Martingale Measures for a Class of Right–Continuous Processes

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Abstract

The subject of the present paper is the following. Suppose that W is a class of adapted, right–continuous processes on the continuous time horizon $[0,1]$, and for every stopping time τ and $\xi \in W$, $\xi(\tau)$ is bounded below. A necessary and sufficient condition will be given for the existence of a probability measure Q which is equivalent to the original measure and each process in W is a martingale under Q. Such a measure is called a martingale measure. This problem has a particular interest in the study of securities markets. If the processes in W represent the discounted prices of available securities, then the condition given here for the existence of a martingale measure can be interpreted as absence of "free lunch" in the securities market. This is a familiar theorem from the finance literature; the novelty of this paper is that the security prices are not required to be in L^p for some $1 \leq p \leq \infty$, neither are assumed to be continuous. Also, the concept of free lunch is invariant under the substitution of the original probability measure by an equivalent probability measure. The assumption that $\xi(\tau)$ is bounded below for every $\xi \in W$ and stopping time τ is quite natural since prices are non–negative.

We shall define a class of admissible subjective probability measures and assume that each agent in the economy has selected a subjective probability measure from that class. Subjective free lunch for an agent will be defined using his/her subjective probability measure. It will be shown that under an additional condition the existence of free lunch is equivalent to the existence to common subjective free lunch simultaneously for all possible agents in the economy.

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1. Introduction

Let W be a class of adapted stochastic processes with continuous index set $[0, 1]$ on a given probability space. The probability measure P of this probability space is called the reference measure. We suppose that for each $\xi \in W$ and stopping time τ, $\xi(\tau)$ is bounded below. We are interested in a sufficient and necessary condition for the existence of a probability measure Q, equivalent to the reference measure, such that each process in W is a martingale under Q. Such a measure is called a martingale measure.

There is a known significance of this problem in the stochastic models of finance. Suppose that W represents the discounted price processes of the available securities. Harrison and Pliska (1981) examined a model with discrete, finite index set and found that under the assumption that Ω is finite, existence of martingale measure is equivalent to the non–existence of arbitrage opportunities in the financial market. This result was significantly generalized by Dalang, Morton & Willinger (1990) since they eliminated the assumption that Ω is finite.

However, with continuous or discrete but infinite index set, absence of arbitrage is not a sufficient condition for the existence of a martingale measure. A stronger condition than absence of arbitrage, called the absence of "free lunch", was introduced by Kreps (1981). It involves the topological closure of the set of achievable gains by certain admissible trading policies. A generalized exposition of the basic result of Kreps can be found in Schachermayer (1992), where it is called the "Kreps–Yan Theorem". Originally Kreps assumed an extra separability condition which was eliminated by Schachermayer. In the Kreps–Yan Theorem the processes in W (the discounted price processes) are assumed to be bounded at every point of the time horizon. However, such boundedness condition seems to be artificial in the financial context. In the present paper only boundedness *below* is assumed, which is very natural since prices are non–negative. The result of the present paper can be thus considered a generalization of the Kreps–Yan Theorem, without imposing any conditions on the security prices besides their natural non–negativity.

The main result in Schachermayer (1992) covers the case when the index set is discrete and infinite. It is shown there that in this case the topological closure in the concept of free lunch can be replaced by almost sure convergence of sequences which are uniformly bounded below. In Delbaen (1992) the index set is continuous and processes in W are assumed to be continuous and bounded. In this setting Delbaen arrived at a sequential characterization with uniform bond.

Another line of research is followed by Duffie & Huang (1986), Ansel & Stricker (1990), and Stricker (1990). It is assumed in these papers that the processes are in $L^p(P)$ for some $1 \le p < \infty$. This assumption is quite general and harmless. However, in these papers the concepts of martingale measures and free lunch are not invariant under the substitution of the reference measure by an equivalent probability measure. Martingale measure is defined as an equivalent probability measure Q such that the processes in W are Q–martingales and $\frac{dQ}{dP} \in L^q(P)$ where q is conjugate to p. In the definition of free lunch $L^p(P)$ topology is used. This setup appropriates a special significance to the reference measure P. If we regard equivalent probability measures as subjective probability measures then agents who selected P have a special status since they decide what is a free lunch. This problem will be resolved in Section 4 of the present paper.

In the present paper the concepts of free lunch and martingale measure are invariant under the substitution of the reference measure by an equivalent probability measure, and neither boundedness nor continuity of the discounted security prices is assumed. The "Kreps–Yan Theorem" is a special case of Theorem 4 of the present paper. However, one can not derive Delbaen's result from Theorem 4. Due to the generality of the model, the topology used in defining free lunch is quite abstract; it is a weak topology compatible with a certain duality. However, this topology can be replaced by a more tangible one under the assumption that there exists a probability measure $\tilde{P}$, equivalent to the reference measure, such that $\xi(\tau)$ is $\tilde{P}$–integrable for every $\xi \in W$ and stopping time τ. This is basically the same as the condition imposed by Duffie & Huang (1986), Ansel & Stricker (1990), and Stricker (1990). However, the above mentioned invariance of free lunches and martingale measures is maintained here. Under this condition we shall derive our topology from $L^1(Q)$ topologies where Q is an admissible subjective probability measure. We shall define *subjective free lunch* using the $L^1(Q)$ topology of an admissible subjective probability measure Q. It will be shown that the existence of free lunch is equivalent to the existence of common subjective free lunch simultaneously for all admissible subjective probability measures, i.e., to a common subjective free lunch for all possible agents in the economy. The main result of Section 4 is that martingale measure exists if and only if there is no common subjective free lunch for *all* possible agents in the economy.

2. Security prices, martingale measures and free lunch

Let $(\Omega, \mathcal{F}, P)$ be a complete probability triplet and $\{\mathcal{F}_t \,:\, 0 \le t \le 1\}$ a filtration in $\mathcal{F}$. We suppose that $\mathcal{F}_0$ is trivial and complete. We do not need to assume the right–continuity of the filtration. Since our time–horizon is $[0,1]$, we define a stopping time as a random time τ such that $0 \le \tau(\omega) \le 1$ and $\{\tau > t\} \in \mathcal{F}_t$ for all $t \in [0,1]$.

Let W be a class of adapted , right–continuous processes on $[0,1]$, such that for each stopping time τ and $\xi \in W$ the random variable $\xi(\tau)$ is bounded below. This assumption is satisfied when the processes in W are non–negative, which is a natural assumption if they model discounted security price processes. However, one can easily find a process satisfying our assumption which is not bounded below, so our assumption is indeed more general than assuming boundedness below. We do not need to assume that the processes in W are cadlag.

A probability measure Q on $\mathcal{F}$ is called a *martingale measure* if it is equivalent to P and each process in W is a martingale under Q with respect to the filtration $\{\mathcal{F}_t \ : \ 0 \leq t \leq 1\}$.

Let $\mathcal{S}$ be the set of all stopping times and

$$S(W) \stackrel{\triangle}{=} \{\xi^\tau \ : \ \tau \in \mathcal{S}, \ \xi \in W\} \ , \tag{1}$$

where ξ^τ is the process ξ stopped at τ. Finally, we denote by V the linear space generated by $S(W)$, i.e.,

$$V \stackrel{\triangle}{=} \mathrm{Lin}(S(W)) \ . \tag{2}$$

A class of processes is called stable for stopping if for each process ξ in that class and each stopping time $\tau \in \mathcal{S}$ the stopped process ξ^τ is also in that class.

1. Lemma: V is the smallest linear space containing W which is stable under stopping.

The proof is trivial and omitted.

An equivalent probability measure Q is a martingale measure for W if and only if it is a martingale measure for V (Karatzas and Shreve, (1988), Ch. I, Problem 3.24).

We define

$$V^0 \stackrel{\triangle}{=} \{\xi \in V \ : \ \xi(0) = 0, \text{a.s.}\} \tag{3}$$

and

$$V_1^0 \stackrel{\triangle}{=} \{\xi(1) \ : \ \xi \in V^0\} \ . \tag{4}$$

These notations agree with the notations of Delbaen (1991). It is easy to see that a process is in V^0 if and only if it has the form

$$\sum_{i=1}^n \lambda_i(\xi_i^{\tau_i} - \xi_i(0)) \ , \quad n = 1, 2, \ldots, \ \lambda_i \in \Re, \ \xi_i \in W, \ \tau_i \in \mathcal{S} \ . \tag{5}$$

It follows that a random variable is in V_1^0 if and only if it has the form

$$\sum_{i=1}^n \lambda_i(\xi_i(\tau_i) - \xi_i(0)) \ , \quad n = 1, 2, \ldots, \ \lambda_i \in \Re, \ \xi_i \in W, \ \tau_i \in \mathcal{S} \ . \tag{6}$$

The following lemma is a trivial consequence of (6) and the assumption that $\mathcal{F}_0$ is trivial.

2. Lemma: Every $z \in V_1^0$ is a linear combination

$$z = \sum_{i=1}^{n} \lambda_i z_i \ , \tag{7}$$

where z_i is in V_1^0 and bounded below for each $i = 1, \ldots, n$.

We recall the definition of a very simple process from Delbaen (1991). For convenience we quote the definition.

3. Definition: A process $\{\theta_t \ ; \ 0 \leq t \leq 1\}$ is called a *very simple process* if there exists a sequence of stopping times $0 = \tau_0 \leq \tau_1 \leq \ldots \leq \tau_n \leq \tau_{n+1} = 1$ and elementary (step) functions $g_0, \ldots, g_n$, g_i being $\mathcal{F}_{\tau_i}$ measurable such that

$$\theta(t) = g_0 1_{\{t=0\}} + \sum_{i=0}^{n} g_i 1_{\{t \in (\tau_i, \tau_{i+1}]\}} \ .$$

If θ is a very simple process and $\xi \in V$ we denote by $\theta \bullet \xi$ the stochastic integral of θ with respect to ξ . It follows from (5) and the representation of $\theta \bullet \xi$ in [Delbaen, 1991] that

$$V^0 = \left\{ \sum_{i=1}^{n} \theta_i \bullet \xi_i \ : \ n = 1, 2, \ldots, \ \theta_i \text{ is very simple, } \ \xi_i \in W \right\} \tag{8}$$

and

$$V_1^0 = \left\{ \sum_{i=1}^{n} (\theta_i \bullet \xi_i)(1) \ : \ n = 1, 2, \ldots, \ \theta_i \text{ is very simple, } \ \xi_i \in W \right\} \ . \tag{9}$$

We denote by L^∞ the class of essentially bounded random variables, and by $L^1(P)$ the class of P–integrable random variables. The basic linear space we shall work with is denoted by N, and it is the linear space generated by $V_1^0 \cup L^\infty$, i.e.,

$$N \stackrel{\triangle}{=} \text{Lin}(V_1^0 \cup L^\infty) \ . \tag{10}$$

Next we are going to define a topology on N. First we introduce the set

$$M(P) = M \stackrel{\triangle}{=} \{ y \in L^1(P) \ : \ E_P|yz| < \infty \text{ for all } z \in N \} \ .$$

Every $y \in M$ also represents a linear functional on N given by the mapping $z \mapsto E_P[yz]$. We shall identify $y \in M$ and the corresponding linear functional on N, thus M will be regarded as a linear subspace of $L^1(P)$ and a linear subspace of the algebraic dual of N at the same time. There is an other representation of the linear functionals in M: it contains all signed bounded measures μ on $\mathcal{F}$ such that μ is absolutely continuous with respect to P

and $\int |x| d|\mu| < \infty$ for all $x \in N$. Here $|\mu|$ is the total variation of μ. The correspondence between such measures and elements of M is given by $y = \frac{d\mu}{dP}$.

We introduce the topology $\sigma(N, M)$ which is the weakest topology on N such that all linear functionals in M are continuous. It follows from the above representation of M by a class of absolutely continuous bounded measures that this topology is invariant under the substitution of P by an equivalent probability measure. We shall write M instead of $M(P)$ except in Section 4 where the reference measure will be changed.

This topology is not necessarily Hausdorff, because N is not necessarily separated in M. (However, M is separated in N, which will be used in Section 4.) The fact that the topology may not be Hausdorff does not cause any problems. We shall use as reference book on topological vector spaces Grothendieck's book, in which the definition of duality does not require that the members of the duality separate each other, and the theorems quoted in this paper are stated and proved in that book without such assumption. However, in Section 4 we shall impose a mild condition on W under which our topology is Hausdorff.

With the $\sigma(N.M)$ topology N is a locally convex topological vector space, and the set of continuous linear forms on N (the topological dual of N) is exactly M (Grothendieck (1973), Ch. 2, Section 8). For any subset $G \subset N$ we denote by $\overline{G}$ the closure of G in the $\sigma(N, M)$ topology. Let L^∞_+ be the class of essentially bounded, almost surely non–negative random variables, and N_+ be the class of almost surely non–negative random variables in N.

Now we are ready to state the main theorem of the paper.

4. Theorem: There exists a martingale measure for V (or, equivalently, for W) if and only if

$$\overline{V_1^0 - L^\infty_+} \cap N_+ \ = \ \{0\} \ . \tag{11}$$

Remark: The closure of the convex set $V_1^0 - L^\infty_+$ is the same in all topologies compatible with the duality $\langle N, M \rangle$ (Grothendieck (1973), Ch.2, Sec.8, Corollary to Thrm. 4 and Sec.13, Definition 11). Note that N is not necessarily separated im M, but M is separated in N, thus Definition 11 of the above reference applies. Thus we could replace our topology with any other topologies compatible with the duality. We shall consider one of these topologies in Section 4 which offers a clear economic interpretation.

Proof of the "only if"part: Suppose that Q is a martingale measure. We define the Radon–Nikodym derivative

$$y = \frac{dQ}{dP} \ , \tag{12}$$

and notice that $y \in M$. The set

$$R \stackrel{\triangle}{=} \{x \in N \ : \ E_Q x \leq 0\} \tag{13}$$

is closed in the $\sigma(N, M)$ topology because the mapping $x \mapsto E_Q[x] = E_P[yx]$ is continuous. $V_1^0 - L_+^\infty \subset R$ and $R \cap N_+ = \{0\}$, thus (11) follows.

The proof of the "if" part is deferred to the next section.

The above theorem has the following interpretation in the stochastic models of securities markets. Suppose that W contains the prices of the available securities. We call a selection of a finite number of securities $\xi_1, \ldots, \xi_n$ and very simple processes $\theta_1, \ldots, \theta_n$ a *very simple trading policy*. The class of very simple trading policies is denoted by $\mathcal{A}$. If $a = (\xi_1, \ldots, \xi_n, \theta_1, \ldots, \theta_n)$ is a very simple trading policy then the gains at the terminal time are given by

$$G(a) \triangleq \sum_{i=1}^{n} (\theta_i \bullet \xi_i)(1) \ . \tag{14}$$

From (9) follows that V_1^0 is the class achievable gains at the terminal time by very simple trading policies. We define *free lunch* as a net (generalized sequence) of pairs $(a_\alpha, b_\alpha) \subset \mathcal{A} \times L_+^\infty$ and an element $c \in N_+$, $P(c > 0) > 0$, such that $G(a_\alpha) - b_\alpha \to c$ in the $\sigma(N, M)$ topology (or, in any other topology compatible with the duality $\langle N, M \rangle$). Now we cast Theorem 4 into the familiar form: free lunch is absent in the securities market if and only if a martingale measure exists. For an economic interpretation of the topology and convergence of sequences and nets we refer to Section 4.

One can derive as a special case of Theorem 4 a version of the "Kreps–Yan Theorem" (a name given to it by Schachermayer (1992)). Suppose that the elements of N are bounded. This amounts to assuming that $\xi(\tau)$ is bounded above as well as below for all stopping times τ and $\xi \in W$ (recall (6) and that $\mathcal{F}_0$ is assumed to be trivial). In that case $N = L^\infty$, $M = L_1(P)$, so the we get the same topology which is considered in the "Kreps–Yan Theorem". In Schachermayer (1992) this theorem is proved under the assumption that $\xi(t)$ is bounded for all deterministic time $t \in [0, 1]$ and $\xi \in W$. We can characterize the closure of $V_1^0 - L_+^\infty$ in the $\sigma(L^\infty, L^1(P))$ topology as the smallest convex set $A \subset L^\infty$ containing $V_1^0 - L_+^\infty$ satisfying the following property: if $(x_n)_{n \geq 1} \subset A$ such that $(\|x_n\|_\infty)_{n \geq 1}$ is bounded and $x_n \to x$ almost surely, then $x \in A$. This follows from Lemma 7 of the next section of this paper. Thus if the elements of V_1^0 are bounded then one can derive the closure operator from almost sure convergence.* The main contribution of Theorem 4 is the fact that only boundedness below is assumed for members of V_1^0.

3. Lemmas and proofs

To complete the proof of Theorem 4 we need several lemmas.

* However, one must be careful here. The $\sigma(L^\infty, L^1(P))$–closure of a convex subset of L^∞ in general can not be obtained by taking all limits of $\sigma(L^\infty, L^1(P))$–convergent sequences.

5. Lemma: Suppose that Q is a probability measure, equivalent to P. Then Q is a martingale measure if and only if

$$E_Q\big[z\big] = 0 \ \text{ for all } z \in V_1^0 \ . \tag{15}$$

Proof: The "only if" part is trivial. To prove the "if" part, suppose that (15) holds and consider an arbitrary process $\xi \in W$ and a fixed time $u \in [0,1]$. By (6), $\xi(u) - \xi(0)$ is in V_1^0 and $\xi(0)$ is almost surely constant, thus $\xi(u)$ is Q–integrable. Next, let $A \in \mathcal{F}_u$ be an arbitrary event. We define the stopping time

$$\tau(\omega) = \begin{cases} u & \text{if } \omega \in A; \\ 1 & \text{otherwise.} \end{cases}$$

From (9) follows that $\xi(1) - \xi(\tau)$ is in V_1^0. But it is equal to $(\xi(1) - \xi(u))1_A$, and the martingale property for ξ now follows. Q.E.D.

We need to introduce some topologies on L^∞. The essential supremum of L^∞–variables is denoted by $\|\cdot\|_\infty$, and the corresponding topology is the norm or strong topology. We denote by $\sigma(L^\infty, L^1)$ the weak L_1–topology on L^∞, i.e., the weakest topology such that all linear functionals represented by L^1 random variables are continuous. Similarly, $\sigma(L^\infty, M)$ is the weakest topology on L^∞ such that all linear functionals in M are continuous.

In the following lemma we are going to use the notion of compact sets in non–Hausdorff topological spaces. We shall use some facts related to compact sets which remain true even in this generality, since they are straightforward consequences of the definition of compactness. One of these is that a compact set remains compact if we pass to a weaker topology. Another fact is that the image of a compact set by a continuous mapping from one topological space to an other is compact. We refer to Dunford and Schwartz (1958), Vol. I, page 17, Definition 5 and Lemma 7b. The topology there is not assumed to be Hausdorff. It may be worth mentioning that a compact set in a non–Hausdorff topology is not necessarily closed. However, one can see easily that adding to a $\sigma(N, M)$–compact subset of N all points that can not be distinguished (by M) from points in that compact subset, gives a $\sigma(N, M)$–compact closed set.

6. Lemma: Suppose that $K \subset L^\infty$ is compact in the $\sigma(L^\infty, L^1)$ topology. Then it is also compact as a subset of N in the $\sigma(N, M)$ topology.

Proof: Since $M \subset L^1$ it follows that $\sigma(L^\infty, M)$ is weaker than $\sigma(L^\infty, L^1)$, thus K is compact in the $\sigma(L^\infty, M)$ topology. By Grothendieck (1973), Ch. 2, Sec. 15, Proposition 20, $\sigma(L^\infty, M)$ is the topology induced on L^∞ by $\sigma(N, M)$. (The polar of L^∞ in M is the set $\{0\}$ thus $M/(L^\infty)^0 = M$.) Therefore, the natural embedding of L^∞ to N is $\sigma(L^\infty, M)$–$\sigma(N, M)$ continuous, and the $\sigma(N, M)$–compactness of K follows. Q.E.D.

Convex sets which are closed in the $\sigma(L^\infty, L^1)$ topology can be characterized in terms of sequences instead of nets (Grothendieck, 1973, Ch. 5, Part 3, p. 240, Exercise 1). For convenience we quote this result.

7.Lemma: A convex set $H \subset L^\infty$ is closed in the $\sigma(L^\infty, L^1)$ topology if and only if for all sequences $(x_n)_{n\geq 1} \subset H$ such that $(\|x_n\|)_{n\geq 1}$ is bounded and $x_n \to x$, a.s., we have $x \in H$.

See, however, the footnote at the end of Section 2.

Proof of the "if" part of Theorem 4: Suppose that (11) is true. For all $\delta \in (0, 1]$ real number we define

$$C_\delta \triangleq \{x \in N_+ \ : \ \|x\|_\infty \leq 1 \ , \ E_P x \geq \delta\} \ . \tag{16}$$

The set C_δ is convex and, by Lemma 7 and the Dominated Convergence Theorem, closed as a subset of L^∞ in the $\sigma(L^\infty, L^1)$ topology. Since it is bounded in the norm topology of L^∞, it must be compact in the $\sigma(L^\infty, L^1)$ topology (Dunford and Schwartz, (1958), Ch. V, Section 4, Corollary 3). From Lemma 6 we conclude that C_δ, as a subset of N, is compact in the $\sigma(N, M)$ topology.

From Grothendieck (1973), Ch.2, Sec.5, Proposition 8 follows that $\overline{V_1^0 - L_+^\infty}$ is convex and by condition (11) it is disjoint from C_δ , thus a separation theorem (Grothendieck, (1973), Ch.2, Sec.7, Proposition 10) guarantees the existence of $y_\delta \in M$ such that

$$E_P\big[zy_\delta\big] < E_P\big[xy_\delta\big] \ , \text{ for all } z \in \overline{V_1^0 - L_+^\infty} \ , \ x \in C_\delta \ . \tag{17}$$

Notice that in the above propositions of Grothendieck it is not assumed that the topology is Hausdorff, i.e., that N is separated in M.

Since V_1^0 is a linear subspace, it follows that

$$E_P\big[y_\delta z\big] = 0 \ , \quad \text{for all } z \in V_1^0 \tag{18}$$

and

$$E_P\big[y_\delta x\big] > 0 \ , \quad \text{for all } x \in C_\delta \ . \tag{19}$$

It follows that $y_\delta \geq 0$, because for $\lambda \in \Re_+$ the variable $-\lambda 1_{\{y_\delta < 0\}}$ is in $\overline{V_1^0 - L_+^\infty}$ thus

$$E_P\big[-\lambda 1_{\{y_\delta<0\}} y_\delta\big] \ < \ E_P\big[y_\delta x\big] \tag{20}$$

for all $x \in C_\delta$. However, the left–hand side converges to infinity as $\lambda \to \infty$, unless $P(y_\delta < 0) = 0$.

From (19) follows that $1_{\{y_\delta=0\}}$ is not in C_δ, thus $P(y_\delta = 0) < \delta$ and $E_P[y_\delta] > 0$. Let w_δ be the normalized y_δ:

$$w_\delta \triangleq \frac{1}{E_P[y_\delta]} y_\delta \ . \tag{21}$$

We summarize what we have achieved so far: for every $\delta \in (0,1]$ we found $w_\delta \in M$ such that $w_\delta \geq 0$, a.s., $E_P[w_\delta] = 1$, $P(w_\delta = 0) < \delta$, and $E_P[w_\delta z] = 0$ for all $z \in V_1^0$. Now we introduce

$$w \triangleq \sum_{n=1}^{\infty} \frac{1}{2^n} w_{\frac{1}{n}} \ , \tag{22}$$

where the infinite sum is a pointwise limit. By the Monotone Convergence Theorem $E_P w = 1$ and $w < \infty$, a.s. Clearly we have $w > 0$, a.s. We need to show that $E_P[wz] = 0$ for all $z \in V_1^0$. By Lemma 2 we may assume that z is bounded below. Let $z = z_+ - z_-$ where z_+ and z_- are non–negative and z_- is bounded. By the Monotone Convergence Theorem

$$E_P\left[z_+ w\right] = E_P\left[\sum_{n=1}^{\infty} \frac{1}{2^n} w_{\frac{1}{n}} z_+\right] = \sum_{n=1}^{\infty} \frac{1}{2^n} E_P\left[w_{\frac{1}{n}} z_+\right]$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n} E_P\left[w_{\frac{1}{n}} z_-\right] = E_P\left[w z_-\right] \ < \ \infty \ ,$$

thus $E_P[wz] = 0$. We define the equivalent measure Q by

$$\frac{dQ}{dP} = w \ ,$$

which is a martingale measure by Lemma 5. Q.E.D.

The assumption that $\xi(\tau)$ is bounded below for all stopping times τ and $\xi \in W$ has only been used at the end of the above proof. This assumption guarantees that if w_δ has property (18) than the infinite convex linear combination of (22) satisfies the same property.

4. Selecting a topology with an economic interpretation

In this section we are going to assume that there exists a probability measure $\tilde{P}$ equivalent to P such that V_1^0 (thus also N) is included in $L^1(\tilde{P})$. Notice that if such probability measure does not exist then no martingale measure exists either. With this assumption we are not back to the cases studied by earlier papers (Ansel & Stricker (1990), Stricker (1990), Duffie & Huang (1986)) because in the present paper the concepts of martingale measure and free lunch are invariant under the substitution of the reference measure by an equivalent probability measure.

We define the class

$$\Pi = \{Q \sim P : \ N \subset L^1(Q)\}, \tag{23}$$

where $Q \sim P$ means that Q and P are equivalent. By our assumption $\tilde{P} \in \Pi$.

8. Lemma: If Π is non-empty, then $\sigma(N, M)$ is a Hausdorff topology on N.

Proof: The topologies $\sigma(N, M)$ and $\sigma(N, M(\tilde{P}))$ are identical. However, $L^\infty \subset M(\tilde{P})$, $N \subset L^1(\tilde{P})$, and L^∞ separates the elements of $L^1(\tilde{P})$. Thus $M(\tilde{P})$ separates the elements of N. Q.E.D.

From the above lemma follows that any topology on N stronger than $\sigma(N, M)$ is also Hausdorff.

We denote by T the weakest topology on N which is stronger than each norm-topology $L^1(Q)$ for all $Q \in \Pi$, i.e., the upper bound of these topologies.

9. Proposition: The topology T is a locally convex Hausdorff topology on N compatible with the duality $\langle N, M \rangle$.

Proof: A neighborhood base of the origin in T is given by finite intersections of sets $D(a, Q) = \{x \in N : \ E_Q|x| \le a\}$, where $Q \in \Pi$ and a is a positive real number. The local convexity of T follows. One can see easily that $C(\frac{a}{2}, Q) \subset D(a, Q) \subset C(a, Q)$ for all $Q \in \Pi$ and $a > 0$, where

$$
\begin{aligned}
C(b, Q) &\stackrel{\triangle}{=} \big\{x \in N : \ |E_P xy| \le b \text{ for all } y \in M \text{ such that } |y| \le \frac{dQ}{dP}\big\} \\
&= \big\{x \in N : \ |E_Q xy| \le b \text{ for all } y \in B_1\big\}, \qquad (24)
\end{aligned}
$$

and B_1 is the unit ball of L^∞. From the first line of (24) follows that T is the topology of uniform convergence on finite unions of subsets of M of the form $I(Q) \stackrel{\triangle}{=} \{y \in M : \ |y| \le \frac{dQ}{dP}\}$ where Q runs through Π. As each $y \in M$ is contained in $u \times I(Q)$ for some positive real u and $Q \in \Pi$, T is stronger than $\sigma(N, M)$.

Let us fix now an arbitrary $Q \in \Pi$ and $b > 0$. We have to show that $C(b, Q)$ is a neighborhood of zero in the Mackey topology. The Mackey topology $\tau(N, M) = \tau(N, M(Q))$ is the topology of uniform convergence on $\sigma(M(Q), N)$–compact, convex, circled subsets of $M(Q)$. The unit ball B_1 is $\sigma(L^\infty, L^1(Q))$–compact thus also compact in the $\sigma(M(Q), N)$ topology (one can imitate the proof of Lemma 6). Therefore, by the second line of (24), $C(b, Q)$ is indeed a neighborhood of zero in the Mackey topology.
Q.E.D.

We can cast Theorem 4 in the following form: a martingale measure exists if and only if the T–closure of $V_1^0 - L_+^\infty$ is disjoint from $N_+ \setminus \{0\}$. The economic interpretation of this result is the following. We call Π the set of admissible subjective probability measures. Thus a probability measure Q is admissible if it is equivalent to P, and gains associated with very simple trading strategies have finite Q–expectation. We suppose that each agent in this economy has an admissible subjective probability measure, and each of these measures may be selected by an agent. Now what is an agreeable topology on N for every possible agent in the economy? Of course an agent with subjective probability measure Q would propose his/her subjective norm–topology $L^1(Q)$. However, we do not appropriate any special significance to an individual agent versus the rest of the agents.

In our topology T a sequence (or generalized sequence) $(x_\alpha) \subset N$ converges to $x \in N$ if and only if it converges in the norm-topology $L^1(Q)$ for *all* admissible Q. Thus T is a topology in which two elements of N are "close" if they are "close" according to the subjective opinion of each individual agent. The topology T is based on the consensus of all agents. It is derived from the L^1–topologies which are more commonly used in the finance literature than weak topologies.

In the rest of this section we define the concepts of "subjective free lunch" and "global free lunch", and give a condition for the existence of a martingale measure in terms of these concepts. We recall the set of very simple trading policies $\mathcal{A}$ and the gains corresponding to a very simple trading policy given by (14).

10. Definition: A *subjective free lunch* for an agent with admissible subjective probability measure $Q \in \Pi$ is a generalized sequence (net) $(a_\alpha, b_\alpha) \subset \mathcal{A} \times L^\infty_+$, an element $c \in N_+$ such that $P(c > 0) > 0$ and $G(a_\alpha) - b_\alpha \to c$ in $L^1(Q)$.

Next we define "global free lunch" in terms of the topology T.

11. Definition: A *global free lunch* is a generalized sequence $(a_\alpha, b_\alpha) \subset \mathcal{A} \times L^\infty_+$, an element $c \in N_+$ such that $P(c > 0) > 0$ and $G(a_\alpha) - b_\alpha \to c$ in the T topology.

Recall the definition of free lunch at the end of Section 2. The closure of a convex set is the same in all topologies compatible with the duality $\langle N, M \rangle$, thus free lunch exists if and only if global free lunch exists.

It follows from the definition of T that a generalized sequence $(a_\alpha, b_\alpha) \subset \mathcal{A} \times L^\infty_+$ and an element $c \in N_+$ represents a global free lunch if and only if it is a subjective free lunch for every possible individual agent in the economy, i.e., if there is a consensus among all possible agents.

Now Theorem 4 and the above definitions imply the following

12. Proposition: The following statements are equivalent:

i) there exists a martingale measure,

ii) there is no global free lunch in the market;

iii) there is no common subjective free lunch for all possible agents in the market simultaneously .

Thus using the topology T the concept of free lunch has clear economic interpretation. An appropriate generalized sequence is a free lunch if and only if it is regarded as a subjective free lunch by all possible agents. The other advantage of this formulation is the fact that the concept of subjective free lunch is based on an $L^1(Q)$ topology which is more conventional in the finance literature.

5. References

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