissible impulse controls.

Example 2.1. Let us consider the strategy of no intervention, that is $P\{\tau_1=\infty\}=1$. If Y would denote the inventory level in that special case, we would have

$$Y_t=x+\int_0^t k(\rho-Y_s)ds+\int_0^t\sigma dW_s.$$

For each $t\in[0,\infty)$, $Y(t)$ would be normally distributed with expected value

$$E[Y(t)]\,=\,e^{-kt}Y(0)+\rho\left(1-e^{-kt}\right)$$

and variance

$$\mathrm{VAR}[Y(t)]\,=\,\frac{\sigma^2}{2k}\left(1-e^{-2kt}\right).$$

Then, Y would satisfy all conditions (9)-(11).

Remark 2.1. We can study a slightly more general problem in which the dynamics (1) are generalized to

$$X_t=x+\int_0^t[\mu+k(\rho-X_s)]\,ds+\int_0^t\sigma dW_s+\sum_{i=1}^\infty I_{\{\tau_i<t\}}\xi_i, \tag{12}$$

where $\mu \in (-\infty, \infty)$. If $k=0$, then we recover the inventory dynamics studied in Constantinides (1976), Constantinides and Richard (1978), Harrison, Sellke and Taylor (1983), Ormeci, Dai and Vande Vate (2008), and Sulem (1986). In the case $k>0$, we can write

$$\begin{aligned} X_t &= x + \int_0^t \left[k(\rho + \frac{\mu}{k} - X_s) \right] ds + \int_0^t \sigma dW_s + \sum_{i=1}^{\infty} I_{\{\tau_i < t\}} \xi_i \\ &= x + \int_0^t \left[k(\tilde{\rho} - X_s) \right] ds + \int_0^t \sigma dW_s + \sum_{i=1}^{\infty} I_{\{\tau_i < t\}} \xi_i, \end{aligned}$$

where $\tilde{\rho} := \rho + \frac{\mu}{k}$. If $\mu \neq 0$, then $\tilde{\rho} \neq \rho$. This mean that if $\mu \neq 0$, then the inventory level would be mean-reverting to a level $\tilde{\rho}$ which would be different from the preferred level ρ. However, that would contradict important inventory models that say that the inventory is mean-reverting towards the desired inventory level (see, for instance, subsection 2.2 of Manaster and Mann 1996, chapter 11 of Hasbrouck 2007, and Madhavan and Smidt 1993). We can obtain analytical solutions to Problem 2.1 when the inventory dynamics (1) is slightly generalized to (12). In that case, H of (37) should be slightly generalized to

$$\begin{aligned} \tilde{H}(y) &= A \sum_{n=0}^{\infty} a_{2n} (y - \tilde{\rho})^{2n} + B \sum_{n=0}^{\infty} b_{2n+1} (y - \tilde{\rho})^{2n+1} \\ &\quad + \left\{ \frac{1}{\lambda + 2k} \right\} y^2 + \left\{ \frac{2k\tilde{\rho} - 4\rho k - 2\rho\lambda}{(\lambda + k)(\lambda + 2k)} \right\} y + \frac{\sigma^2}{\lambda(\lambda + 2k)} + \frac{\rho^2}{\lambda + 2k} + \frac{2\mu(\mu - \rho\lambda)}{\lambda(\lambda + k)(\lambda + 2k)}, \end{aligned}$$

where $\tilde{A}$ and $\tilde{B}$ are real numbers, and the coefficients of the series are defined below equations (38)-(39). However, motivated by some empirical work on inventory, we will assume that $\mu = 0$, or equivalently that $\tilde{\rho} = \rho$.

Although there is an extensive literature that observes empirically that inventories are mean reverting, ours is the first paper that applies the theory of stochastic control to obtain analytically the optimal inventory policy when the inventory dynamics is mean-reverting. For instance, the classical works of Constantinides (1976), Constantinides and Richard (1978), Harrison, Sellke and Taylor (1983), Ormeci, Dai and Vande Vate (2008), and Sulem (1986) assume that the uncontrolled inventory dynamics follows a Brownian motion with drift, and therefore diverges. On the other hand, these papers assume that the shortage and storage costs are linear, while we assume that the cost for the inventory level to be far away from the target is quadratic. The recent paper of Ormeci, Dai and Vande Vate (2008) has an interesting innovation: it assumes that the inventory level and/or the sizes of the interventions are bounded. We allow the inventory level to be unbounded, but the running cost represented by $f(x) = (x - \rho)^2$ indicates that it cannot be optimal that the inventory level be too far away from the target ρ.

3. The Value Function

Let us denote the value function by V. That is, for every $x \in (-\infty, \infty)$:

$$V(x) := \inf \{ J(x; T, \xi);\ (T, \xi) \in \mathcal{A}(x) \}. \tag{13}$$

We define the *minimum cost operator* M by

$$MV(x) := \inf \{ V(x + \eta) + g(\eta) :\ \eta \in \mathbf{R}, \eta + x \in (-\infty, \infty) \}. \tag{14}$$

$MV(x)$ represents the value of the strategy that consists in starting with the best immediate intervention, and then following an optimal strategy. Let us consider the operator $\mathcal{L}$ defined by

$$\mathcal{L}\psi(x) := \frac{1}{2}\sigma^2 \frac{d^2\psi(x)}{dx^2} + k(\rho - x)\frac{d\psi(x)}{dx} - \lambda\psi(x). \tag{15}$$

Now we intend to find the value function and an associated optimal strategy.

Suppose there exists an optimal strategy for each initial point. Then, if the process starts at x and follows the optimal strategy, the cost function associated with this optimal strategy is $V(x)$. On the other hand, if the process starts at x, selects the best immediate intervention, and then follows an optimal strategy, then the cost associated with this second strategy is $MV(x)$. Since the first strategy is optimal, its cost function is smaller than the cost function associated with the second strategy. Furthermore, these two costs are equal when it is optimal to jump. Hence, $V(x) \le MV(x)$, with equality when it is optimal to intervene. In the continuation region, that is, when the management does not intervene, we must have $\mathcal{L}V(x) = -f(x)$ (this is an heuristic application of the dynamic programming principle to the problem we are considering). These intuitive observations can be applied to give a characterization of the value function. We formalize this intuition in the next two definitions and theorem.

Definition 3.1 (QVI). We say that a function $v : (-\infty,\infty) \mapsto [0,\infty)$ satisfies the quasi-variational inequalities for Problem 2.1 if for every $x \in (-\infty,\infty)$:

$$\mathcal{L}v(x) + f(x) \ge 0, \tag{16}$$

$$v(x) \le Mv(x), \tag{17}$$

$$(v(x) - Mv(x))\,(\mathcal{L}v(x) + f(x)) = 0. \tag{18}$$

We observe that a solution v of the QVI separates the interval $(-\infty,\infty)$ into two disjoint regions: a continuation region

$$\mathcal{C} := \{x \in (-\infty,\infty) : v(x) < Mv(x) \text{ and } \mathcal{L}v(x) + f(x) = 0\}$$

and an intervention region

$$\Sigma := \{x \in (-\infty,\infty) : v(x) = Mv(x) \text{ and } \mathcal{L}v(x) + f(x) > 0\}.$$

From a solution to the QVI it is possible to construct the following stochastic impulse control.

Definition 3.2. Let v be a continuous solution of the QVI. Then the following impulse control is called the QVI-control associated with v (if it exists):

$$\tau_1^v := \inf\{t \ge 0 :\ v(X^v(t)) = Mv(X^v(t))\} \tag{19}$$

$$\xi_1^v := \arg\inf\{v(X^v(\tau_1^v) + \eta) + g(\eta) :\ \eta \in \mathbf{R}, X(\tau_1^v) + \eta \in (-\infty,\infty)\}. \tag{20}$$

and, for every $n \ge 2$:

$$\tau_n^v := \inf\left\{t > \tau_{n-1}^v :\ v(X^v(t)) = Mv(X^v(t))\right\} \tag{21}$$

$$\xi_n^v := \arg\inf\{v(X^v(\tau_n^v) + \eta) + g(\eta) :\ \eta \in \mathbf{R}, X^v(\tau_n^v) + \eta \in (-\infty,\infty)\}. \tag{22}$$

Here, X^v represents the process generated by $(\tau_1^v, \tau_2^v, \cdots, \tau_n^v, \cdots; \xi_1^v, \xi_2^v, \cdots, \xi_n^v, \cdots)$. We also denote $\tau_0^v = 0$ and $\xi_0^v = 0$.

This means that the management intervenes whenever v and Mv coincide and the size of the intervention is the solution to the optimization problem corresponding to $Mv(x)$.

To the best of our knowledge, Cadenillas, Sarkar and Zapatero (2007) is the only other paper that obtains a solution for a stochastic impulse control problem in which the dynamics of the uncontrolled process is a mean-reverting process. However, those results cannot be applied directly to our problem, because the interventions in that paper occur only on side. Furthermore, that paper considers a random horizon while we consider an infinite horizon model. In addition, our paper has a running cost which does not appear in that paper.

Examples of solutions to stochastic impulse control problems include Cadenillas et al. (2006), Cadenillas and Zapatero (1999, 2000), Constantinides and Richard (1978), Harrison, Sellke, and Taylor (1983), Ormeci, Dai and Vande Vate (2008), and Sulem (1986), but the theory developed in these papers cannot be applied directly to the inventory problem that we study in this paper.

Now, we present a verification theorem.

THEOREM 3.1. *Let $v \in C^1((-\infty,\infty);(0,\infty))$ be a solution of the QVI and let $\mathcal{N}$ be a finite subset of $(-\infty,\infty)$ such that $v \in C^2((-\infty,\infty)-\mathcal{N};(0,\infty))$. Suppose that there exists $-\infty < L < U < \infty$ such that v is linear in $(-\infty,L)$ and in (U,∞). Then, for every $x \in (-\infty,\infty)$:*

$$V(x) \geq v(x). \tag{23}$$

Furthermore, if the QVI-control corresponding to v is admissible then it is an optimal impulse control, and for every $x \in (-\infty,\infty)$:

$$V(x) = v(x). \tag{24}$$

Proof. The proof is similar to that of Theorem 3.1 of Cadenillas and Zapatero (1999). We note that the differentiability of v implies its continuity, and therefore its boundedness in the compact interval $[L,U]$. Furthermore, v' is bounded in $(-\infty,\infty)$, because it is continuous in $[L,U]$ and a constant in $(-\infty,L)$ and in (U,∞). Let (T,ξ) be an admissible policy, and denote by $X = X^{(T,\xi)}$ the trajectory determined by (T,ξ). We observe that condition (11), the boundedness of v in the compact interval $[L,U]$, and its linearity in $(-\infty,L)\cup(U,\infty)$, imply that

$$\lim_{T\longrightarrow\infty} E\left[e^{-\lambda T} v(X(T+))\right] = 0. \tag{25}$$

Furthermore, the boundedness of v' implies that

$$E\left[\int_0^\infty \{e^{-\lambda t} v'(X(t))\}^2 dt\right] < \infty. \tag{26}$$

We can write for every $t > 0$ and $n \in \mathbf{N}$,

$$\begin{aligned}
& e^{-\lambda(t\wedge\tau_n)} v(X_{(t\wedge\tau_n)+}) - v(X_0) \\
&= \sum_{i=1}^{n} \left\{ e^{-\lambda(t\wedge\tau_i)} v(X_{t\wedge\tau_i}) - e^{-\lambda(t\wedge\tau_{i-1})} v(X_{(t\wedge\tau_{i-1})+}) \right\} \\
&\quad + \sum_{i=1}^{n} I_{\{\tau_i \le t\}} e^{-\lambda\tau_i} \left\{ v(X_{\tau_i+}) - v(X_{\tau_i}) \right\}.
\end{aligned}$$

Since X is a *continuous* semimartingale in the stochastic interval $(\tau_{i-1},\tau_i]$ and v is twice continuously differentiable in $(-\infty,\infty)-\mathcal{N}$, where $\mathcal{N}$ is a finite subset of $(-\infty,\infty)$, we may apply an appropriate version of Itô's formula (see, for instance, section IV.45 of Rogers and Williams (1987)). Thus, for every $i \in \mathbf{N}$,

$$\begin{aligned}
& e^{-\lambda(t\wedge\tau_i)} v(X_{t\wedge\tau_i}) - e^{-\lambda(t\wedge\tau_{i-1})} v(X_{(t\wedge\tau_{i-1})+}) \\
&= \int_{(t\wedge\tau_{i-1},t\wedge\tau_i]} e^{-\lambda s}\{v'(X_s)k(\rho - X_s) + \frac{1}{2}\sigma^2 v''(X_s) - \lambda v(X_s)\} ds \\
&\quad + \int_{(t\wedge\tau_{i-1},t\wedge\tau_i]} e^{-\lambda s} v'(X_s)\sigma dW_s \\
&= \int_{(t\wedge\tau_{i-1},t\wedge\tau_i]} e^{-\lambda s}\mathcal{L}v(X_s) ds + \int_{(t\wedge\tau_{i-1},t\wedge\tau_i]} e^{-\lambda s} v'(X_s)\sigma dW_s.
\end{aligned}$$

According to inequality (16),

$$\begin{aligned} & e^{-\lambda(t\wedge\tau_i)}v(X_{t\wedge\tau_i}) - e^{-\lambda(t\wedge\tau_{i-1})}v(X_{(t\wedge\tau_{i-1})+}) \\ & \geq \int_{(t\wedge\tau_{i-1},t\wedge\tau_i]} e^{-\lambda s}\{-f(X_s)\}ds + \int_{(t\wedge\tau_{i-1},t\wedge\tau_i]} e^{-\lambda s}v'(X_s)\sigma dW_s. \end{aligned}$$

We note that this inequality becomes an equality for the QVI-control associated to v (see Definition 3.2). According to inequality (17), in the event $\{\tau_i \leq t\}$ we have

$$e^{-\lambda\tau_i}\left\{v(X_{\tau_i+}) - v(X_{\tau_i})\right\} \geq -e^{-\lambda\tau_i}g(\xi_i).$$

This inequality becomes an equality for the QVI-control associated to v (see Definition 3.2). Combining the above inequalities, and taking expectations, we obtain

$$\begin{aligned} & v(x) - E\left[e^{-\lambda(t\wedge\tau_n)}v(X_{(t\wedge\tau_n)+})\right] \\ & \leq E\left[\sum_{i=1}^{n}\left\{I_{\{\tau_i\leq t\}}e^{-\lambda\tau_i}g(\xi_i)\right.\right. \\ & \qquad \left.\left. + \int_{t\wedge\tau_{i-1}}^{t\wedge\tau_i} e^{-\lambda s}f(X(s))ds - \int_{t\wedge\tau_{i-1}}^{t\wedge\tau_i} e^{-\lambda s}v'(X_s)\sigma dW_s\right\}\right], \end{aligned}$$

with equality for the QVI-control associated to v. From condition (10),

$$\lim_{n\longrightarrow\infty}\left\{v(x) - E\left[e^{-\lambda(t\wedge\tau_n)}v(X_{(t\wedge\tau_n)+})\right]\right\} = v(x) - E\left[e^{-\lambda t}v(X_{t+})\right].$$

According to (26),

$$\lim_{n\longrightarrow\infty} E\left[\int_0^{t\wedge\tau_n} e^{-\lambda s}v'(X_s)\sigma dW_s\right] = 0.$$

Thus,

$$\begin{aligned} & v(x) - E\left[e^{-\lambda t}v(X_{t+})\right] \\ & \leq E\left[\sum_{i=1}^{\infty}\left\{I_{\{\tau_i\leq t\}}e^{-\lambda\tau_i}g(\xi_i) + \int_{t\wedge\tau_{i-1}}^{t\wedge\tau_i} e^{-\lambda s}f(X(s))ds\right\}\right], \end{aligned}$$

with equality for the QVI-control associated to v.

According to (25),

$$\lim_{t\longrightarrow\infty}\left\{v(x) - E\left[e^{-\lambda t}v(X_{t+})\right]\right\} = v(x).$$

Furthermore,

$$\begin{aligned} & E\left[\sum_{i=1}^{\infty}\left\{I_{\{\tau_i\leq t\}}e^{-\lambda\tau_i}g(\xi_i) + \int_{t\wedge\tau_{i-1}}^{t\wedge\tau_i} e^{-\lambda s}f(X(s))ds\right\}\right] \\ & \xrightarrow{t\to\infty} E\left[\sum_{i=1}^{\infty} I_{\{\tau_i<\infty\}}e^{-\lambda\tau_i}g(\xi_i) + \int_0^{\infty} e^{-\lambda s}f(X(s))ds\right]. \end{aligned}$$

Hence,

$$v(x) \leq E\left[\sum_{i=1}^{\infty} I_{\{\tau_i<\infty\}}e^{-\lambda\tau_i}g(\xi_i) + \int_0^{\infty} e^{-\lambda s}f(X(s))ds\right],$$

with equality for the QVI-control generated by v. Therefore, for every $(T,\xi) \in \mathcal{A}(x)$:

$$v(x) \leq J(x;T,\xi), \tag{27}$$

with equality for the QVI-control generated by v. □

4. The Solution of the QVI

We conjecture that there exists an optimal solution $(\hat{T},\hat{\xi})$ characterized by four parameters a,α,β,b with $-\infty<a<\alpha\le\beta<b<\infty$ such that the optimal strategy is to stay in the band $[a,b]$ and jump to α (respectively, β) when reaching a (respectively, b). That is, we conjecture that

$$\hat{\tau}_i=\inf\{t>\hat{\tau}_{i-1}:X_t\notin(a,b)\} \tag{28}$$

and

$$X_{\hat{\tau}_{i+}}=X_{\hat{\tau}_i}+\hat{\xi}_i=\beta I_{\{X_{\hat{\tau}_i}=b\}}+\alpha I_{\{X_{\hat{\tau}_i}=a\}}. \tag{29}$$

In addition, we would expect that if $x>b$, then the optimal strategy would be to jump to β; while if $x<a$, then the optimal strategy would be to jump to α. Thus, the value function would satisfy

$$\forall x\in(-\infty,a]:\quad V(x)=V(\alpha)+C+c(\alpha-x) \tag{30}$$

and

$$\forall x\in[b,\infty):\quad V(x)=V(\beta)+D+d(x-\beta). \tag{31}$$

If V were differentiable in $\{a,b\}$, then from equations (30)-(31) we would get

$$V'(a)\;=\;-c \tag{32}$$
$$V'(b)\;=\;d. \tag{33}$$

If V were differentiable in $\{\alpha,\beta\}$, then

$$V'(\alpha)\;=\;-c \tag{34}$$
$$V'(\beta)\;=\;d. \tag{35}$$

In fact, the minimum of $V(y)+C+c(y-a)$ (respectively, $V(y)+D+d(b-y)$) is attained at $y=\alpha$ (respectively $y=\beta$). We also conjecture that the continuation region is the interval (a,b), so

$$\mathcal{L}V(x)=-f(x)=-(x-\rho)^2,\quad\forall x\in(a,b). \tag{36}$$

The general solution of this ordinary differential equation is

$$H(y)\;=\;AF(y)+BG(y)+\frac{1}{\lambda+2k}(y-\rho)^2+\frac{\sigma^2}{\lambda(\lambda+2k)}, \tag{37}$$

where A and B are real numbers,

$$F(y)\;:=\;\sum_{n=0}^{\infty}a_{2n}(y-\rho)^{2n}, \tag{38}$$
$$G(y)\;:=\;\sum_{n=0}^{\infty}b_{2n+1}(y-\rho)^{2n+1}, \tag{39}$$

and the coefficients of the series F and G are given by

$$a_{2n}\;=\;\begin{cases}1 & \text{if } n=0\\ \left(\frac{2}{\sigma^2}\right)^n\frac{1}{(2n)!}\prod_{i=0}^{n-1}(2ik+\lambda) & \text{if } n\ge1\end{cases}$$

and

$$b_{2n+1}\;=\;\begin{cases}1 & \text{if } n=0\\ \left(\frac{2}{\sigma^2}\right)^n\frac{1}{(2n+1)!}\prod_{i=0}^{n-1}((2i+1)k+\lambda) & \text{if } n\ge1.\end{cases}$$

Then,

$$F'(y) := \sum_{n=0}^{\infty} p_{2n+1}(y-\rho)^{2n+1} \quad \text{and} \quad G'(y) := \sum_{n=0}^{\infty} q_{2n}(y-\rho)^{2n},$$

where the coefficients of these series are given by

$$p_{2n+1} = \left(\frac{2}{\sigma^2}\right)^{n+1} \frac{1}{(2n+1)!} \prod_{i=0}^{n}(2ik+\lambda), \text{ if } n \geq 0,$$

and

$$q_{2n} = \begin{cases} 1 & \text{if } n = 0 \\ \left(\frac{2}{\sigma^2}\right)^n \frac{1}{(2n)!} \prod_{i=0}^{n-1}((2i+1)k+\lambda) & \text{if } n \geq 1. \end{cases}$$

We note that the power series F converges absolutely in any interval of the form $(\rho - M, \rho + M)$, where $M < \infty$. Similarly, the power series G also converges absolutely in any bounded interval. We also observe that $F(\rho) = 1$, $G(\rho) = 0$, $F'(\rho) = 0$, and $G'(\rho) = 1$, so

$$A = H(\rho) - \frac{\sigma^2}{\lambda(\lambda+2k)} \qquad \text{and} \qquad B = H'(\rho).$$

In summary, we conjecture that the solution is described by (28)-(29), and that the six unknowns $A, B, a, \alpha, \beta, b$ are a solution to the system of six equations

$$H(a) = H(\alpha) + C + c(\alpha - a), \tag{40}$$
$$H(b) = H(\beta) + D + d(b - \beta), \tag{41}$$
$$H'(a) = -c, \tag{42}$$
$$H'(b) = d, \tag{43}$$
$$H'(\alpha) = -c, \tag{44}$$
$$H'(\beta) = d, \tag{45}$$

where

$$H(y) = AF(y) + BG(y) + \frac{1}{\lambda+2k}(y-\rho)^2 + \frac{\sigma^2}{\lambda(\lambda+2k)}. \tag{46}$$

Lemma 4.1. *There exists a solution to the system of equations (40)-(45). Furthermore, if we define the function* $v : (-\infty, \infty) \longmapsto [0, \infty)$ *by*

$$v(x) := \begin{cases} H(\alpha) + C + c(\alpha - x) & \text{if } x < a \\ H(x) & \text{if } a \leq x \leq b. \\ H(\beta) + D + d(x - \beta) & \text{if } x > b \end{cases} \tag{47}$$

then the following conditions are satisfied.

$$\forall x < a: \quad -kc(\rho - x) - \lambda[H(\alpha) + C + c(\alpha - x)] + (x-\rho)^2 > 0, \tag{48}$$

$$\forall x > b: \quad kd(\rho - x) - \lambda[H(\beta) + D + d(x - \beta)] + (x-\rho)^2 > 0, \tag{49}$$

$$\forall \alpha < x < \beta: \qquad -c < v'(x) < d, \tag{50}$$

$$\forall \beta \leq x \leq b: \qquad v'(x) \geq d, \tag{51}$$

and

$$\forall a \leq x \leq \alpha: \qquad v'(x) \leq -c. \tag{52}$$

Proof. See the Appendix. □

By making some conjectures, we have found a candidate for optimal control (28)-(29) and a candidate for value function (47). Now we are going to prove rigorously that the above conjectures are valid, and as a consequence that the optimal control is given by (28)-(29) and the value function is given by (47).

THEOREM 4.1. *Let $A, B, a, b, \alpha, \beta$, with $-\infty < a < \alpha \le \beta < b < \infty$ be a solution of the system of equations (40)-(45). Then the function v defined in (47) is the value function of Problem 2.1. That is,*

$$v(x) = V(x) = \inf\{J(x;T,\xi);(T,\xi)\in\mathcal{A}(x)\}. \tag{53}$$

Furthermore, the optimal strategy is given by (28)-(29).

Proof. It is obvious that if v were a solution to the QVI then, according to Theorem 3.1, v would be the value function and the optimal strategy would be given by (28)-(29). Indeed, v is twice continuously differentiable in $(-\infty,a)\cup(a,b)\cup(b,\infty)$, and once continuously differentiable in $\{a,b\}$. Furthermore, v is linear in $(-\infty,a)$ and in (b,∞). In addition, the QVI-control associated with v would be admissible. In fact, the trajectory X generated by the QVI-control associated with v behaves like a mean-reverting process in each random interval (τ_n,τ_{n+1}) and satisfies $P\{\forall t\in(0,\infty): X(t)\in[a,b]\}=1$. Thus, the conditions (9)-(11) would be satisfied, and the QVI-control associated to v would be admissible. Hence, it only remains to verify that v is a solution to the QVI.

We observe that

$$\mathcal{L}v(x)+f(x)=\begin{cases}-ck(\rho-x)-\lambda[H(\alpha)+C+c(\alpha-x)]+(x-\rho)^2 & \text{if } x<a\\ \mathcal{L}H(x)+(x-\rho)^2 & \text{if } a\le x\le b.\\ dk(\rho-x)-\lambda[H(\beta)+D+d(x-\beta)]+(x-\rho)^2 & \text{if } x>b\end{cases}$$

Thus,

$$\mathcal{L}v(x)+f(x)$$

is equal to zero in the interval $[a,b]$, is positive in $(-\infty,a)$ because of condition (48), and is positive in (b,∞) because of condition (49). We note that

$$Mv(x)=\begin{cases}H(\alpha)+C+c(\alpha-x) & \text{if } x\le\alpha\\ H(x)+\min(C,D) & \text{if } \alpha<x<\beta.\\ H(\beta)+D+d(x-\beta) & \text{if } x\ge\beta\end{cases}$$

We have used condition (50) to obtain Mv in the interval (α,β). Thus,

$$v(x)-Mv(x)$$

is equal to zero in the intervention region $(-\infty,a]\cup[b,\infty)$, and is negative in the continuation region (a,b) because of conditions (51)-(52). Hence, v is a solution of the QVI. This proves the theorem. □

REMARK 4.1. We observe that the above expressions would simplify dramatically in the special case $k=0$. Indeed, if $k=0$ then (3) remains the same while (1) simplifies. After some computations, we see that $k=0$ implies $F(y)=\cosh(\theta(x-\rho))$ and $G(y)=\frac{1}{\theta}\sinh(\theta(x-\rho))$, where $\theta=\sqrt{\frac{2\lambda}{\sigma^2}}$.

5. Times to Increase or Decrease the Inventory Level

We consider a stochastic process Y that satisfies the dynamics

$$dY_t = k(\rho - Y_t)dt + \sigma\, dW_t \tag{54}$$

$$Y_0 = y, \tag{55}$$

where $k > 0$, $\rho \in (-\infty, \infty)$, and $\sigma > 0$ are constants, and W is a standard Brownian motion.

For $-\infty < a < y < b < \infty$, we define the stopping time

$$\tau(a,b) := \inf\{t \in [0,\infty) : Y(t) \notin (a,b)\}. \tag{56}$$

$\tau(a,b)$ represents the first time that the management is going to intervene. The notation $\{Y_{\tau(a,b)} = a\}$ represents the event that the management will first increase the inventory level, and $\{Y_{\tau(a,b)} = b\}$ represents the event that the management will first decrease the inventory level.

We define the Gamma function for every $\nu > 0$ by

$$\Gamma(\nu) := \int_0^\infty u^{\nu-1} e^{-u} du,$$

and the parabolic cylinder function by

$$D_{-\nu}(x) := \exp\left\{-\frac{x^2}{4}\right\} 2^{-\frac{\nu}{2}} \sqrt{\pi} \left\{ \frac{1}{\Gamma\left((\nu+1)/2\right)} \left(1 + \sum_{k=1}^\infty \frac{\nu(\nu+2)\cdots(\nu+2k-2)}{3\cdot 5\cdots(2k-1)k!} \left(\frac{x^2}{2}\right)^k\right) - \frac{x\sqrt{2}}{\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \sum_{k=1}^\infty \frac{(\nu+1)(\nu+3)\cdots(\nu+2k-1)}{3\cdot 5\cdots(2k+1)k!} \left(\frac{x^2}{2}\right)^k\right)\right\}.$$

We also define for every $\nu > 0$:

$$S(\nu,x,y) := \frac{\Gamma(\nu)}{\pi} \exp\left\{\frac{x^2+y^2}{4}\right\} \Big(D_{-\nu}(-x)D_{-\nu}(y) - D_{-\nu}(x)D_{-\nu}(-y)\Big).$$

In addition, we define

$$\mathrm{Erfi}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{v^2} dv = \frac{2}{\sqrt{\pi}} \sum_{k=0}^\infty \frac{x^{2k+1}}{k!(2k+1)}$$

and

$$\mathrm{Erfid}(x,y) := \lim_{\nu \to 0} S(\nu,x,y) = \mathrm{Erfi}\left(\frac{x}{\sqrt{2}}\right) - \mathrm{Erfi}\left(\frac{y}{\sqrt{2}}\right).$$

We know that (see, for instance, chapter 7.3 of Borodin and Salminen (1996))

$$P_y\{Y_{\tau(a,b)} = a\} = \frac{\mathrm{Erfid}\left(\frac{(b-\rho)\sqrt{2k}}{\sigma}, \frac{(y-\rho)\sqrt{2k}}{\sigma}\right)}{\mathrm{Erfid}\left(\frac{(b-\rho)\sqrt{2k}}{\sigma}, \frac{(a-\rho)\sqrt{2k}}{\sigma}\right)} \tag{57}$$

$$P_y\{Y_{\tau(a,b)} = b\} = \frac{\mathrm{Erfid}\left(\frac{(y-\rho)\sqrt{2k}}{\sigma}, \frac{(a-\rho)\sqrt{2k}}{\sigma}\right)}{\mathrm{Erfid}\left(\frac{(b-\rho)\sqrt{2k}}{\sigma}, \frac{(a-\rho)\sqrt{2k}}{\sigma}\right)}. \tag{58}$$

Let us define the functionals $\tilde{f}$, $\tilde{g}$, and $\tilde{h}$ by

$$\tilde{f}(y) := E_y\left[\tau(a,b)\right] \tag{59}$$

$$\tilde{g}(y) := E_y\left[\tau(a,b) I_{\{Y_{\tau(a,b)}=b\}}\right] \tag{60}$$

$$\tilde{h}(y) := P_y\{Y_{\tau(a,b)}=b\}. \tag{61}$$

THEOREM 5.1. *The functional $\tilde{f}$ is given by*

$$\tilde{f}(y) = \tilde{A} + \tilde{B}\int_{\frac{\sqrt{2k}}{\sigma}(a-\rho)}^{\frac{\sqrt{2k}}{\sigma}(y-\rho)} \exp\left\{\frac{w^2}{2}\right\} dw - \frac{1}{k}\int_{\frac{\sqrt{2k}}{\sigma}(y-\rho)}^{\frac{\sqrt{2k}}{\sigma}(b-\rho)} \left[\int_w^{\frac{\sqrt{2k}}{\sigma}(b-\rho)} \exp\left\{-\frac{u^2}{2}\right\} du\right] \exp\left\{\frac{w^2}{2}\right\} dw, \tag{62}$$

where the constants $\tilde{A}$ and $\tilde{B}$ can be found from the equations

$$\tilde{f}(a) = 0 \qquad \text{and} \qquad \tilde{f}(b) = 0. \tag{63}$$

Proof. See Appendix B of Cadenillas, Sarkar and Zapatero (2007). □

THEOREM 5.2. *The functional $\tilde{g}$ is given by*

$$\tilde{g}(y) = \tilde{C} + \tilde{D}\int_{\frac{\sqrt{2k}}{\sigma}(a-\rho)}^{\frac{\sqrt{2k}}{\sigma}(y-\rho)} \exp\left\{\frac{w^2}{2}\right\} dw + \sum_{n=0}^{\infty} c_n (y-\rho)^n. \tag{64}$$

The sequence $\{c_n; n \in \{0,1,2,\ldots\}\}$ must satisfy

$$c_2 = \frac{1}{\sigma^2}\left(\frac{P}{Q}\right),$$

for every $i \in \{2,3,\cdots\}$:

$$c_{2i} = \left(\frac{2}{\sigma^2}\right)^i \frac{(2i-2)(2i-4)(2i-6)\cdots(2)}{(2i)!} k^{i-1}\left(\frac{P}{Q}\right),$$

and for every $i \in \{0,1,2,3,\cdots\}$:

$$\frac{1}{2}\sigma^2(2i+3)(2i+2)c_{2i+3} - k(2i+1)c_{2i+1} = -\frac{2}{\sqrt{\pi}Q}\left(\frac{\sqrt{k}}{\sigma}\right)^{2i+1}\frac{1}{i!(2i+1)}.$$

Here,

$$P := \frac{2}{\sqrt{\pi}}\sum_{i=0}^{\infty}\left(\frac{\sqrt{k}}{\sigma}\right)^{2i+1}\frac{(a-\rho)^{2i+1}}{i!(2i+1)} = -Erfi\left(\frac{\sqrt{k}}{\sigma}(\rho-a)\right)$$

and

$$Q := Erfid\left(\frac{(b-\rho)\sqrt{2k}}{\sigma}, \frac{(a-\rho)\sqrt{2k}}{\sigma}\right) = Erfi\left(\frac{\sqrt{k}}{\sigma}(b-\rho)\right) + Erfi\left(\frac{\sqrt{k}}{\sigma}(\rho-a)\right).$$

The constants $\tilde{C}$ and $\tilde{D}$ can be found from the boundary conditions

$$\tilde{g}(a) = 0 \qquad \text{and} \qquad \tilde{g}(b) = 0. \tag{65}$$

Proof. See Appendix B of Cadenillas, Sarkar and Zapatero (2007). □

We note that we have the freedom to choose c_0 and c_1. In particular, we can select $c_0 = 0$ and $c_1 = 0$.

Remark 5.1. Following some of the ideas of Remark 2.1, we observe that can easily generalize the results of this section to the case in which the dynamics of Y are given by

$$dY_t = [\mu + k(\rho - Y_t)]\,dt + \sigma\,dW_t \tag{66}$$

$$Y_0 = y, \tag{67}$$

instead of simply (54)-(55). If $k = 0$, then the stopping time (56) has been widely studied in the literature. If $k > 0$, then we can rewrite (66) as

$$dY_t = \left[k\left(\frac{\mu}{k} + \rho - Y_t\right)\right] dt + \sigma\,dW_t = k\,(\tilde{\rho} - Y_t)\,dt + \sigma\,dW_t,$$

where $\tilde{\rho} = \frac{\mu}{k} + \rho$. Then the formulas of this section are still valid when the process Y follows the dynamics (66)-(67) instead of simply (54)-(55). In that slightly more general case, we would just need to replace ρ by $\tilde{\rho} = \frac{\mu}{k} + \rho$ in the formulas of this section.

6. Numerical Solutions and Comparative Statics Analysis

We have written a computer code in the R programming language to solve the system (40)-(45). Using a standard notebook computer (laptop computer) we have obtained some numerical examples. For instance, let us consider the baseline parameters

$$\rho = 2,\ \ \sigma = 1.2,\ \ k = 0.2,\ \ \lambda = 0.06,\ \ C = 5.0,\ \ D = 5.0,\ \ c = 2.0,\ \ d = 2.0.$$

In this case we necessarily have $B = 0$ (see the discussion following the proof of Proposition 1 in the Appendix). In addition, we find from (40)-(45)

$$a = -1.1625,\ \ \alpha = 1.3515,\ \ \beta = 2.6485,\ \ b = 5.1625,\ \ A = -10.4484.$$

Applying equations (57)-(58) for the baseline parameters with $a = -1.1625$, $y = \rho = 2$, and $b = 5.1625$, we see that

$$P_\rho\{Y_{\tau(a,b)} = a\} = 0.5 \qquad \text{and} \qquad P_\rho\{Y_{\tau(a,b)} = b\} = 0.5.$$

This means that, starting at the target level $y = \rho = 2$, the probability of increasing the inventory before reducing it is 0.5, and the probability of reducing the inventory before increasing it is 0.5. Similarly, we compute

$$P_\alpha\{Y_{\tau(a,b)} = a\} = 0.56 \qquad \text{and} \qquad P_\alpha\{Y_{\tau(a,b)} = b\} = 0.44,$$

and

$$P_\beta\{Y_{\tau(a,b)} = a\} = 0.44 \qquad \text{and} \qquad P_\beta\{Y_{\tau(a,b)} = b\} = 0.56.$$

Hence, immediately after increasing the inventory, the probability of increasing the inventory before reducing it is 0.56, and the probability of reducing the inventory before increasing it is 0.44. Furthermore, immediately after decreasing the inventory, the probability of increasing the inventory before reducing it is 0.44, and the probability of reducing the inventory before increasing it is 0.56.

From Theorem 5.1, we see that for the baseline parameters,

$$E_\alpha\,[\tau(a,b)] = 11.519, \qquad E_\rho\,[\tau(a,b)] = 11.817, \qquad \text{and} \qquad E_\beta\,[\tau(a,b)] = 11.519.$$

Hence, starting at the target level $\rho = 2$, it will take on average 11.817 time units to increase or decrease the inventory level. In addition, immediately after increasing the inventory, it will take on average 11.519 time units to change (increase or decrease) the inventory level. Similarly, immediately after decreasing the inventory, it will take on average 11.519 time units to change the inventory level.

From Theorem 5.2, we see that for the baseline parameters,

$$E_\alpha\left[\tau(a,b)I_{\{Y_{\tau(a,b)}=b\}}\right] = 5.618, \; E_\rho\left[\tau(a,b)I_{\{Y_{\tau(a,b)}=b\}}\right] = 5.9086, \text{ and } E_\beta\left[\tau(a,b)I_{\{Y_{\tau(a,b)}=b\}}\right] = 5.901.$$

Combining these expected values with $P_\cdot\{Y_{\tau(a,b)} = a\}$ and $P_\cdot\{Y_{\tau(a,b)} = b\}$, gives the expected number of time units until the next change of the inventory level given that we know the direction of that change. Starting at the target level $\rho = 2$, it will take on average $5.9086/0.5 = 11.8172$ units of time to change the inventory level provided that the manager will reduce the inventory level before increasing it. In addition, immediately after increasing the inventory, it will take on average $5.618/0.44 = 12.7681$ units of time to change the inventory level provided that the manager will reduce the inventory level before increasing it. Similarly, immediately after decreasing the inventory, it will take on average $5.901/0.56 = 10.5375$ units of time to change the inventory level provided that the manager will reduce the inventory level before increasing it.

From Theorems 5.1 and 5.2, it is obvious that we can also compute $E_y\left[\tau(a,b)I_{\{Y_{\tau(a,b)}=a\}}\right]$ for every value of y.

In Table 1, we consider different sets of parameter values (σ, k, C, c, D, d, and λ) to study the effect of the parameters on the optimal strategy. Some of the results are to be expected, while other results are less intuitive.

We first analyze the effects of a change in volatility σ. We observe that the larger the volatility, the higher the level of intervention b, and the lower the level of intervention a. Indeed, since there is a fixed cost of intervention, as the volatility increases, the manager waits longer to intervene, and the sizes of the interventions will be larger.

We observe that, if the fixed cost C increases, then a decreases, b increases, $\alpha - a$ increases, and $b - \beta$ decreases. Furthermore, if the fixed cost D increases, then a decreases, b increases, $\alpha - a$ decreases, and $b - \beta$ increases. In other words, as expected, when a fixed cost increases, it is optimal to wait longer before intervening in any direction. In addition, the size of the intervention that pays that fixed cost increases, and the size of the intervention which does not pay that fixed cost decreases. It is interesting that changes in the fixed cost of intervention only on one side of the target ρ, will also affect the optimal strategy on the other side of the target.

We note that if the proportional costs c and d increase, then a decreases, b increase, $\alpha - a$ decreases, and $b - \beta$ decreases. In other words, when proportional costs increase, it is optimal to wait longer before intervening but the interventions will be smaller. Increases in proportional costs on one side of the target affect the optimal inventory strategy on both sides.

In Table 1, we observe that if the speed of mean reversion k increases, then a decreases and b increases. In other words, as the speed of mean reversion increases, it is optimal to wait longer before intervening.

Finally, as the discount rate λ increases, a decreases and b increases. This is to be expected since λ basically represents the inflation rate, hence by increasing λ future interventions expressed in today's dollars become cheaper, so we would rather postpone intervention.

7. Additional Remarks

Some remarks are still in order. It is quite simple to see what effect the change of ρ has on the solution of our optimization problem. Suppose that $H(y), a, \alpha, \beta, b, A, B$ is a solution for a particular value of ρ. Let us suppose that we change ρ to $\bar{\rho}$ but leave all other parameters the same and

Table 1 Effect of the parameters σ, k, C, c, D, d, and λ.

σ	a	α	β	b	$\alpha - a$	$b - \beta$
1.0	-0.9245	1.4105	2.5895	4.9245	2.3350	2.3350
1.2	-1.1625	1.3515	2.6485	5.1625	2.5140	2.5140
1.4	-1.3885	1.2915	2.7085	5.3885	2.6800	2.6800
k	a	α	β	b	$\alpha - a$	$b - \beta$
0.1	-1.0145	1.4415	2.5585	5.0145	2.4560	2.4560
0.2	-1.1625	1.3515	2.6485	5.1625	2.5140	2.5140
0.3	-1.3415	1.2375	2.7625	5.3415	2.5790	2.5790
C	a	α	β	b	$\alpha - a$	$b - \beta$
5.0	-1.1625	1.3515	2.6485	5.1625	2.5140	2.5140
7.0	-1.3875	1.4265	2.6815	5.1815	2.8140	2.5000
D	a	α	β	b	$\alpha - a$	$b - \beta$
5.0	-1.1625	1.3515	2.6485	5.1625	2.5140	2.5140
7.0	-1.1815	1.3185	2.5735	5.3875	2.5000	2.8140
c	a	α	β	b	$\alpha - a$	$b - \beta$
2.0	-1.1625	1.3515	2.6485	5.1625	2.5140	2.5140
4.0	-1.5365	0.9095	2.7265	5.2065	2.4460	2.4800
d	a	α	β	b	$\alpha - a$	$b - \beta$
2.0	-1.1625	1.3515	2.6485	5.1625	2.5140	2.5140
4.0	-1.2065	1.2735	3.0905	5.5365	2.4800	2.4460
λ	a	α	β	b	$\alpha - a$	$b - \beta$
0.01	-1.0955	1.3805	2.6195	5.0955	2.4760	2.4760
0.06	-1.1625	1.3515	2.6485	5.1625	2.5140	2.5140
0.11	-1.2305	1.3215	2.6785	5.2305	2.5520	2.5520

The default parameters in the calculations are $\rho = 2.0$, $\sigma = 1.2$, $k = 0.2$, $C = 5.0$, $D = 5.0$, $c = 2.0$, $d = 2.0$, and $\lambda = 0.06$.

denote the solution of the control problem under $\bar{\rho}$ by $\bar{H}(y), \bar{a}, \bar{\alpha}, \bar{\beta}, \bar{b}, \bar{A}, \bar{B}$. The change of ρ will have the following effect. The multipliers A and B will remain the same, i.e., $\bar{A} = A$, $\bar{B} = B$. $\bar{H}(y)$ can be derived from $H(y)$ by the simple transition $\bar{H}(y) = H(y + \rho - \bar{\rho})$. The optimal boundary values under $\bar{\rho}$ will be $\bar{a} = a + \bar{\rho} - \rho$, $\bar{\alpha} = \alpha + \bar{\rho} - \rho$, $\bar{\beta} = \beta + \bar{\rho} - \rho$, and $\bar{b} = b + \bar{\rho} - \rho$.

It may be interesting to note that it is possible to have $a < \alpha < \beta < \rho < b$ or $a < \rho < \alpha < \beta < b$. In the first case whenever the inventory level hits b the manager will reduce it so that the inventory drops below the target level ρ. This is a bit surprising since this means that the manager is paying an additional cost (compared to just reducing the inventory to the target level) to reduce the inventory from the level b (which is above the target) to a level below the target. In the second case the manager is increasing the inventory level from a (which is below the target) to a level α that is above the target. The first case happens when $B > 0$ and the variable cost d is small relative to B. The second case happens when $B < 0$ the variable cost c is small relative to $-B$. We can construct an example for the first case, i.e., for $a < \alpha < \beta < \rho < b$, as follows. Select first $A > 0, B > 0$ arbitrary. These two constants determine $H(\cdot)$ via (46). Now select d such that $0 < d < B$ but otherwise arbitrary. The selection of d will determine b and β such that $\beta < \rho < b$ (see Figure 1) by (43) and (45). Next we select D so that it satisfies (41). In order to guarantee that D will be positive we have to be sure that $H(b) > H(\beta) + d(b - \beta)$. But this follows from the fact that between β and b the graph of $H'(\cdot)$ lies above the level d. We can now select $c > 0$ such that $-c > \min\{H'(y);\ y \leq \rho\}$ but otherwise arbitrary (see Figure 1). Then a and α will be determined by (42) and (44). Finally we determine C by (40). Again it is easy to see that C will be positive, because between a and α the graph of $H'(\cdot)$ lies below the level $-c$.

More on the intuitive level, it is quite clear that $a < \alpha < \beta < \rho < b$ happens when the fixed cost D is large compared to the other costs c, C, d, because in that case when decreasing the inventory the manager would rather decrease it below the target, in order to reduce the chance of having to decrease the inventory too soon and paying the large fixed cost D again. We expect the other scenario, that is $a < \rho < \alpha < \beta < b$, when the fixed cost C is large compared to the other costs c, d, D.

Case I: a<α<β<ρ<b

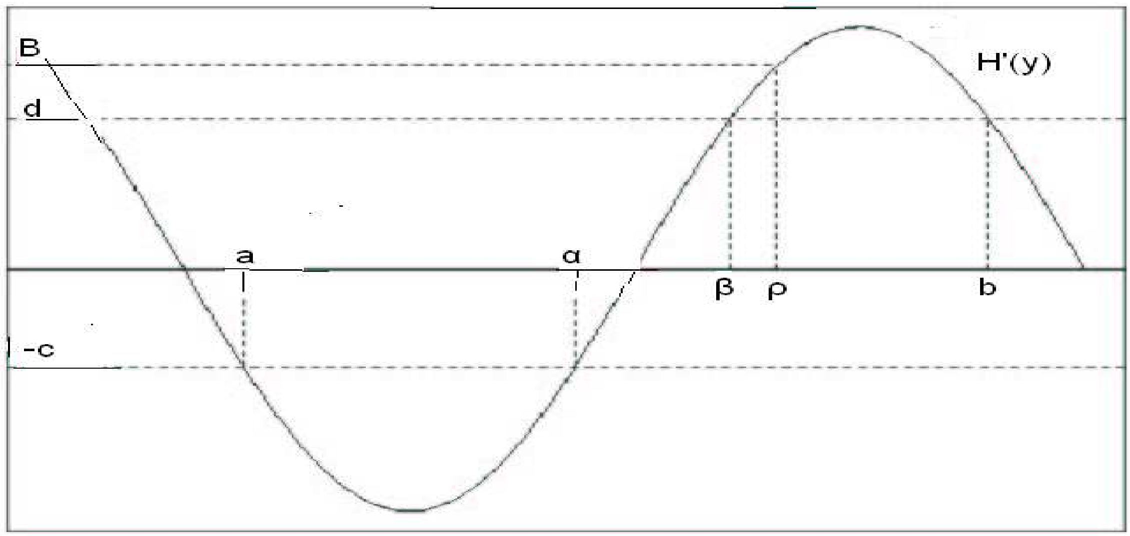

Figure 1

We omit the details of the example illustrating the possibility of $a < \rho < \alpha < \beta < b$ since it is quite similar to the example above. Instead we include Figure 2 which clearly illustrates this scenario.

Case II: a<ρ<α<β<b

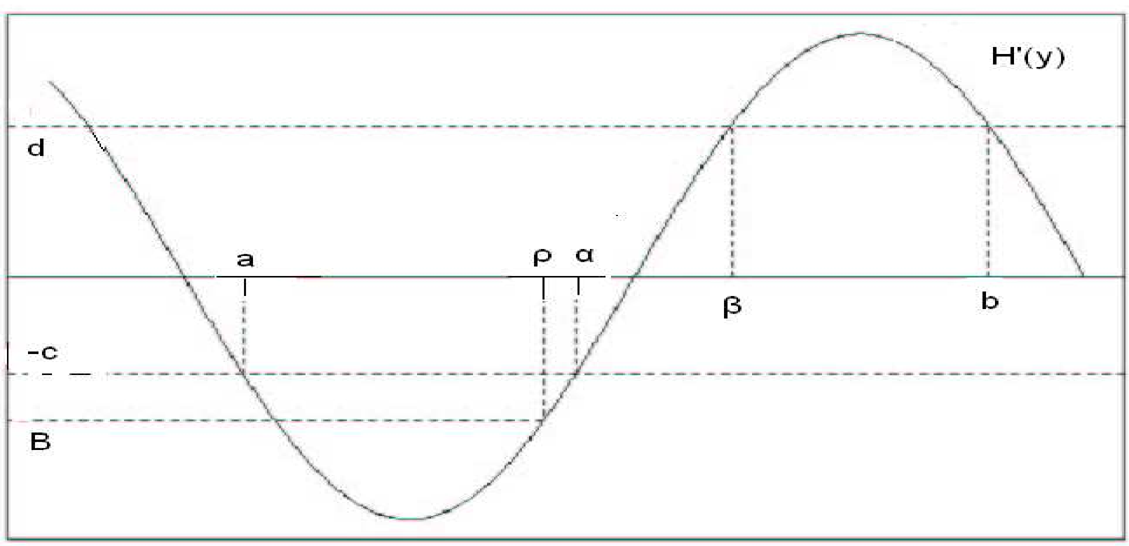

Figure 2

8. Conclusions

Motivated by some practical applications, we have assumed that the inventory level of a company follows a mean-reverting process. The management's objective is to keep the inventory level as close as possible to a target, so there is a running cost associated with the difference between the inventory level and the target. In addition, there are fixed and proportional costs for increasing or reducing the inventory level. The objective of the management is to minimize the total cost. We have solved this problem analytically. We have also considered some examples, and analyzed the effects of the parameters on the solution. In particular, we have studied cases in which asymmetric costs generate solutions of the form $a < \alpha < \beta < \rho < b$ and $a < \rho < \alpha < \beta < b$.

As stated in the introduction, our problem with the uncontrolled inventory process following a Brownian motion with zero drift and mean reversion is a generalization of the model where the uncontrolled inventory process just follows a Brownian motion. It would be possible to extend our results further by assuming that the underlying Brownian motion has a drift as well. This problem remains tractable; however, the resulting equations are then more intricate without providing much additional insight.

Appendix. Proof of Lemma 4.1 under the assumption that the cost function is symmetric

In this section we are going to prove Lemma 4.1 under the assumption that the cost function is symmetric, i.e., $c = d$ and $C = D$.

Proposition .1. *If the cost function is symmetric then the value function $v(\cdot)$ is symmetric around ρ.*

Proof. Let x be an arbitrary initial inventory value and $(T, \xi) = (\tau_0, \tau_1, \dots, \xi_0, \xi_1, \dots)$ be an admissible control for the initial inventory level x. We define the Brownian motion $\tilde{w} = -w$ and the reflected process $\tilde{X}_t = 2\rho - X_t$. So $\tilde{X}$ is identical to X reflected at the level ρ. Now follows that

$$\begin{aligned}\tilde{X}_t &= 2\rho - X_t = 2\rho - x - \int_0^t k(\rho - X_s)ds - \sigma w_t - \sum_{i=1}^{\infty} 1_{\{\tau_i < t\}}\xi_i \\ &= \tilde{x} + \int_0^t k(\rho - \tilde{X}_s)ds + \sigma \tilde{w}_t + \sum_{i=1}^{\infty} 1_{\{\tau_i < t\}}\tilde{\xi}_i\end{aligned}$$

where $\tilde{x} = 2\rho - x$ and $\tilde{\xi}_i = -\xi_i$. It follows that for every admissible control (T, ξ) with initial value x there is a corresponding admissible control $(T, -\xi)$ for the initial value $\tilde{x}$. The costs of these two controls are the same because $(X_t - \rho)^2 = (\tilde{X}_t - \rho)^2$ and the cost function is assumed to be symmetric. It follows that the infimum of these costs over all admissible controls also coincide, i.e., $v(x) = v(2\rho - x)$. □

Let's analyze now $H(\cdot)$ given by (37) which is our candidate for $v(\cdot)$ in the interval $[a, b]$. The symmetry of $v(\cdot)$ around ρ implies that $H'(\rho) = 0$ and this could happen only if $B = 0$. We can thus reduce the 6 equations (40)-(45) to three equations:

$$H(b) = H(\beta) + D + d(b - \beta) \tag{68}$$

$$H'(b) = d \tag{69}$$

$$H'(\beta) = d, \tag{70}$$

where

$$H(y) = AF(y) + \frac{1}{\lambda + 2k}(y-\rho)^2 + \frac{\sigma^2}{\lambda(\lambda+2k)}. \tag{71}$$

The function $F(\cdot)$ is given in (38). We have three unknowns, b, β, A. Once these are determined, a and α can be determined by symmetry, i.e., $a = 2\rho - b$ and $\alpha = 2\rho - \beta$.
Equation (68) implies that $\beta < b$. Notice that in order to satisfy (68)-(70) we must have $A < 0$. Indeed, $A \geq 0$ would imply that $H(\cdot)$ would be convex (in particular, $H(\cdot)$ would be convex in $[\beta, b]$). Then by (69) and (70) $H(\cdot)$ would be linear on $[\beta, b]$ which is impossible because (36) has no linear solution.

Proposition .2. *In the symmetric case there exist constants $a < \alpha < \beta < b$ and $A < 0$ satisfying (40)-(45).*

Proof. As shown above it is sufficient to find constants β, b and A satisfying (68)-(70). Notice that $H'(\cdot)$ is strictly concave on $[\rho, \infty)$ because $H''(\cdot)$ is strictly decreasing on the same half-line (remember that $A < 0$). In the following we shall show in the notation the dependence of H, H' on A by writing $H(y, A)$ and $H'(y, A)$ instead of $H(y)$ and $H'(y)$. We define the function $g : (-\infty, 0) \mapsto [0, \infty)$ by

$$g(A) := \max_{y \geq \rho} H'(y, A), \quad A < 0.$$

It is clear that

$$\lim_{A \uparrow 0} g(A) = \infty \tag{72}$$

and $g(\cdot)$ is increasing on $(-\infty, 0)$. To be more precise, there exists a point $A_0 < 0$ such that g is zero on $(-\infty, A_0]$ and positive on $(A_0, 0)$. The value of A_0 is determined by $H''(\rho, A_0) = A_0 F''(\rho) + \frac{2}{\lambda+2k} = 0$. This happens because for $A < A_0$ the second derivative $H''(\rho, A)$ is negative, so $H'(\cdot, A)$ is decreasing on $[\rho, \infty)$. It follows that there exists a unique value $A_1 < 0$ such that $g(A_1) = d$. In order to satisfy $H'(b, A) = H'(\beta, A) = d$ we must have $A \in [A_1, 0)$. For any $A \in [A_1, 0)$ we define the functions $\beta(A)$ and $b(A)$ by

$$H'(\beta(A), A) = H'(b(A), A) = d. \tag{73}$$

Notice that $\beta(A_1) = b(A_1)$, but for all $A_1 < A < 0$ we have $\beta(A) < b(A)$. Now we define the function

$$J(A) = H(b(A), A) - H(\beta(A), A) - d(b(A) - \beta(A)). \tag{74}$$

The identity $\beta(A_1) = b(A_1)$ implies $J(A_1) = 0$. We need to show that

$$\lim_{A \uparrow 0} J(A) = \infty. \tag{75}$$

Notice that

$$\lim_{A \uparrow 0} \beta(A) = \beta_0 \quad \text{and} \quad \lim_{A \uparrow 0} b(A) = \infty, \tag{76}$$

where β_0 is determined by $\frac{2}{\lambda+2k}(\beta_0 - \rho) = d$ (see 71). By the concavity of $H'(\cdot)$ we have

$$J(A) = \int_{\beta(A)}^{b(A)} H'(u, A) du - d(b(A) - \beta(A)) \geq \left(\frac{1}{2} g(A) - d\right)(b(A) - \beta(A)) \tag{77}$$

and this converges to infinity as $A \uparrow 0$ by (72) and (76).

¿From (75) follows the existence of a constant $\bar{A} \in (A_1, 0)$ such that $J(\bar{A}) = D$. It follows from the above construction that $b = b(\bar{A})$, $\beta = \beta(\bar{A})$, and $A = \bar{A}$ satisfy (68)-(70). □

Proposition .3. *In the symmetric case conditions (48)-(52) are satisfied.*

Proof. Suppose now that A, b, β satisfy 68-70. By the symmetry of $H(\cdot)$ around ρ instead of (48)-(52) it is sufficient to show that

$$kd(\rho - x) - \lambda [H(\beta) + D + d(x-\beta)] + (x-\rho)^2 > 0, \quad \text{for all } x > b, \tag{78}$$

$$0 < H'(x) < d, \quad \text{for all } \rho < x < \beta \tag{79}$$

$$H'(x) \geq d \quad \text{for all } \beta \leq x \leq b. \tag{80}$$

(79)-(80) follow immediately from (69)-(70), and the concavity of $v'(\cdot)$, so we only need to prove (78). Since $H'(\beta) = H'(b) = d$ and H' is a strictly concave function on (ρ, ∞), we conclude that H' is decreasing in a neighborhood of b hence $H''(b) \leq 0$. Let us denote the left-hand side of (78) by $\psi(x)$. Now by (68) we have

$$\begin{aligned}
\psi(b) &= k(\rho - b)d - \lambda H(b) + (b-\rho)^2 \\
&\geq \frac{1}{2}\sigma^2 H''(b) + k(\rho - b)H'(b) - \lambda H(b) + (b-\rho)^2 \\
&= \mathcal{L}H(b) + (b-\rho)^2 \\
&= 0,
\end{aligned}$$

because (36) is satisfied. Since $\psi(\cdot)$ is a quadratic function, it suffices to show that $\psi'(b) > 0$. This inequality becomes

$$b \geq \rho + d\frac{k+\lambda}{2}. \tag{81}$$

By (71) we have

$$H'(x) \;=\; AF'(x) + \left(\frac{2}{\lambda + 2k}\right)(x-\rho) \;\leq\; \left(\frac{2}{\lambda+2k}\right)(x-\rho),$$

because $A < 0$. Hence b must be larger that x_1 determined by $\left(\frac{2}{\lambda+2k}\right)(x_1 - \rho) = d$, i.e., $b > \rho + \frac{d(\lambda+2k)}{2}$. This implies (81). □

We omit the proof for the existence of $A, B, a, \alpha, \beta, b$ satisfying Lemma 4.1 in the asymmetric case, and constrain ourselves to mentioning that all the numerical examples that we have considered (see Table 1) suggest that such solution always exists.

Appendix. Comparison with the case in which the dynamics of the uncontrolled inventory is a Brownian motion with drift

In this Appendix we study Problem 2.1 but replacing the dynamics (1) by

$$X_t = x + \int_0^t \mu ds + \int_0^t \sigma dW_s + \sum_{i=1}^{\infty} I_{\{\tau_i < t\}} \xi_i. \tag{82}$$

In other words, we study the case in which the dynamics of the uncontrolled inventory is a Brownian motion with drift.

As in section 4, we conjecture that there exists an optimal solution $(\hat{T}, \hat{\xi})$ characterized by four parameters $\breve{a}, \breve{\alpha}, \breve{\beta}, \breve{b}$ with $-\infty < \breve{a} < \breve{\alpha} \leq \breve{\beta} < \breve{b} < \infty$ such that the optimal strategy is to stay in the band $[\breve{a}, \breve{b}]$ and jump to $\breve{\alpha}$ (respectively, $\breve{\beta}$) when reaching $\breve{a}$ (respectively, $\breve{b}$). That is, we conjecture that

$$\hat{\tau}_i = \inf\left\{t > \hat{\tau}_{i-1} : X_t \notin (\breve{a}, \breve{b})\right\} \tag{83}$$

and

$$X_{\hat{\tau}_{i+}} = X_{\hat{\tau}_i} + \hat{\xi}_i = \breve{\beta} I_{\{X_{\hat{\tau}_i} = \breve{b}\}} + \breve{\alpha} I_{\{X_{\hat{\tau}_i} = \breve{a}\}}. \tag{84}$$

In addition, we would expect that if $x > \breve{b}$, then the optimal strategy would be to jump to $\breve{\beta}$; while if $x < \breve{a}$, then the optimal strategy would be to jump to $\breve{\alpha}$. Thus, the value function $\breve{V}$ would satisfy

$$\forall x \in (-\infty, \breve{a}] : \quad \breve{V}(x) = \breve{V}(\breve{\alpha}) + C + c(\breve{\alpha} - x) \tag{85}$$

and

$$\forall x \in [\breve{b}, \infty) : \quad \breve{V}(x) = \breve{V}(\breve{\beta}) + D + d(x - \breve{\beta}). \tag{86}$$

If V were differentiable in $\{\breve{a}, \breve{b}\}$, then from equations (85)-(86) we would get

$$\breve{V}'(\breve{a}) = -c \tag{87}$$
$$\breve{V}'(\breve{b}) = d. \tag{88}$$

If $\breve{V}$ were differentiable in $\{\breve{\alpha}, \breve{\beta}\}$, then

$$\breve{V}'(\breve{\alpha}) = -c \tag{89}$$
$$\breve{V}'(\breve{\beta}) = d. \tag{90}$$

We also conjecture that the continuation region is the interval $(\breve{a}, \breve{b})$, so

$$\mathcal{L}\breve{V}(x) = \frac{1}{2}\sigma^2 \frac{d^2\breve{V}(x)}{dx^2} + \mu \frac{d\breve{V}(x)}{dx} - \lambda \breve{V}(x) = -f(x) = -(x-\rho)^2, \quad \forall x \in (\breve{a}, \breve{b}). \tag{91}$$

The general solution of this ordinary differential equation is

$$\breve{H}(y) \;=\; \breve{A}e^{r_1 y} + \breve{B}e^{r_2 y} + \frac{1}{\lambda}y^2 + \left\{ \frac{2(\mu - \rho\lambda)}{\lambda^2} \right\} y + \frac{\sigma^2\lambda + 2\mu^2 - 2\rho\lambda\mu + \rho^2\lambda^2}{\lambda^3}, \tag{92}$$

where $\breve{A}$ and $\breve{B}$ are real numbers, and

$$r_1 := \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2\lambda}}{\sigma^2}$$
$$r_2 := \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2\lambda}}{\sigma^2}.$$

We observe that $\breve{H}$ is mathematically and computationally simpler than H of (37).

In summary, we conjecture that the solution is described by (83)-(84), and that the six unknowns $\breve{A}, \breve{B}, \breve{a}, \breve{\alpha}, \breve{\beta}, \breve{b}$ are a solution to the system of six equations

$$\breve{H}(\breve{a}) = \breve{H}(\breve{\alpha}) + C + c(\breve{\alpha} - \breve{a}), \tag{93}$$
$$\breve{H}(\breve{b}) = \breve{H}(\breve{\beta}) + D + d(\breve{b} - \breve{\beta}), \tag{94}$$
$$\breve{H}'(\breve{a}) = -c, \tag{95}$$
$$\breve{H}'(\breve{b}) = d, \tag{96}$$
$$\breve{H}'(\breve{\alpha}) = -c, \tag{97}$$
$$\breve{H}'(\breve{\beta}) = d, \tag{98}$$

where $\breve{H}$ is defined in (92). As in Lemma 4.1, it is possible to prove that there exists a solution to the system of equations (93)-(98). Then, we can define the function $\breve{v} : (-\infty, \infty) \longmapsto [0, \infty)$ by

$$\breve{v}(x) := \begin{cases} \breve{H}(\breve{\alpha}) + C + c(\breve{\alpha} - x) & \text{if } x < \breve{a} \\ \breve{H}(x) & \text{if } \breve{a} \le x \le \breve{b}. \\ \breve{H}(\breve{\beta}) + D + d(x - \breve{\beta}) & \text{if } x > \breve{b} \end{cases} \tag{99}$$

The proof of the following theorem is similar to the proof of Theorem 4.1.

Table 2 Effect of the parameter k.

k	a	α	β	b	$\alpha - a$	$b-\beta$
0.1	-3.0145	-0.5585	0.5585	3.0145	2.4560	2.4560
0.2	-3.1625	-0.6485	0.6485	3.1625	2.5140	2.5140
0.3	-3.3415	-0.7625	0.7625	3.3415	2.5790	2.5790

The default parameters in the calculations are $\rho = 0.0$, $\sigma = 1.2$, $C = 5.0$, $D = 5.0$, $c = 2.0$, $d = 2.0$, and $\lambda = 0.06$.

THEOREM .1. *Let $\check{A}, \check{B}, \check{a}, \check{b}, \check{\alpha}, \check{\beta}$, with $-\infty < \check{a} < \check{\alpha} \leq \check{\beta} < \check{b} < \infty$ be a solution of the system of equations (93)-(98). Then the function $\check{v}$ defined in (99) is the value function of Problem 2.1 when the dynamics of the inventory is given by (82). That is,*

$$\check{v}(x) = \check{V}(x) = \inf\{J(x;T,\xi);(T,\xi)\in\mathcal{A}(x)\}. \tag{100}$$

Furthermore, the optimal strategy is given by (83)-(84).

For instance, for the parameters

$$\rho = 0.0, \quad \mu = 0.0, \quad \sigma = 1.2, \quad \lambda = 0.06, \quad C = 5.0, \quad D = 5.0, \quad c = 2.0, \quad d = 2.0,$$

we obtain from (93)-(98) the following values:

$$\check{a} = -2.89175, \quad \check{\alpha} = -0.48775, \quad \check{\beta} = 0.48775, \quad \check{b} = 2.89175, \quad \check{A} = -174.8262, \quad \check{B} = -174.8262.$$

We observe that $\check{b} - \check{\beta} = \check{\alpha} - \check{a} = 2.404$.

To compare numerically the Brownian-motion-with-drift model with the mean-reverting model, we consider the symmetric case for both models. In other words, we consider a Brownian motion with drift model in the case when the drift $\mu = 0$, and set the target level ρ equal to zero as well. We compare this model to the mean reverting model with $\rho = 0$ and positive values for k. Notice that in the Brownian motion with zero drift case the mean of the uncontrolled inventory at any time is zero, whereas in the mean reverting case with $\rho = 0$ and $k > 0$ the limit (as t goes to infinity) of the corresponding mean (the long term mean) is zero (see Example 2.1). Table 2 shows the solution for the mean-reverting model for the baseline parameters

$$\rho = 0.0, \quad \sigma = 1.2, \quad \lambda = 0.06, \quad C = 5.0, \quad D = 5.0, \quad c = 2.0, \quad d = 2.0.$$

This facilitates the numerical comparison with the above example. Since we are assuming $\rho = 0$, the extreme case $k = 0$ can be interpreted as the Brownian motion with drift $\mu = 0$. We observe that, for every $k > 0$, the value b for the mean-reverting model is larger than the value $\check{b}$ for the Brownian-motion model. Furthermore, $b = b(k)$ converges to $\check{b}$ as k decreases to 0. This agrees with our intuition. Indeed, in the mean-reverting model ($k > 0$) the uncontrolled inventory will not go too far away from the target ρ, so it is unnecessary to intervene as often as in the case of the Brownian motion model. Thus, the value b for the mean-reverting model should be larger than the value $\check{b}$ for the Brownian motion model. Furthermore, as k converges to 0, we expect that b converges to $\check{b}$. Similarly, we observe that $a = a(k) < \check{a}$ for every $k > 0$, and $a = a(k)$ converges to $\check{a}$ as k decreases to 0. This behavior is also a consequence of the fact that in the mean-reverting model the uncontrolled inventory will not go too far away from the target ρ, so it is unnecessary to intervene as often as in the Brownian motion model.

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