OPTIMAL TRADING STRATEGY FOR AN INVESTOR: THE CASE OF PARTIAL INFORMATION

PETER LAKNER*
New York University, Department of Statistics and Oper. Res.
44 W. 4th St., New York, NY 10012

Abstract:

We shall address here the optimization problem of an investor who wants to maximize the expected utility from terminal wealth. The novelty of this paper is that the drift process and the driving Brownian Motion appearing in the stochastic differential equation for the security prices are not assumed to be observable for investors in the market. Investors observe security prices and interest rates only. The drift process will be modelled by a Gaussian process, which in a special case becomes a multi-dimensional mean-reverting process. The main result of the paper is an explicit representation for the optimal trading strategy for a wide range of utility functions.

Keywords: Utility function, security prices and their filtration, trading strategy, optimization, gradient operator, Clark's formula.

* e-mail: plakner@stern.nyu.edu phone: (212) 998-0476

1. Introduction

In this paper we solve a utility maximization problem of an investor who wants to maximize the expected utility from the terminal value of his/her portfolio on the finite time interval $[0, T]$. We assume that there are N risky securities $(S_1(t), \ldots, S_N(t))$ available in the market whose dynamics are given by (2.1), and there is a fixed interest rate r. This problem has been widely studied, for example by Cox & Huang (1989), Cox at al. (1985), Duffie & Zame (1989), Karatzas at al. (1991), Karatzas at al. (1987) or Ocone & Karatzas (1991). The special feature of this paper is that we shall *not* assume that investors can observe the drift process μ_t and the Brownian Motion appearing in the stochastic differential equation for the security prices. We shall call this situation the case of *partial information* to distinguish it from the case of "full information" studied in the above papers. Clearly, it is more realistic to assume that investors have only partial information since prices and interest rates are published and available to the public, but drifts and paths of Brownian Motions are mere mathematical tools for model creation, but certainly not observable. The fact that investors have only partial information will be modelled by requiring that trading strategies are adapted to the filtration generated by the security prices, which is smaller than the original filtration.

The problem of partial information was discussed already in Lakner (1995) where a formula was presented for the optimal level of terminal wealth, and the existence of a corresponding trading strategy has been shown. The main objective of the present paper is to work out explicit formula for the optimal trading strategy as well. The drift process μ will be a Gaussian process modelled by a system of linear stochastic differential equations where the driving Brownian Motion is independent from the one appearing in the equation for the security prices, and in a special case μ becomes a multidimensional mean-reverting process. Having specified the dynamics of the drift this way, we can make the general formula of Lakner (1995) for the optimal level of terminal wealth more explicit by computing m_t, which is the conditional expectation of the drift μ_t given the available information. The formula for the optimal trading strategy will involve the process m as well. Two specific examples will be worked out, one for the logarithmic and the other for the power utility function. With the logarithmic utility function the optimal trading strategy can be written in a feedback form which can be formally "derived" from the corresponding formula in the full information case by substituting m for μ. However, it will be shown that with the power utility function the formal substitution of m for μ in the feedback form of the optimal trading strategy in the full information case does not yield the correct formula for the optimal trading strategy in the partial information case. (See also Browne & Whitt (1996) for similar example in a discrete time model.)

The computation of the optimal trading strategy basically amounts to finding the integrand in the stochastic integral representation of the optimal terminal wealth. The technique used here involves the gradient operator D, as in Ocone & Karatzas (1991), in which the optimal trading strategy under full information is computed using the same technique. We are using that paper as our basic reference for information on the gradient operator.

The optimal trading strategy has been worked out for the "Bayesean" case by Browne & Whitt (1996) for the logarithmic utility, and by Lakner (1994) for general utility functions. The word Bayesean means here that μ is an unobserved random variable with a known prior distribution.

The organization and basic content of the paper is the following. In Section 2 we describe the market model and recall the general formula for the optimal terminal wealth. This will involve a process ζ which is the conditional expectation of the Radon-Nicodym derivative of the "martingale measure" with respect to the original probability measure. In Section 3 we show that ζ satisfies a stochastic differential equation and write down the solution. This will now involve the above mentioned conditional expectation m_t of the drift μ_t given the available information. However, we can not compute m_t without specifying the dynamics of the unobserved drift process, which will be done in the beginning of Section 4. Then we can write down the filtering equations for the conditional mean vector m and the conditional covariance matrix γ. It turns out that γ is deterministic, and is the solution of a system of first-order nonlinear ordinary differential equations. We have an explicit solution for this in the case when $N = 1$. In the multidimensional case we have to rely on numerical methods for computing γ. The equation for m is linear thus we can solve it in terms of γ. Next the main theorem is stated, which presents our formula for the optimal trading strategy. This formula involves the previously described processes ζ and m and the deterministic covariance function γ. We specialize the formula for the optimal trading strategy for the logarithmic and the power utility functions. In Section 5 the proof of the main theorem will be presented. We start with recalling the definition of the gradient operator and quote the relevant version of Clark's formula from Karatzas, Ocone, & Li (1991). The proof itself will be broken down to several lemmas. Section 6 (Appendix) contains the proof of a lemma in Section 4.

2. Description of the model and recollection of earlier results

Let (Ω, F, P), $\mathcal{F} = \{\mathcal{F}_t;\ 0 \leq t \leq T\}$ be a complete filtered probability space with a fixed terminal time $T > 0$. There are N risky securities on this space with the N–dimensional price process $S = \{S_t = (S_1(t), \ldots, S_N(t))^*;\ t \in [0, T]\}$ (the asterisk signifies transposition). The dynamics of these processes are determined by the system of stochastic differential equations

$$dS_i(t) = \mu_i(t) S_i(t) dt + S_i(t) \sum_{j=1}^{N} \sigma_{ij} dw_j^{(1)}(t) \ . \tag{2.1}$$

In the above equation the drift $\mu = \{\mu_t = (\mu_1(t), \ldots, \mu_N(t))^*;\ t \in [0, T]\}$ is an adapted, measurable N-dimensional process such that

$$\int_0^T \|\mu_u\|^2 du < \infty \ , \quad \text{a.s.}, \tag{2.2}$$

where $\|\cdot\|$ is the Euclidean norm. The process $w^{(1)} = \{w_t^{(1)} = (w_1^{(1)}(t), \ldots, w_N^{(1)}(t))^*; \ t \in [0,T]\}$ is an N-dimensional Brownian Motion, and $\sigma = (\sigma_{ij})_{i,j=1,N}$ is a nonsingular matrix of constants. Let r be a constant deterministic interest rate. We suppose that the initial prices $S_i(0), i = 1, \ldots, N$ are deterministic positive constants. Let $\mathcal{F}^S = \{\mathcal{F}_t^S \ ; \ t \leq T\}$ be the augmented filtration generated by the price process S. In this paper we shall assume that only $\mathcal{F}^S$-adapted processes are observable, so agents in this market do *not* observe the Brownian Motion $w^{(1)}$ and the drift process μ. The constant interest rate r, the initial price vector S_0 and the volatility matrix σ are known to all agents acting in the market. We define the positive local martingale $Z = \{Z_t \ ; \ t \leq T\}$ by the equation

$$dZ_t = -(\mu_t - r\mathbf{1})^*(\sigma^*)^{-1} Z_t dw_t^{(1)}, \tag{2.3}$$

$$Z_0 = 1 \ , \tag{2.4}$$

where $\mathbf{1}$ is the N-dimensional vector with all entries equal to 1. Equations (2.3)–(2.4) have the unique solution

$$Z_t = \exp\{-\int_0^t (\mu_u - r\mathbf{1})^*(\sigma^*)^{-1} dw_u^{(1)} - \frac{1}{2}\int_0^t \|\sigma^{-1}(\mu_u - r\mathbf{1})\|^2 du \ . \tag{2.5}$$

2.1 Assumption: We shall assume that Z is a martingale.

Next we shall define a trading strategy for an agent acting in this market. Let $\pi_i(t)$ be the amount of money invested in the i-th security at time t.

2.2 Definition: A trading strategy $\pi = \{\pi_t = (\pi_1(t), \ldots, \pi_N(t))^*; \ 0 \leq t \leq T\}$ is an N-dimensional, measurable, $\mathcal{F}^S$-adapted process such that

$$\int_0^T \|\pi_t\|^2 dt < \infty \ , \quad \text{a.s.} \tag{2.6}$$

We emphasize that a trading strategy is required to be $\mathcal{F}^S$-adapted, thus investors indeed observe the security prices only, not the drift μ or the Brownian Motion $w^{(1)}$. Let X_t be the wealth at time t of an agent who follows the trading strategy π. The initial wealth $X_0 = x_0$ is a deterministic constant. The process $X = \{X_t; \ t \in [0,T]\}$ is assumed to evolve according to the dynamics

$$dX_t = \pi_t^* \mu_t dt + \pi_t^* \sigma dw_t^{(1)} + (X_t - \pi_t^* \mathbf{1}) r dt. \tag{2.7}$$

Ito's rule implies that the discounted wealth $e^{-rt}X_t$ has the form

$$d(e^{-rt}X_t) = e^{-rt}\pi_t^*\sigma d\tilde{w}_t, \tag{2.8}$$

where

$$\tilde{w}_t = w_t^{(1)} + \int_0^t \sigma^{-1}(\mu_u - r\mathbf{1})du. \tag{2.9}$$

By Girsanov's Theorem and Assumption 2.1, the N-dimensional process $\tilde{w} = \{\tilde{w}_t = (\tilde{w}_1(t), \ldots, \tilde{w}_N(t))^*; 0 \leq t \leq T\}$ is a Brownian Motion under the probability measure $\tilde{P}$ where

$$\frac{d\tilde{P}}{dP} = Z_T. \tag{2.10}$$

We denote by $\tilde{E}$ the expectation operator corresponding to the measure $\tilde{P}$.

2.3 Definition: A trading strategy π is called *admissible* if $X_t \geq 0$, a.s., $t \in [0,T]$.

2.4 Definition: A function $U : [0,\infty) \mapsto \Re \cup \{-\infty\}$ is called a *utility function* if it is continuous, strictly increasing, strictly concave on its domain, continuously differentiable on $(0,\infty)$ with derivative function $U'(\cdot)$ satisfying the relation

$$\lim_{x\to\infty} U'(x) = 0\ . \tag{2.11}$$

Our *optimization problem* is to maximize the expected utility from terminal wealth, i.e.,

$$\max E[U(X_T)]$$

over all admissible trading strategies.

We define the N–dimensional return process $R = \{R_t = (R_1(t), \ldots, R_N(t))^*;\ t \in [0,T]\}$ by

$$dS_i(t) = S_i(t)dR_i(t), \quad i = 1, \ldots, N, \tag{2.12}$$

so we have the following decompositions for the return process:

$$dR_t = \mu_t dt + \sigma dw_t^{(1)}, \tag{2.13}$$

and

$$dR_t = r\mathbf{1}dt + \sigma d\tilde{w}_t. \tag{2.14}$$

Relations (2.12) and (2.14) imply that S, R, and $\tilde{w}$ each generate the same filtration. Thus $\mathcal{F}^S$ is continuous (Karatzas & Shreve (1988), Corollary 2.7.8).

Let $\zeta = \{\zeta_t, t \in [0,T]\}$ be the optional projection of the P–martingale Z to $\mathcal{F}^S$, so

$$\zeta_t = E[Z_t|\mathcal{F}^S], \quad \text{a.s.}, t \in [0,T]. \tag{2.15}$$

We note that ζ is a martingale with respect to $(P, \mathcal{F}^S)$, and for every $\mathcal{F}_t^S$-measurable random variable V, $\mathcal{F}_u$-measurable random variable Y , and $\mathcal{F}_u^S$-measurable random variable W with $0 \le t \le u \le T$

$$\tilde{E}V = E\zeta_t V, \tag{2.16}$$

$$\tilde{E}[Y|\mathcal{F}_t^S] = \frac{1}{\zeta_t} E[Z_u Y|\mathcal{F}_t^S] \ , \tag{2.17}$$

and

$$\tilde{E}[W|\mathcal{F}_t^S] = \frac{1}{\zeta_t} E[\zeta_u W|\mathcal{F}_t^S] \ . \tag{2.18}$$

The last identity implies that $\frac{1}{\zeta}$ is a $(\tilde{P}, \mathcal{F}^S)$–martingale. Since $\mathcal{F}^S$ is generated by $\tilde{w}$, so $\frac{1}{\zeta}$, and also ζ, must be continuous.

Let the function $I : (0, \infty) \mapsto [0, \infty)$ be the pseudo inverse function of the strictly decreasing derivative of the utility function:

$$I(y) = \inf\{x \ge 0 : \ U'(x) \le y\}. \tag{2.19}$$

The above defined function I actually becomes the inverse function of U' if $\lim_{x\to 0} U'(x) = \infty$. However, we did not make this assumption.

We recall the following theorem from Lakner (1995).

2.5 Theorem: Suppose that for every constant $x \in (0, \infty)$

$$\tilde{E}[I(x\zeta_T)] < \infty \ . \tag{2.20}$$

Then the optimal level of terminal wealth is

$$\hat{X}_T = I(ye^{-rT}\zeta_T) \ , \tag{2.21}$$

where the constant y is uniquely determined by

$$\tilde{E}[e^{-rT} I(ye^{-rT}\zeta_T)] = x_0 \ . \tag{2.22}$$

The optimal wealth process $\hat{X}$ and the trading strategy $\hat{\pi}$ is implicitly determined by

$$e^{-rt}\hat{X}_t = \tilde{E}\left[e^{-rT} I(ye^{-rT}\zeta_T) \ \middle| \ \mathcal{F}_t^S\right] = x_0 + \int_0^T e^{-rt}\hat{\pi}_t^* \sigma d\tilde{w}_t \ . \tag{2.23}$$

3. Explicit representation of the optimal terminal wealth level

Formula (2.21) for the optimal level of terminal wealth involves the random variable ζ_T and we shall find a way to compute it in this section. We introduce the conditional mean vector and covariance matrix of μ_t

$$m_t = E[\mu_t|\mathcal{F}_t^S], \tag{3.1}$$

and

$$\gamma(t) = E[(\mu_t - m_t)(\mu_t - m_t)^* | \mathcal{F}_t^S], \tag{3.2}$$

with the understanding that the processes m and γ are measurable versions of the appropriate conditional expectations. The result for the representation of ζ_t will be formulated in the following theorem.

3.1 Theorem: Suppose that

$$E[\|\mu_t\| < \infty, \quad t \in [0,T] \tag{3.3}$$

and the N–dimensional process m is continuous. Then the process $\frac{1}{\zeta}$ satisfies the stochastic differential equation

$$d\left(\frac{1}{\zeta_t}\right) = \frac{1}{\zeta_t}(m_t - r\mathbf{1})^*(\sigma^*)^{-1} d\tilde{w}_t \ , \tag{3.4}$$

and we have the representation

$$\zeta_t = \exp\{-\int_0^t (m_u - r\mathbf{1})^*(\sigma^*)^{-1} d\tilde{w}_u + \frac{1}{2}\int_0^t \|\sigma^{-1}(m_u - r\mathbf{1})\|^2 du\} \ . \tag{3.5}$$

Proof: From (2.5) and (2.9) follows that

$$Z_t = \exp\{-\int_0^t (\mu_u - r\mathbf{1})^*(\sigma^*)^{-1} d\tilde{w}_u + \frac{1}{2}\int_0^t \|\sigma^{-1}(\mu_u - r\mathbf{1})\|^2 du\} \ , \tag{3.6}$$

thus the process $\frac{1}{Z}$ satisfies the equation

$$d\left(\frac{1}{Z_t}\right) = \frac{1}{Z_t}(\mu_t - r\mathbf{1})^*(\sigma^*)^{-1} d\tilde{w}_t \ , \tag{3.7}$$

Liptser & Shiryayev I (1977), p.185, Theorem 5.14 guarantees that

$$\tilde{E}\left[\int_0^t (\mu_u - r\mathbf{1})^*(\sigma^*)^{-1}\frac{1}{Z_u} d\tilde{w}_u \ \middle| \ \mathcal{F}_t^S\right] = \int_0^t \tilde{E}\left[(\mu_u - r\mathbf{1})^*(\sigma^*)^{-1}\frac{1}{Z_u} \ \middle| \ \mathcal{F}_u^S\right] d\tilde{w}_u \ , \tag{3.8}$$

provided that the following two conditions hold:

$$\tilde{E}\left[\Big|\sum_{i=1}^N (\mu_i(u) - r)s_{ij}\frac{1}{Z_u}\Big|\right] < \infty, \quad j = 1,\dots,N, \ u \in [0,T], \tag{3.9}$$

and

$$\int_0^T \left(\tilde{E}\left[\frac{1}{Z_u}\sum_{i=1}^N (\mu_i(u) - r)s_{ij} \ \middle| \ \mathcal{F}_u^S\right]\right)^2 du < \infty, \quad \text{a.s.}, \ j = 1,\dots,N, \tag{3.10}$$

where s_{ij} is the (i,j)-entry of the matrix $(\sigma^*)^{-1}$. However, (3.9) follows from (3.3) because

$$\tilde{E}\left|\sum_{i=1}^{N}(\mu_i(u)-r)s_{ij}\frac{1}{Z_u}\right| = E\left|\sum_{i=1}^{N}(\mu_i(u)-r)s_{ij}\right| < \infty, \quad j=1,\ldots,N,\ u\in[0,T]. \quad (3.11)$$

The left-hand side of (3.10) can be written as

$$\int_0^T \frac{1}{\zeta_u^2}\left(E\left[\sum_{i=1}^{N}(\mu_i(u)-r)s_{ij}\ \Big|\ \mathcal{F}_u^S\right]\right)^2 du = \int_0^T \frac{1}{\zeta_u^2}\left(\sum_{i=1}^{N}(m_i(u)-r)s_{ij}\right)^2 du\ , \quad (3.12)$$

and this last expression is almost surely finite because of the continuity of m and ζ. Now (3.7) and (3.8) imply that

$$\tilde{E}\left[\frac{1}{Z_t}\ \Big|\ \mathcal{F}_t^S\right] = 1 + \int_0^t \tilde{E}\left[(\mu_u - r\mathbf{1})^*(\sigma^*)^{-1}\frac{1}{Z_u}\ \Big|\ \mathcal{F}_u^S\right]d\tilde{w}_u\ , \quad (3.13)$$

By (2.17) the left–hand side of this last identity is equal to $\frac{1}{\zeta_t}$, and by (2.17) and (3.1) the right–hand side is equal to

$$1 + \int_0^t \frac{1}{\zeta_u}(m_u - r\mathbf{1})^*(\sigma^*)^{-1}d\tilde{w}_u\ ,$$

so (3.4) follows. Identity (3.5) is an obvious consequence of (3.4), thus our proof is complete.

Now (3.5) represents a formula for ζ, but it is still not explicit enough because it involves the process m for which we still don't have a computable representation. We can say more about m only if we specify the dynamics of the drift μ, and this will be done in the next section.

4.Explicit formula for the optimal trading strategy

For the rest of this paper we shall assume that the N–dimensional drift process μ is the solution of the stochastic differential equation:

$$d\mu_t = \alpha(\delta - \mu_t)dt + \beta dw_t^{(2)}\ , \quad (4.1)$$

where $w^{(2)}$ is an N-dimensional Brownian Motion with respect to $(\mathcal{F},P)$, independent of $w^{(1)}$ under P, α and β are known $N\times N$ matrices of real numbers, and δ is a known N-dimensional vector of real numbers.We shall assume that β is invertible, and that μ_0 follows an N–dimensional normal distribution with mean vector m_0 and covariance matrix γ_0. The vector m_0 and the matrix γ_0 are assumed to be known to all agents in the market. We note that if α is a diagonal matrix with positive entries in the diagonal, then μ will be an N-dimensional "mean-reverting" process.

We shall also assume that $\mathrm{tr}(\gamma_0)$ and $\|\beta\|$ are "small". To be more rigorous, we shall assume that

$$\mathrm{tr}(\gamma_0) + T\|\beta\|^2 < K_1 \tag{4.2}$$

where

$$K_1 = \frac{1}{360T\|\sigma^{-1}\|^2 K} \tag{4.3}$$

and

$$K = \max_{t \leq T} \|e^{-\alpha t}\|^2. \tag{4.4}$$

Assumption (4.2) roughly means that the variances of the components of the drift μ_t are "small" compared to the variances of the components of the return process R, which is defined in (2.12)-(2.13) (see (6.5) in the Appendix for the covariance matrix of μ_t). We note that if α is a positive semidefinite symmetric matrix then $K = N$ because in that case we can write α in the form $\alpha = T\Lambda T^*$, where T is an orthogonal matrix and Λ is diagonal with the non-negative entries $\lambda_1, \ldots, \lambda_N$ in the diagonal. Now using elementary matrix algebra we can compute

$$\left\| e^{-\alpha t} \right\|^2 = \mathrm{tr}\Big(e^{-\alpha t} e^{-\alpha t} \Big) = \mathrm{tr}\Big(T e^{-t\Lambda} e^{-t\Lambda} T^* \Big)$$

$$= \mathrm{tr}\Big(e^{-t\Lambda} e^{-t\Lambda} \Big) = \sum_{i=1}^{N} e^{-2t\lambda_i} \leq N.$$

4.1 Lemma: With the above specified drift process μ, Assumption 2.1. is satisfied. Furthermore,

$$E\Big[Z_T^5 + Z_T^{-4} \Big] + \tilde{E}\Big[\zeta_T^4 + \zeta_T^{-5} \Big] < \infty \ . \tag{4.5}$$

We defer the proof to the Appendix.

We can use the return process R of (2.12) as the "observation"process since it generates the same filtration as the price process S. If we do so then we are exactly in the framework of Liptser & Shiryayev II (1978), Theorem 12.7, which guarantees that the pair m, γ is the unique $\mathcal{F}^S$-measurable solution of the system

$$dm_t = [-\alpha - \gamma(t)(\sigma\sigma^*)^{-1}] m_t dt + \gamma(t)(\sigma\sigma^*)^{-1} dR_t + \alpha\delta dt \tag{4.6}$$

$$\dot{\gamma}(t) = -\gamma(t)(\sigma\sigma^*)^{-1}\gamma(t) - \alpha\gamma(t) - \gamma(t)\alpha^* + \beta\beta^* \tag{4.7}$$

with the initial conditions (m_0, γ_0). It follows that the conditional covariance matrix $\gamma(t)$ is deterministic. In the case when $N > 1$ we don't have an explicit formula for γ.

However, in terms of γ we can solve for the conditional mean m in the following way. Let $\phi : [0,T] \mapsto \Re^{N\times N}$ the fundamental solution of the deterministic system

$$\dot{\phi}(t) = [-\alpha - \gamma(t)(\sigma\sigma^*)^{-1}]\phi(t) \ , \tag{4.8}$$

i.e., ϕ is an $N \times N$-matrix valued function satisfying (4.8) with the initial condition that $\phi(0)$ is the $N \times N$ identity matrix. Then m_t is determined in terms of γ and ϕ as

$$m_t = \phi(t)\left[m_0 + \int_0^t \phi^{-1}(s)\gamma(s)(\sigma\sigma^*)^{-1}dR_s + \int_0^t \phi^{-1}(s)ds \ \alpha\delta\right] . \tag{4.9}$$

4.2 Remark: In the case of $N = 1$ we do have an explicit representation for $\gamma(t)$. Equation (4.7) becomes

$$\dot{\gamma}(t) = -\frac{1}{\sigma^2}\gamma^2(t) - 2\alpha\gamma(t) + \beta^2 \tag{4.10}$$

which has the solution

$$\gamma(t) = \sqrt{C}\sigma\frac{C_1\exp\{2\frac{\sqrt{C}}{\sigma}t\} + C_2}{C_1\exp\{2\frac{\sqrt{C}}{\sigma}t\} - C_2} - \alpha\sigma^2 \tag{4.11}$$

where

$$C = \alpha^2\sigma^2 + \beta^2 \tag{4.12}$$

$$C_1 = \sqrt{C}\sigma + \gamma_0 + \alpha\sigma^2 \tag{4.13}$$

and

$$C_2 = -\sqrt{C}\sigma + \gamma_0 + \alpha\sigma^2 \ . \tag{4.14}$$

Also, in the one-dimensional case

$$\phi(t) = \exp\{-\alpha t - \frac{1}{\sigma^2}\int_0^t \gamma(u)du\} \tag{4.15}$$

and m_t is given explicitly by

$$m_t = \phi(t)\left[m_0 + \frac{1}{\sigma^2}\int_0^t \frac{\gamma(s)}{\phi(s)}dR_s + \alpha\delta\int_0^t \frac{1}{\phi(s)}ds\right] . \tag{4.16}$$

We return now to the discussion of the N-dimensional case. Notice that m_t is computable via (4.9) once the deterministic functions γ and ϕ are computed, which are solutions systems of ordinary first order differential equations. For our drift process μ specified at the beginning of this section, the conditions of Theorem 3.1 are satisfied so (3.4) and (3.5) must hold.

According to Liptser & Shiryayev II (1978), formula (12.65), the process

$$\bar{w}_t = \tilde{w}_t - \int_0^t \sigma^{-1}(m_u - r\mathbf{1})du \tag{4.17}$$

is a Brownian Motion with respect to $(P, \mathcal{F}^S)$. Now we are ready to state the main theorem of the paper:

4.3 Theorem: Suppose that U is twice continuously differentiable on $(0, \infty)$ and

$$I(x) < K_2(1 + x^{-5}) \tag{4.18}$$

$$-I'(x) < K_2(1 + x^{-2}) \tag{4.19}$$

for some $K_2 > 0$. Then the optimal trading strategy is

$$\hat{\pi}_t = H(t)\frac{1}{\zeta_t}E\Big[I'(ye^{-rT}\zeta_T)\zeta_T^2\Big\{-\gamma(t)(\phi^*(t))^{-1}\int_t^T \phi^*(u)(\sigma^*)^{-1}d\bar{w}_u - m_t + r\mathbf{1}\Big\}\Big|\mathcal{F}_t^S\Big] \tag{4.20}$$

where

$$H(t) = e^{r(t-2T)}y(\sigma\sigma^*)^{-1}, \tag{4.21}$$

ζ is given in (3.5), m is given in (4.9), and the constant y is uniquely determined by (2.22).

We defer the proof of this theorem to the next section and instead examine two special cases.

4.4 Example: Suppose that

$$U(x) = \log x \ , \tag{4.22}$$

which satisfies conditions (4.18) and (4.19). Condition (2.20) is satisfied by Lemma 4.1. Formulas (4.20), (4.21) and (2.22) yield that the optimal trading strategy is given by

$$\hat{\pi}_t = e^{rt}x_0(\sigma\sigma^*)^{-1}\frac{1}{\zeta_t}(m_t - r\mathbf{1}). \tag{4.23}$$

We can write this in a "feedback form"on the current level of wealth. Formula (2.21) becomes

$$e^{-rT}\hat{X}_T = x_0\frac{1}{\zeta_T} \tag{4.24}$$

and since both $\frac{1}{\zeta_t}$ and $e^{-rt}\hat{X}_t$ are $(\tilde{P}, \mathcal{F}^S)$-martingales, this implies

$$e^{-rt}\hat{X}_t = x_0\frac{1}{\zeta_t} \ . \tag{4.25}$$

Substituting this last expression into (4.23) we get the feedback form

$$\hat{\pi}_t = (\sigma\sigma^*)^{-1}(m_t - r\mathbf{1})X_t. \tag{4.26}$$

Notice that one can formally "derive"(4.26) in the following way. Consider the case of full information when the drift and the Brownian Motion appearing in the equation for the security prices are observable and the utility function is given by (4.22). In this case the optimal trading strategy has the feedback form (Ocone & Karatzas, (1991), formula (4.20))

$$(\sigma\sigma^*)^{-1}(\mu_t - r\mathbf{1})X_t, \tag{4.27}$$

and we can formally "derive"(4.26) if we substitute μ_t in (4.27) by its conditional mean m_t. Next another example will be shown which, besides having interest on its own, shows that formal substitution of m for μ in the feedback form of the optimal trading strategy in the full information case does not necessarily yield the correct formula for the optimal trading strategy in the partial information case. (We refer for another counterexample to Browne & Whitt (1996), for the Bayesean case in discrete time, i.e., when the drift is an unobservable random variable selected at time zero with a known "prior"distribution.)

4.5 Example: Suppose now that the utility function is

$$U(x) = \frac{1}{\lambda}x^{\lambda}\ , \tag{4.28}$$

where $\lambda < 0$. In this case

$$I(x) = x^{\frac{1}{\lambda-1}} \tag{4.29}$$

and

$$-I'(x) = \frac{1}{1-\lambda}x^{\frac{1}{\lambda-1}-1} \tag{4.30}$$

and it is clear that (4.18) and (4.19) hold. Condition (2.20) is also satisfied because

$$\tilde{E}\big[I(x\zeta_T)\big] = x^{\frac{1}{\lambda-1}}\tilde{E}\left[\zeta_T^{\frac{1}{\lambda-1}}\right] = x^{\frac{1}{\lambda-1}}E\left[\zeta_T^{\frac{\lambda}{\lambda-1}}\right] \le x^{\frac{1}{\lambda-1}}\Big(E\big[\zeta_T\big]\Big)^{\frac{\lambda}{\lambda-1}} = x^{\frac{1}{\lambda-1}} < \infty\ .$$

Now our formula for the optimal trading strategy becomes

$$\hat{\pi}_t = \frac{1}{1-\lambda}(\sigma\sigma^*)^{-1}(m_t - r\mathbf{1})\hat{X}_t \ + \ G_t \tag{4.31}$$

where

$$\begin{aligned} G_t = y^{\frac{1}{\lambda-1}}\frac{1}{1-\lambda}\exp\left\{r\left(t+\frac{T\lambda}{1-\lambda}\right)\right\}\frac{1}{\zeta_t}(\sigma\sigma^*)^{-1}\gamma(t)(\phi(t))^{-1} \\ \times E\left[\zeta_T^{\frac{\lambda}{\lambda-1}}\int_t^T \phi^*(u)(\sigma^*)^{-1}d\bar{w}_u \,\middle|\, \mathcal{F}_t^S\right] \end{aligned} \tag{4.32}$$

The optimal trading strategy for this utility function under full information is

$$\frac{1}{1-\lambda}(\sigma\sigma^*)^{-1}(\mu_t - r\mathbf{1})X_t$$

(Ocone & Karatzas, (1991), formula (4.22)), and our formula (4.31) for the case of partial information can not be derived from this by substituting m for μ because of the additional non-zero term G_t in (4.31).

One may find the constraint $\lambda < 0$ in (4.28) too restrictive. The problem with power utility functions with positive λ is that they do not satisfy (4.18) and (4.19). In the next proposition we overcome this problem by strengthening (4.2).

4.6 Proposition: Let $\theta \in (0,1)$ arbitrary, and instead of (4.2) assume the stronger

$$\mathrm{tr}(\gamma_0) + T\|\beta\|^2 < K_4 \tag{4.33}$$

where

$$K_4 = \frac{1}{8K\|\sigma^{-1}\|^2 T} \min\left\{ \frac{1}{45} \, , \, \frac{(1-\theta)^2}{(\theta+3)(\theta+7)} \right\} \tag{4.34}$$

and K is given in (4.4). Then for the power utility function of the form (4.28) with $0 < \lambda \leq \theta$, formulae (4.31) and (4.32) still yield the optimal trading strategy.

We defer the proof of this proposition to Section 6.

5. Proof of Theorem 4.3

For the convenience of the reader we recall the concept of the gradient operator D from Ocone & Karatzas (1991), and a generalized version of Clark's formula from Karatzas, Ocone & Li (1991). Let $\mathcal{S}$ be the class of "smooth functionals" of $\tilde{w}$, i.e., the class of $\mathcal{F}_T^S$-measurable random variables of the form

$$A = f(\tilde{w}_1(t_1), \ldots, \tilde{w}_m(t_m)) \ , \tag{5.1}$$

where $(t_1, \ldots, t_m) \in [0,T]^m$ and $f : \Re^{N\times m} \mapsto \Re$ is bounded and has bounded derivatives of all orders. For $A \in \mathcal{S}$, the gradient $DA = (D^1 A, \ldots, D^N A)^*$ is an $\big(L^2([0,T])\big)^N$-valued random variable with components

$$D^i A = \sum_{j=1}^{m} \frac{\partial}{\partial x^{ij}} f(\tilde{w}(t_1), \ldots, \tilde{w}(t_m)) 1_{[0,t_j]}(t) \ , \quad i = 1, \ldots, N. \tag{5.2}$$

The norm $\|\cdot\|_{1,1}$ is defined on $\mathcal{S}$ as

$$\|A\|_{1,1} = \tilde{E}\left[|A| + \left(\sum_{i=1}^{N} \int_0^T |D^i A(\omega)(t)|^2 dt \right)^{\frac{1}{2}} \right] . \tag{5.3}$$

Shigekawa (1980) showed that D is well-defined on the space $D_{1,1}$ by closure, so $D_{1,1}$ is a Banach space with the norm $\|\cdot\|_{1,1}$, and $\mathcal{S}$ is dense in $D_{1,1}$. According to the discussion in Ocone and Karatzas (1991) on the top of page 190, for every $A \in D_{1,1}$ there exists a measurable process $(t,\omega) \mapsto D_t A(\omega)$ such that $D_t A(\omega) = DA(\omega)(t)$ holds for almost every $(t,\omega) \in [0,T] \times \Omega$. Following the notation of that paper, we shall identify $DA(\omega)(t)$ and $D_t A(\omega)$ without further comment.

In order to compute a formula for the optimal trading strategy, we need the following version of Clark's formula (Karatzas, Ocone & Li (1991) or Karatzas & Ocone (1991), Proposition 2.1 and Remark 2.2). For every random variable $A \in D_{1,1}$ we have the stochastic integral representation

$$\tilde{E}[A \mid \mathcal{F}_t^S] = \tilde{E}A + \int_0^t \tilde{E}[(D_u A)^* \mid \mathcal{F}_u^S] d\tilde{w}_u \ . \tag{5.4}$$

For an N-dimensional random variable $A \in (D_{1,1})^N$, we define DA as an $N \times N$ dimensional matrix with components

$$(DA)_{i,j} = D^i A_j \ .$$

5.1 Definition: Let $\mathcal{L}$ be the class of $\Re^N$-valued, measurable, $\mathcal{F}^S$-adapted continuous processes $(s,\omega) \mapsto u(s,\omega)$ such that
(i) for almost every $s \in [0,T]$, $u(s,\cdot) \in (D_{1,1})^N$;
and $(t,s,\omega) \mapsto D_t u(s,\omega)$ admits a measurable version such that

$$(ii) \qquad \tilde{E}\left[\left(\int_0^T \|u(s)\|^2 ds\right)^{\frac{1}{2}} + \left(\int_0^T \int_0^T \|D_t u(s)\|^2 dt ds\right)^{\frac{1}{2}}\right] < \infty$$

(iii) $s \mapsto D_t u(s,\omega)$ is left (or right) continuous for almost every $(t,\omega) \in [0,T] \times \Omega$
(iv) for every $s \in [0,T]$, $(t,\omega) \mapsto D_t u(s,\omega)$ is $\mathcal{B}\big([0,T]\big) \otimes \mathcal{F}_s^S$-measurable.

According to the discussion on page 191 of Ocone & Karatzas (1991), $\mathcal{L}$ is included in $L_{1,1}^a$ defined on page 190 of the same paper.

We continue the section with the proof of Theorem 4.3. The main idea is to show that $\hat{X}_T = I(ye^{-rT}\zeta_T)$ is included in $D_{1,1}$, and to compute its gradient. This will be accomplished through several lemmas.

5.2 Lemma: Let $(A_n)_{n \geq 1} \subset D_{1,1}$ be a sequence of random variables, $\{h(t,\omega);\ t \leq T\}$ a measurable, N-dimensional process, and A a real random variable. Suppose that

$$\tilde{E}|A_n - A| \to 0 \text{ as } n \to \infty \tag{5.5}$$

and

$$\tilde{E}\left(\int_0^T \|D_t A_n - h(t)\|^2 dt\right)^{\frac{1}{2}} \to 0 \text{ as } n \to \infty \ . \tag{5.6}$$

Then $A \in D_{1,1}$ and $D_t A = h(t)$.

Proof: $(A_n)_{n\geq 1}$ is a Cauchy sequence in $D_{1,1}$ thus it converges to a limit which, by (5.5), can only be A. Hence $A \in D_{1,1}$ and $\|A_n - A\|_{1,1} \to 0$ as $n \to \infty$. This implies that

$$\tilde{E}\bigg(\int_0^T \|D_t A_n - D_t A\|^2 dt\bigg)^{\frac{1}{2}} \to 0 \text{ as } n \to \infty,$$

and in the light of (5.6) $D_t A = h(t)$ follows.

The following lemma spells out conditions under which the gradient operator and the ordinary Lebesgue-integral are exchangeable.

5.3 Lemma: Let $\{u(s,\omega);\ s \leq T\}$ be a real valued, continuous, measurable process such that $u(s) \in D_{1,1}$ for every $s \in [0,T]$,

$$\sup_{s\leq T} \tilde{E}\Big[|u(s)|^q\Big] < \infty \tag{5.7}$$

for some $q > 1$, and

$$\sup_{s\leq T} \tilde{E}\bigg[\int_0^T |D_t^j u(s)|^4 dt\bigg] < \infty\ ;\ j = 1,\dots,N. \tag{5.8}$$

Furthermore, we suppose that $s \mapsto D_t u(s,\omega)$ is left (or right) continuous for almost every $(t,\omega) \in [0,T] \times \Omega$. Then $\int_0^T u(s)ds \in D_{1,1}$ and

$$D_t \int_0^T u(s)ds = \int_0^T D_t u(s)ds\ . \tag{5.9}$$

Proof: We define the random variables

$$A_n = \sum_{i=1}^n \frac{T}{n} u\Big(\frac{iT}{n}\Big) \tag{5.10}$$

and

$$A = \int_0^T u(s)ds\ . \tag{5.11}$$

By the continuity of u we have

$$\lim_{n\to\infty} A_n = A\ , \quad \text{a.s.} \tag{5.12}$$

Jensen's inequality and (5.7) yield

$$\tilde{E}\big[|A_n|^q\big] \leq \frac{T^q}{n} \sum_{i=1}^n \tilde{E}\Big|u\Big(\frac{iT}{n}\Big)\Big|^q < \infty\ , \tag{5.13}$$

thus $(A_n)_{n\le 1}$ is uniformly integrable under $\tilde{P}$ and condition (5.5) of Lemma 5.2 is satisfied. Next we show that with

$$h(t) = \int_0^T D_t u(s)ds \tag{5.14}$$

condition (5.6) also holds. By the left or right continuity of $s \mapsto D_t u(s,\omega)$ we have

$$\lim_{n\to\infty} D_t A_n = h(t) \ . \tag{5.15}$$

In order to prove (5.6) it suffices to show that

$$\sup_n \tilde{E}\Big[\int_0^T \big|D_t^j A_n\big|^4 dt\Big] < \infty \ , \ j = 1, \ldots, N. \tag{5.16}$$

Using the elementary inequality $|\sum_{i=1}^n x_i|^4 \le n^3 \sum_{i=1}^n |x_i|^4$ we bound the left-hand side of the above inequality by

$$\sup_n \frac{T^4}{n}\tilde{E}\Big[\int_0^T \sum_{i=1}^n \Big|D_t^j u\Big(\frac{iT}{n}\Big)\Big|^4 dt\Big]$$

which is finite by condition (5.8). Hence both conditions (5.5) and (5.6) of Lemma 5.2 are satisfied which implies the statement of the present lemma.

We cast the conditional mean m_u of (4.9) in the form

$$m_u = \phi(u)\Big[m_0 + \Big(\int_0^u \phi^{-1}(s)ds\Big)\alpha\delta + \int_0^u \phi^{-1}(s)\gamma(s)(\sigma^*)^{-1}d\tilde{w}_s + \\ +r\Big(\int_0^u \phi^{-1}(s)\gamma(s)ds\Big)(\sigma\sigma^*)^{-1}\mathbf{1}\Big] \tag{5.17}$$

where $\phi(\cdot)$ is the fundamental solution of the system (4.8).

5.4 Lemma: For every $u \in [0,T]$, $m_u \in (D_{1,1})^N$ and

$$D_t m_u = \sigma^{-1}\gamma(t)\big(\phi^*(t)\big)^{-1}\phi^*(u)1_{\{t\le u\}} \ . \tag{5.18}$$

Proof: This follows from (5.17) and Ocone & Karatzas, (1991), Proposition 2.3.

5.5 Lemma: The following four relations hold:

$$\sup_{u\le T}\tilde{E}\Big[\|\sigma^{-1}(m_u - r\mathbf{1})\|^4\Big] < \infty \ , \tag{5.19}$$

$$\|\sigma^{-1}(m_u - r\mathbf{1})\|^2 \in D_{1,1} \ , \tag{5.20}$$

$$D_t\|\sigma^{-1}(m_u - r\mathbf{1})\|^2 = 2(D_t m_u)(\sigma\sigma^*)^{-1}(m_u - r\mathbf{1}) \ , \tag{5.21}$$

and

$$\sup_{u,t\le T} \tilde{E}\left[\left|D_t^j\|\sigma^{-1}(m_u - r\mathbf{1})\|^2\right|^4\right] < \infty \ ; \quad j = 1,\dots,N. \tag{5.22}$$

Proof: Using the elementary inequalities $\big(\sum_{i=1}^N a_i\big)^2 \le N\sum_{i=1}^N a_i^2$ and $(a-b)^4 \le 8\big((a-c)^4 + (c-b)^4\big)$ we compute

$$\tilde{E}\left[\|\sigma^{-1}(m_u - r\mathbf{1})\|^4\right] \le \|\sigma^{-1}\|^4 \tilde{E}\left[\|m_u - r\mathbf{1}\|^4\right] \le \|\sigma^{-1}\|^4 N \sum_{i=1}^N \tilde{E}(m_i(u) - r)^4$$

$$\le 8\|\sigma^{-1}\|^4 N \sum_{i=1}^N \left\{ \tilde{E}\big(m_i(u) - \tilde{E}m_i(u)\big)^4 + \big(\tilde{E}m_i(u) - r\big)^4 \right\} .$$

By (5.17), $m_i(u)$ follows normal distribution under $\tilde{P}$ with mean

$$\tilde{E}m_i(u) = \left(\phi(u)\left(m_0 + \left(\int_0^u \phi^{-1}(s)ds\right)\alpha\delta + r\left(\int_0^u \phi^{-1}(s)\gamma(s)ds\right)(\sigma\sigma^*)^{-1}\mathbf{1}\right)\right)_i \tag{5.23}$$

and variance

$$\mathrm{Var}(m_i(u)) = \int_0^u \left\| \left(\phi(u)\phi^{-1}(s)\gamma(s)(\sigma^*)^{-1}\right)_i \right\|^2 ds \ , \tag{5.24}$$

where a matrix followed by a subscript i is a notation for the i-th row vector of the matrix. These imply that

$$\tilde{E}\left[\left(m_i(u) - \tilde{E}m_i(u)\right)^4\right] = 3\left(\int_0^u \left\| \left(\phi(u)\phi^{-1}(s)\gamma(s)(\sigma^*)^{-1}\right)_i \right\|^2 ds\right)^2 \ , \tag{5.25}$$

thus both $u \mapsto \tilde{E}(m_i(u) - \tilde{E}m_i(u))^4$ and $u \mapsto (\tilde{E}m_i(u) - r)^4$ are continuous on $[0,T]$, which implies now (5.19). In order to prove (5.20) and (5.21) we shall apply Lemma A1 of Ocone & Karatzas (1991). Let $\psi_1 : \Re^n \mapsto \Re$ be defined as

$$\psi_1(m) = \|\sigma^{-1}(m - r\mathbf{1})\|^2 = \sum_{i=1}^N \left(\sum_{j=1}^N s_{j,i}(m_j - r)\right)^2 \tag{5.26}$$

where $s_{j,i}$ is the entry in the j-th row and i-th column of the matrix $(\sigma^{-1})^*$. Both (5.20) and (5.21) would follow from Lemma A1 of Ocone & Karatzas (1991), if we show that the condition of that lemma is satisfied. The partial derivatives of ψ_1 are

$$\frac{\partial\psi_1}{\partial m_k}(m) = 2\sum_{i,j=1}^N s_{j,i}(m_j - r)s_{k,i} \ , \tag{5.27}$$

thus the condition of that lemma translates to

$$\tilde{E}\|\sigma^{-1}(m_u - r\mathbf{1})\|^2 < \infty \tag{5.28}$$

and

$$\tilde{E}\Big(\int_0^T \|2\sum_{i,j,k=1}^N s_{j,i}s_{k,i}(m_j(u)-r)D_t m_k(u)\big\|^2 dt\Big)^{\frac{1}{2}} < \infty \ . \tag{5.29}$$

Inequality (5.28) follows from (5.19). By Jensen's inequality it suffices to show (5.29) without the square-root. By (5.18)and the elementary inequality $\|\sum_{i=1}^n v_i\|^2 \le n\sum_{i=1}^n \|v_i\|^2$ for $v_i \in \Re^N, n = 1,\ldots,n$ we have

$$\tilde{E}\Big[\int_0^T \|2\sum_{i,j,k=1}^N s_{j,i}s_{k,i}(m_j(u)-r)D_t m_k(u)\big\|^2 dt\Big]$$

$$\le 4N^3\sum_{i,j,k=1}^N \Big\{s_{j,i}^2 s_{k,i}^2 \tilde{E}\big(m_j(u)-r\big)^2 \int_0^u \Big\|\Big(\sigma^{-1}\gamma(t)(\phi^*(t))^{-1}\phi^*(u)\Big)^{(k)}\Big\|^2 dt\Big\} < \infty,$$

where the matrix followed by superscript (k) in the last expression represents the k-th column vector of the matrix.
Next we are going to show (5.22). By (5.21) and (5.18), for all $t \le u \le T$

$$\tilde{E}\Big[\big|D_t^j\|\sigma^{-1}(m_u - r\mathbf{1})\|^2\big|^4\Big] \le \tilde{E}\Big[\big\|D_t\|\sigma^{-1}(m_u - r\mathbf{1})\|^2\big\|^4\Big]$$

$$= 16\tilde{E}\|\sigma^{-1}\gamma(t)(\phi^*(t))^{-1}\phi^*(u)(\sigma\sigma^*)^{-1}(m_u - r\mathbf{1})\|^4$$

$$\le 16\|\sigma^{-1}\gamma(t)(\phi^*(t))^{-1}\phi^*(u)(\sigma^*)^{-1}\ |^4\tilde{E}\|\sigma^{-1}(m_u - r\mathbf{1})\|^4$$

and (5.22) now follows from (5.19).

We note that $m_u \in (D_{1,1})^N$ implies that

$$\sigma^{-1}(m_u - r\mathbf{1}) \in D_{1,1}\ , \tag{5.30}$$

and

$$D_t\sigma^{-1}(m_u - r\mathbf{1}) = (D_t m_u)(\sigma^*)^{-1}\ . \tag{5.31}$$

5.6 Lemma: The following two relations hold:

$$\int_0^T \|\sigma^{-1}(m_u - r\mathbf{1})\|^2 du \in D_{1,1}\ , \tag{5.32}$$

and

$$D_t \int_0^T \|\sigma^{-1}(m_u - r\mathbf{1})\|^2 du = 2\int_0^T (D_t m_u)(\sigma\sigma^*)^{-1}(m_u - r\mathbf{1}) du \ . \tag{5.33}$$

Proof: This follows from Lemma 5.3, formulas (5.30),(5.19) and (5.22).

5.7 Lemma: The following two relations hold:

$$\int_0^T (m_u - r\mathbf{1})^*(\sigma^*)^{-1} d\tilde{w}_u \in D_{1,1} \tag{5.34}$$

and

$$D_t \int_0^T (m_u - r\mathbf{1})^*(\sigma^*)^{-1} d\tilde{w}_u = \int_t^T (D_t m_u)(\sigma^*)^{-1} d\tilde{w}_u + \sigma^{-1}(m_t - r\mathbf{1}) \ . \tag{5.35}$$

Proof: This would follow from Proposition 2.3 of Ocone & Karatzas (1991) once we verify that the process $\{\sigma^{-1}(m_t - r\mathbf{1});\ t \le T\}$ is a member of the class $\mathcal{L}$ of Definition 5.1. Condition (i) of Definition 5.1 is exactly (5.30), and (iii), (iv) follow from (5.33), (5.31) and (5.18). The only work to do is to show condition (ii). The finiteness of the first term on the left-hand side of the expression in (ii) follows from (5.19). To show the finiteness of the second component, by (5.31) and (5.18) we can compute

$$\tilde{E}\int_0^T \int_0^T \|D_t \sigma^{-1}(m_u - r\mathbf{1})\|^2 dt\ du = \int_0^T \int_0^u \|\sigma^{-1}\gamma(t)(\phi^*(t))^{-1}\phi^*(u)(\sigma^*)^{-1}\|^2 dt\ du$$

which is finite, thus the statement of the lemma follows.

5.8 Lemma: The random variable ζ_T is a member of $D_{1,1}$ and

$$D_t\zeta_T = \zeta_T\left[-\int_t^T (D_t m_u)(\sigma^*)^{-1} d\tilde{w}_u - \sigma^{-1}(m_t - r\mathbf{1}) + \int_t^T (D_t m_u)(\sigma\sigma^*)^{-1}(m_u - r\mathbf{1}) du\right] . \tag{5.36}$$

Proof: We shall apply again Lemma A1 of Ocone & Karatzas (1991). By (3.5) we can write ζ_T in the form

$$\zeta_T = \psi_2(V_1, V_2) \ , \tag{5.37}$$

where $\psi_2 : \Re^2 \mapsto \Re$ is defined as

$$\psi_2(v_1, v_2) = \exp\{v_1 + v_2\} \tag{5.38}$$

and

$$V_1 = -\int_0^T (m_u - r\mathbf{1})^*(\sigma^*)^{-1} d\tilde{w}_u \tag{5.39}$$

$$V_2 = \frac{1}{2}\int_0^T \|\sigma^{-1}(m_u - r\mathbf{1})\|^2 du \ . \tag{5.40}$$

We already know from Lemmas 5.7 and 5.6 that V_1 and V_2 are in $D_{1,1}$, so we have to show only the condition of Lemma A1 of Ocone & Karatzas (1991), which in our case becomes

$$\tilde{E}\zeta_T < \infty \tag{5.41}$$

and

$$\tilde{E}\Big(\int_0^T \Big\| \frac{\partial \psi_2}{\partial v_1}(V_1, V_2) D_t V_1 + \frac{\partial \psi_2}{\partial v_2}(V_1, V_2) D_t V_2 \Big\|^2 dt\Big)^{\frac{1}{2}} < \infty \tag{5.42}$$

Inequality (5.41) follows from Lemma 4.1, thus we have to show (5.42) only. By Jensen's inequality for concave functions it suffices to show that the left-hand side of (5.42) is finite without the square-root, and this becomes

$$\tilde{E}\int_0^T \zeta_T^2 \Big\| D_t V_1 + D_t V_2 \Big\|^2 dt < \infty \ . \tag{5.43}$$

By Holders's inequality and Lemma 4.1 it suffices to show that

$$\tilde{E}\int_0^T \|D_t V_1\|^4 dt < \infty \tag{5.44}$$

and

$$\tilde{E}\int_0^T \|D_t V_2\|^4 dt < \infty \tag{5.45}$$

We start with (5.44). By (5.35)

$$\tilde{E}\int_0^T \|D_t V_1\|^4 = \tilde{E}\int_0^T \Big\| \int_t^T (D_t m_u)(\sigma^*)^{-1} d\tilde{w}_u + \sigma^{-1}(m_t - r\mathbf{1}) \Big\|^4 dt \ . \tag{5.46}$$

Now by (5.19) it suffices to show that

$$\tilde{E}\int_0^T \Big\| \int_t^T (D_t m_u)(\sigma^*)^{-1} d\tilde{w}_u \Big\|^4 dt < \infty \ . \tag{5.47}$$

With the notation

$$a(t,u) = \sigma^{-1}\gamma(t)(\phi^*(t))^{-1}\phi^*(u)(\sigma^*)^{-1} \tag{5.48}$$

the left-hand side of (5.47) becomes

$$\tilde{E}\int_0^T \Big\| \int_t^T a(t,u) d\tilde{w}_u \Big\|^4 dt = \tilde{E}\int_0^T \Big(\sum_{i=1}^N \Big(\int_t^T \sum_{j=1}^N a_{i,j}(t,u) d\tilde{w}_j(u)\Big)^2\Big)^2 dt$$

$$\leq N \int_0^T \sum_{i=1}^N \tilde{E}\Big(\int_t^T \sum_{j=1}^N a_{i,j}(t,u)d\tilde{w}_j(u)\Big)^4 dt \ . \tag{5.49}$$

However, $\int_t^T \sum_{j=1}^N a_{i,j}(t,u)d\tilde{w}_j(u)$ follows normal distribution with mean zero and variance $\int_t^T \|a_i(t,u)\|^2 du$ where a_i represents the i-th row vector of a, thus the right-hand side of (5.49) becomes

$$N \int_0^T \sum_{i=1}^N 3\Big(\int_t^T \|a_i(t,u)\|^2 du\Big)^2 dt < \infty \ .$$

Next we show (5.45). By (5.33) and (5.21) the left-hand side of (5.45) can be written as

$$\tilde{E}\int_0^T \bigg\| \int_t^T \frac{1}{2} D_t \big\|\sigma^{-1}(m_u - r\mathbf{1})\big\|^2 du \bigg\|^4 dt$$

$$= \tilde{E}\int_0^T \bigg(\sum_{j=1}^N \Big(\int_t^T \frac{1}{2} D_t^j \big\|\sigma^{-1}(m_u - r\mathbf{1})\big\|^2 du\Big)^2\bigg)^2 dt$$

$$\leq N\tilde{E}\int_0^T \sum_{j=1}^N \Big(\int_t^T \frac{1}{2} D_t^j \big\|\sigma^{-1}(m_u - r\mathbf{1})\big\|^2 du\Big)^4 dt$$

$$\leq T^3 N\tilde{E}\int_0^T \sum_{j=1}^N \int_t^T \Big(\frac{1}{2} D_t^j \big\|\sigma^{-1}(m_u - r\mathbf{1})\big\|^2\Big)^4 du \ dt \ ,$$

which is finite by (5.22).

Now we are ready to prove Theorem 4.3. We are going to show that for every $x \in (0,\infty)$

$$I(x\zeta_T) \in D_{1,1} \tag{5.50}$$

and

$$D_t I(x\zeta_T) = xI'(x\zeta_T)D_t\zeta_T \ . \tag{5.51}$$

Both relations follows from Ocone & Karatzas, Lemma A1, provided that the conditions

$$\tilde{E}\int_0^T \|I'(x\zeta_T)D_t\zeta_T\|^2 dt < \infty \tag{5.52}$$

and

$$\tilde{E}I(x\zeta_T) < \infty \tag{5.53}$$

are satisfied. The last inequality is a consequence of assumption (4.18) and Lemma 4.1. The left-hand side of (5.52) becomes by (5.36), (5.35), (5.33), (5.39), (5.40) and (4.19)

$$\tilde{E}\Big[\big|I'(x\zeta_T)\big|^2\zeta_T^2 \int_0^T \|D_tV_1 + D_tV_2\|^2 dt\Big] \leq K_2^2\tilde{E}\Big[\big(1 + x^{-2}\zeta_T^{-2}\big)^2\zeta_T^2 \int_0^T \|D_tV_1 + D_tV_2\|^2 dt\Big]$$

$$\leq 2K_2^2\tilde{E}\Big[\zeta_T^2\int_0^T\|D_tV_1+D_tV_2\|^2dt\Big]+2K_2^2x^{-4}\tilde{E}\Big[\zeta_T^{-2}\int_0^T\|D_tV_1+D_tV_2\|^2dt\Big]. \quad (5.54)$$

The first term of the last expression is finite by (5.43). The second term without the constant multiplier is bounded by Holder's and Jensen's inequalities by

$$\Big(\tilde{E}\zeta_T^{-4}\Big)^{\frac{1}{2}}\Big(\tilde{E}\big(\int_0^T\|D_tV_1+D_tV_2\|^2dt\big)^2\Big)^{\frac{1}{2}}$$

$$\leq\Big(\tilde{E}\zeta_T^{-4}\Big)^{\frac{1}{2}}\sqrt{T}\Big(\tilde{E}\int_0^T\|D_tV_1+D_tV_2\|^4dt\Big)^{\frac{1}{2}},$$

which is finite by (5.44), (5.45), and Lemma 4.1. We can now derive formula (4.20) for the optimal trading strategy by straightforward algebra, putting together (2.23), (5.4), (5.51), (5.36), (5.18), and (4.17), which completes the proof of the theorem.

6. Appendix

We are going to prove Lemma 4.1 through two other lemmas.

6.1 Lemma: Suppose that for some $\gamma\in\Re$

$$\int_0^T E\Big[\exp\Big\{2T\|\sigma^{-1}\|^2\gamma(2\gamma-1)\|\mu_u\|^2\Big\}\Big]du<\infty\ . \quad (6.1)$$

Then

$$EZ_T^\gamma<\infty\ . \quad (6.2)$$

Proof: We note that Z is a positive local martingale thus by Fatou's Lemma it is a supermartingale. Therefore, we can assume without loss of generality that $\gamma<0$ or $\gamma>1$, because otherwise

$$EZ_T^\gamma\leq(EZ_T)^\gamma\leq 1$$

follows from the supermartingale property for Z. By Holder's inequality and (2.5)

$$EZ_T^\gamma=E\Big[\exp\Big\{-\gamma\int_0^T(\mu_u-r\mathbf{1})(\sigma^*)^{-1}dw_u^{(1)}-\gamma^2\int_0^T\|\sigma^{-1}(\mu_u-r\mathbf{1})\|^2du\Big\}$$

$$\times\exp\Big\{\big(\gamma^2-\frac{\gamma}{2}\big)\int_0^T\|\sigma^{-1}(\mu_u-r\mathbf{1})\|^2du\Big\}\Big]$$

$$\leq\Big(E\exp\Big\{-2\gamma\int_0^T(\mu_u-r\mathbf{1})(\sigma^*)^{-1}dw_u^{(1)}-2\gamma^2\int_0^T\|\sigma^{-1}(\mu_u-r\mathbf{1})\|^2du\Big\}\Big)^{\frac{1}{2}}$$

$$\times\Big(E\exp\Big\{\big(2\gamma^2-\gamma\big)\int_0^T|\sigma^{-1}(\mu_u-r\mathbf{1})\|^2du\Big\}\Big)^{\frac{1}{2}}$$

The first factor in the last expression is finite because the process

$$(t,\omega)\mapsto \exp\Big\{-2\gamma\int_0^t(\mu_u-r\mathbf{1})(\sigma^*)^{-1}dw_u^{(1)}-2\gamma^2\int_0^t\|\sigma^{-1}(\mu_u-r\mathbf{1})\|^2du\Big\}$$

is again a positive local martingale thus a supermartingale. The square of the second factor is bounded by

$$E\exp\Big\{(2\gamma^2-\gamma)\Big(\int_0^T 2\|\sigma^{-1}\|^2\|\mu_u\|^2du+2\|\sigma^{-1}\|^2r^2NT\Big)\Big\},$$

and by Jensen's inequality this is bounded by constant multiplier times the left-hand side of (6.1), which completes the proof of the lemma.

6.2 Lemma: Suppose that λ is a positive real number such that

$$\lambda<\frac{1}{4K(\mathrm{tr}(\gamma_0)+T\|\beta\|^2)} \tag{6.3}$$

where the constant K is defined in (4.4). Then

$$\int_0^T E\exp\Big\{\lambda\|\mu_u\|^2\Big\}du<\infty\ . \tag{6.4}$$

Proof: Let $V(t)$ be the covariance matrix of μ_t which has the form

$$V(t)=e^{-\alpha t}\Big[\gamma_0+\int_0^t e^{\alpha s}\beta\beta^*(e^{\alpha s})^*ds\Big](e^{-\alpha t})^* \tag{6.5}$$

(Arnold (1973), formula (8.2.7)). With this notation the random vector

$$V^{-\frac12}(t)(\mu_t-E\mu_t)$$

follows N-dimensional standard normal distribution. Now using the identity $\|A\|^2=\mathrm{tr}(AA^*)$ for any $N\times N$-matrix A, we compute

$$E\exp\Big\{\lambda\|\mu_u\|^2\Big\}\le E\exp\Big\{2\lambda\|\mu_u-E\mu_u\|^2+2\lambda E\|\mu_u\|^2\Big\}$$

$$\le\exp\Big\{2\lambda\|E\mu_u\|^2\Big\}E\exp\Big\{2\lambda\|V^{\frac12}(u)\|^2\|V^{-\frac12}(u)(\mu_u-E\mu_u)\|^2\Big\}\le K_3\varphi\Big(2\lambda\ \mathrm{tr}(V(u))\Big) \tag{6.6}$$

where

$$K_3=\max_{u\le T}\exp\Big\{2\lambda\|E\mu_u\|^2\Big\} \tag{6.7}$$

and $\varphi(\cdot)$ is the moment generating function of the χ^2 distribution with parameter N. We note that $K_3 < \infty$ because

$$E\mu_u = e^{-\alpha u}\left[m_0 + \int_0^u e^{\alpha s}\alpha\delta\ ds\right] \tag{6.8}$$

is a continuous function of $u \in [0,T]$. One can see by looking at the density function of the χ^2 distribution that $\varphi(\cdot)$ is finite and increasing on $(-\infty, \frac{1}{2})$, which implies that $\varphi(2\lambda\ \mathrm{tr}(V(u)))$ is bounded if

$$2\lambda\ \mathrm{tr}(V(u)) < \frac{1}{2} - \epsilon \tag{6.9}$$

for some positive constant ϵ. However, by (6.5)

$$\mathrm{tr}(V(u)) = E\left[\mathrm{tr}\left(e^{-\alpha u}(\mu_0 - m_0)(\mu_0 - m_0)^*(e^{-\alpha u})^*\right)\right] + \int_0^u \mathrm{tr}\left(e^{-\alpha(u-s)}\beta\beta^*(e^{-\alpha(u-s)})^*\right)ds$$

$$= E\left\|e^{-\alpha u}(\mu_0 - m_0)\right\|^2 + \int_0^u \left\|e^{-\alpha(u-s)}\beta\right\|^2 ds \le K\left(\mathrm{tr}(\gamma_0) + T\|\beta\|^2\right) \tag{6.10}$$

where the constant K is given in (4.4). Our condition (6.3) implies that for some $\epsilon > 0$

$$\lambda < (\frac{1}{2} - \epsilon)\frac{1}{2K(\mathrm{tr}(\gamma_0) + T\|\beta\|^2)} \tag{6.11}$$

and (6.11), (6.10) now yield (6.9), which completes the proof.

Proof of Lemma 4.1: The finiteness of EZ_T^5 and EZ_T^{-4} follows from the previous two lemmas and condition (4.2). For $\theta = 4$ or $\theta = -5$, by Jensen's inequality we have

$$\tilde{E}\zeta_T^\theta = E\zeta_T^{\theta+1} = E\left(E\left[Z_T \,\middle|\, \mathcal{F}_T^S\right]\right)^{\theta+1} \le EZ_T^{\theta+1} < \infty$$

which proves (4.5) entirely. We still need to show is that Z is a P-martingale for which it suffices to prove that

$$EZ_T = 1\ . \tag{6.12}$$

We define partition $\{A_n;\ n = 0, 1, \ldots\}$ of Ω by

$$A_n = \{\omega \in \Omega :\ \sup_{t \le T}\|\mu_t\| \in [n, n+1)\}\ ; \quad n = 0, 1 \ldots \tag{6.13}$$

and the index set

$$J = \{n :\ P(A_n) > 0\}\ . \tag{6.14}$$

We also define the probability measure $P_n << P$ for every $n \in J$ as

$$P_n(A) = P(A \mid A_n)\ . \tag{6.15}$$

By the independence of μ and $w^{(1)}$ under P, the process $w^{(1)}$ is still a Brownian Motion under the probability measure P_n and the same filtration $\mathcal{F}$. We introduce the random variables

$$Z_T^{(n)} = \exp\left\{ -\int_0^T (\mu_u - r\mathbf{1})^* (\sigma^*)^{-1} dw_u^{(1)} - \frac{1}{2}\int_0^T \|\sigma^{-1}(\mu_u - r\mathbf{1})\|^2 du \right\} \tag{6.16}$$

where the stochastic integral in the right-hand side of the above formula is computed under the probability measure P_n. The absolute continuity of P_n with respect to P implies that

$$Z_T^{(n)} = Z_T, \quad P_n - \text{a.s.}, \ n \in J \tag{6.17}$$

(Protter, (1990), Theorem II.5.14). Under P_n the drift process μ is almost surely bounded by $n+1$ thus

$$E_n Z_T^{(n)} = E_n Z_T = 1, \quad n \in J \tag{6.18}$$

where E_n is the expectation corresponding to P_n. This implies that

$$E\left[Z_T \,\middle|\, A_n \right] = 1 \ ; \quad n \in J \tag{6.19}$$

and (6.12) follows.

Proof of Proposition 4.6:

Lemmas 6.1, 6.2, and conditions (4.33)-(4.34) now guarantee that

$$E\left[Z_T^{\frac{\theta+3}{\theta-1}} \right] < \infty \ ,$$

thus by Jensen's inequality

$$E\left[\zeta_T^{\frac{\theta+3}{\theta-1}} \right] = \tilde{E}\left[\zeta_T^{\frac{4}{\theta-1}} \right] < \infty \ . \tag{6.20}$$

We don't have now conditions (4.18) and (4.19), thus we have to verify that the proof of Theorem 4.3 remains valid. Those conditions were used to show (5.53) and in (5.54). Inequality (5.53) is a straightforward consequence of (6.20). Instead of (5.54) now we have

$$\tilde{E}\left[|I'(x\zeta_T)|^2 \zeta_T^2 \int_0^T \left\| D_t V_1 + D_t V_2 \right\|^2 dt \right] = \text{const} \times \tilde{E}\left[\zeta_T^{\frac{2}{\lambda-1}} \int_0^T \left\| D_t V_1 + D_t V_2 \right\|^2 dt \right] \ ,$$

and by Holder's inequality, (6.20), (5.44), and (5.45) this last expression is finite.

7. Acknowledgement: The author is indebted to Stanley R. Pliska for the fruitful discussions about the subject of this paper.

8. REFERENCES

ARNOLD, L., *Stochastic Differential Equations: Theory and Applications*, Wiley-Interscience, 1973.

BROWNE, S. & WHITT, W., Portfolio choice and the Bayesian Kelly criterion, *Advances in Applied Probability* 28, (1996), 1145-1176.

COX,J.C. & HUANG,C.F., Optimal consumption and portfolio policies when asset prices follow a diffusion process, *J. Economic Theory* 49, (1989), 33–83.

COX, J.C., INGERSOLL,J.E., & ROSS, S.A., An intertemporal general equilibrium model of asset prices, *Econometrica* 53 (1985), 363–384.

DUFFIE, D. & ZAME, W., The consumption–based capital asset pricing model, *Econometrica* 57 (1989), 1279–1297.

HE, H. & PEARSON, N., Consumption and portfolio policies with incomplete markets: the infinite dimensional case, *Journal of Economic Theory* 54 (1991), 259–305.

KARATZAS, I., LAKNER, P., LEHOCZKY, J.P., & SHREVE, S.E., Equilibrium in a simplified dynamic, stochastic economy with heterogeneous agents, in: E. Mayer–Wolf, E. Merzbach, & A. Schwartz, eds., *Stochastic Analysis* (Academic Press 1991), pp. 245–272.

KARATZAS, I., LEHOCZKY, J.P., & SHREVE, S.E., Optimal portfolio and consumption decisions for a "small investor" on a finite horizon, *SIAM Journal of Control and Optimization* 25 (1987), 1557–1586.

KARATZAS, I., LEHOCZKY, J.P., SHREVE, S.E., & XU, Martingale and duality methods for utility maximization in incomplete markets, *SIAM Journal of Control and Optimization* 29 (1991), 702–730.

KARATZAS, I., OCONE, D.L., & LI, J., An Extension of Clark's Formula, *Stochastics and Stochastic Reports* 37 (1991), 127–131.

KARATZAS, I., & SHREVE, S.E., *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York, 1988.

LAKNER, P., Optimal Investment Processes for Utility Maximization Problems with Restricted Information, (1994), manuscript.

LAKNER, P., Utility maximization with partial information, *Stochastic Processes and their Applications* 56 (1995), 247–273.

LIPTSER, R.S. & SHIRYAYEV, A.N., *Statistics of Random Processes I*, Springer-Verlag, 1977.

LIPTSER, R.S. & SHIRYAYEV, A.N., *Statistics of Random Processes II*, Springer-Verlag, 1978.

OCONE, D.L. & KARATZAS, I., A generalized Clark representation formula, with application to optimal portfolios, *Stochastics and Stochastic Reports* 34 (1991), 187–220.

PROTTER, P., *Stochastic Integration and Differential Equations*, Springer–Verlag, New York 1990.

SHIGEKAWA, I., Derivatives of Wiener Functionals and Absolute Continuity of Induced Measures, *J. Math. Kyoto University* 20 (1980), 263-289.