Optimal cash management using impulse control

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Abstract

We consider the impulse control of Lévy processes under the infinite horizon, discounted cost criterion. Our motivating example is the cash management problem in which a controller is charged a fixed plus proportional cost for adding to or withdrawing from his/her reserve, plus an opportunity cost for keeping any cash on hand. Our main result is to provide a verification theorem for the optimality of control band policies in this scenario. We also analyze the transient and steady-state behavior of the controlled process under control band policies and explicitly solve for the optimal policy in the case in which the Lévy process to be controlled is the sum of a Brownian motion with drift and a compound Poisson process with exponentially distributed jump sizes.

Keywords: impulse control; cash management problem; Lévy processes; Brownian motion.

AMS Subject Classification: 60G51, 60J25, 93E20.

1 Introduction

Impulse control problems have a long history related to applications to the cash management problem. In the present paper, we consider the impulse control of Lévy processes. Our motivating application is the cash management problem in which there exists a system manager who must control the amount of cash he/she has on hand. We assume that the manager's cash on hand fluctuates due to randomly occurring withdrawals from and deposits to his/her account but that the manager is charged a fixed plus proportional cost for any specific, intentional adding to or withdrawing from his/her reserves and that there exists an opportunity cost for keeping too little or too much cash on hand. The manager's objective is to minimize his/her long run opportunity cost of keeping cash on hand plus any cost incurred from depositing or withdrawing from the reserve. An alternative motivating application which is also considered in the literature is a manager who wishes to control his/her inventory level. The manager's inventory level fluctuates randomly and he/she may increase or decrease his/her inventory level at will by expediting or salvaging parts, paying a fixed plus proportional cost to do so. The manager's objective is to minimize his/her long run inventory holding costs plus costs of expediting and salvaging.

Our first main result in the paper is to provide a verification theorem for the optimality of control band policies for the impulse control of Lévy processes. Our result is fairly general and holds for a wide class of opportunity cost functions. We then explicitly calculate the Laplace transform with respect to time and steady-state distribution of any spectrally positive Lévy process controlled under a control band policy. In Section 5, we consider the special case of a Lévy process which is comprised of the sum of a Brownain motion and a compound Poisson process with exponentially distributed jump sizes. In this specific case, we show how one may use the results derived in this paper in order to characterize the value function for the associated control problem, and characterize the band levels in the optimal control policy as a solution of a system of equations. Moreover, we also show how one may determine the steady-state distribution of the controlled process when the underlying Lévy process is the sum of a negative drift and a compound Poisson process with exponentially distributed jump sizes.

The technique of impulse control was originally developed by Bensoussan and Lions [3, 4] and extended by Richard et. al [8, 19]. Harrison, Selke and Taylor [11] and Sulem [21] also consider the impulse control of Brownian motion and explicitly calculate the critical parameters determining the optimal policy. In [10], an iterative computational scheme is to provided in order to determine the optimal policy for the impulse control of Brownain motion. Recently, Ormeci, Dai and Vande Vate [17] have considered impulse control of Brownian motion under the average cost criterion and again show that a control band policy is optimal. Cadenillas, Zapatero, and Sarkar [7] and Cadenillas, Lakner, and Pinedo [6] solved the Brownian case with a mean-reverting drift. None of the above mentioned works allow there to be jumps in the process to be controlled. In [5], the optimality of an (s, S) policy is proven for a process which is the sum of a constant drift, a Brownian motion, and a compound Poisson process. In related work, Bar-Ilan, Perry and Stadje [2] have also considered the problem of impulse control of Lévy processes for the specific case in which the Lévy process is a sum of a Brownian motion and a compound Poisson process. Assuming that a control band policy is optimal, their main results are to evaluate the cost functionals of the resulting policy through a fundamental identity derived from the martingale originally introduced by Kella and Whitt [14]. In [23], Yamazaki uses a scale function approach to study one-sided impulse control problems for spectrally positive Lévy demand processes. In [9], an impulse control problem is studied for a refracted Lévy process where the ruin time is modeled by a Parisian delay. Finally, [22] studies the impulse control of a geometric Lévy process.

2 The Model

In this Section we provide the specifics of the model described in the Introduction. All forthcoming processes are assumed to live on a probability triplet equipped with a filtration $\mathcal{F} = \{\mathcal{F}_t,\ 0 \le t < \infty\}$. We begin by assuming that Y_t is a Lévy process started from x with Lévy measure ν such that

$$\int_{\{|y|\ge 1\}} |y|\nu(dy) < \infty. \tag{1}$$

The process Y_t will be used to represent the cash on hand process assuming that the manager exerts no control by making no deposits to or withdrawal from his/her fund. Let $J(\omega, dt, dy) = J(dt, dy)$ be the jump measure of Y. Then, Y_t has the Itô-Lévy decomposition

$$Y_t = x + \mu t + \sigma w_t + A_t + \varphi_t \tag{2}$$

where φ_t is the following martingale

$$\varphi_t = \int_{\{|x|<1\}} x \left\{J((0,t], dx) - t\nu(dx)\right\}$$

and A is the sum of the "large" jumps

$$A_t = \sum_{0<s\le t} \Delta Y_s 1_{\{|\Delta Y_s|\ge 1\}}.$$

The process w is assumed to be a standard Wiener process and μ is a constant. We do not make any assumption regarding σ, it may be zero or non-zero. Also we allow $\nu(\Re) = 0$ in which case Y is continuous. The case when both $\sigma = 0$ and $\nu(\Re) = 0$ is also included, although this case is trivial (Y is deterministic in this case). In general, we will use P_x to denote the probability measure under which Y_t is started from x and E_x its associated expectation operator. We note that φ is a quadratic pure jump local martingale ([18], Chapter II, Section 6) with quadratic variation

$$[\varphi]_t = \sum_{0<s\le t} (\Delta Y_s)^2 1_{\{|\Delta Y_s|\le 1\}} = \int_{(0,t]\times\Re} y^2 1_{\{|y|<1\}} J(ds, dy),$$

which has expected value

$$E\,[\varphi]_t = t\int_{\Re} y^2 1_{\{|y|<1\}}\nu(dy) < \infty. \tag{3}$$

It follows ([18], Chapter II, Corollary 3 to Theorem 27) that φ is a square-integrable martingale.

We let

$$(T, \Xi) = (\tau_1, \tau_2, \ldots, \tau_n, \ldots,\ \xi_1, \xi_2, \ldots, \xi_n, \ldots,)$$

denote the impulse control policy used by the manager where $0 \le \tau_1 < \tau_2 < \tau_3 \ldots$ are $[0,\infty]$-valued stopping times and ξ_n is an $\mathcal{F}_{\tau_n}$ measurable random variable for each $n \ge 1$. Positive values of ξ_n

represent deposits by the manager into his/her fund and negative values represent withdrawal. We allow the possibility that $P_x(\tau_n = \infty) > 0$. In that case on the event $\{\tau_n = \infty\}$ there are fewer than n interventions on the time horizon $[0, \infty)$. We require that $\xi_n = 0$ on $\{\tau_n = \infty\}$. However,we also require that $\xi_n \neq 0$ on $\{\tau_n < \infty\}$.

As described in the Introduction, the controlled cash on hand process X_t follows the dynamics

$$X_t = Y_t + \sum_{i=1}^{\infty} 1_{\{\tau_i \leq t\}} \xi_i \tag{4}$$

and has RCLL paths. Let $\lambda > 0$ be a fixed *discount factor*.

Definition 2.1. *A control policy (T, Ξ) is called admissible if for some constant $K \in (0, \infty)$ the following relation holds:*

$$e^{-\lambda t}|X_t| < K, \quad a.s.,\ t \in [0, \infty). \tag{5}$$

We emphasize that the constant K is not "universal"; for different policies it may be different.

Let the opportunity cost function be $\phi : \mathbb{R} \mapsto [0, \infty)$, so the cost for cash on hand being at level x is $\phi(x)$. We shall assume that for some constants $K_1, K_2 > 0$

$$K_1 \phi(x) + K_2 \geq |x|, \quad x \in \Re. \tag{6}$$

The manager's total cost is the sum of his/her expected discounted opportunity costs as well as impulse control costs and is given by

$$I(x, T, \Xi) = E_x \left[\int_0^{\infty} e^{-\lambda t} \phi(X_t) dt + \sum_{n=1}^{\infty} e^{-\lambda \tau_n} g(\xi_n) \right] \tag{7}$$

where the manager's impulse control costs are given by

$$g(\xi) = \begin{cases} C + c\xi, & \text{if } \xi > 0, \\ 0, & \text{if } \xi = 0, \\ D - d\xi, & \text{if } \xi < 0. \end{cases} \tag{8}$$

We assume that the fixed costs C and D are positive and the variable costs c and d are non-negative constants. The term $e^{-\lambda \tau_n}$ in (7) is well defined once we set $e^{-\infty} = 0$.

One of our primary objectives in this paper is to identify the optimal impulse control (T, Ξ) that minimizes the above total cost $I(x, T, \Xi)$. The value function of this optimization problem is

$$V(x) = \inf \{ I(x, T, \Xi),\ (T, \Xi) \text{ is an admissible impulse control} \}.$$

Lemma 2.2. *Let (T, Ξ) be a control such that $I(x, T, \Xi) < \infty$. Then*

$$\lim_{n \to \infty} \tau_n(\omega) = \infty, \quad a.s. \tag{9}$$

and

$$\liminf_{t \to \infty} E_x \left[e^{-\lambda t} |X_t| \right] = 0 \tag{10}$$

Proof: We prove first (9). Suppose the opposite, i.e., that $P(G) > 0$ where $G = \{\omega \in \Omega : \ \tau_n \uparrow \tau < \infty\}$. Then

$$\sum_{n=1}^{\infty} E_x \left[e^{-\lambda \tau_n} g(\xi_n) \right] \geq \sum_{n=1}^{\infty} E_x \left[e^{-\lambda \tau_n} g(\xi_n) 1_G \right] \geq \sum_{n=1}^{\infty} E_x \left[e^{-\lambda \tau_n} 1_G \right] \min\{C, D\} \geq$$

$$\sum_{n=1}^{\infty} E_x \left[e^{-\lambda \tau} 1_G \right] \min\{C, D\} = \infty$$

which contradicts our assumption that $I(x, T, \Xi) < \infty$. The second inequality in the above chain holds because on the event G we have $\tau_n < \infty$ for every n, hence $\xi_n \neq 0$. Next we prove (10). By (6)

$$\int_0^{\infty} E_x \left[e^{-\lambda t} |X_t| \right] dt \leq \int_0^{\infty} E_x \left[e^{-\lambda t} \left(K_1 \phi(X_t) + K_2 \right) \right] dt,$$

which is finite by our assumption that $I(x, T, \Xi) < \infty$, and (10) follows. •

In the section that follows we show that the impulse control takes the form of a control band policy which arise frequently as the solution to impulse control problems. Moreover, in Section 5 we provide an example in which we are able to explicitly identify the parameters corresponding to this policy.

3 Main Results

In this Section, we provide the main result of the paper, Theorem 3.1, showing that a solution of an ordinary differential equation that satisfies some additional conditions, must be the value function V. Also included in the statement of Theorem 3.1 is the optimal impulse control policy which turns out to be a double bandwidth control policy. We begin first with some preliminary results before providing the statement of Theorem 3.1.

For a function $f : \Re \mapsto \Re$ we define the operator

$$Mf(x) = \inf\{f(x + \eta) + g(\eta), \ \eta \in \Re \setminus \{0\}\}.$$

We shall also use the linear operator $\mathcal{A}$ associated with the uncontrolled process Y, that is, for $f \in C^2(\Re)$

$$\mathcal{A}f(x) = \frac{1}{2}\sigma^2 f''(x) + \mu f'(x) + \int_{\Re} \left[f(x+y) - f(x) - f'(x) y 1_{\{|y|<1\}} \right] \nu(dy). \tag{11}$$

Our assumption (1) and Taylor's theorem implies that the integral on the right-hand side is finite whenever f' and f'' are bounded. Indeed,

$$\left| \int_{\Re} \left[f(x+y) - f(x) - f'(x) y 1_{\{|y|<1\}} \right] \nu(dy) \right|$$

$$
\begin{aligned}
&\leq \int_{\Re} |f(x+y) - f(x)|\, 1_{\{|y|\geq 1\}}\nu(dy) + \int_{\Re} \left|f(x+y) - f(x) - f'(x)y\right| 1_{\{|y|<1\}}\nu(dy) \\
&\leq \text{const} \times \left[\int_{\Re} |y| 1_{\{|y|\geq 1\}}\nu(dy) + \int_{\Re} y^2 1_{\{|y|<1\}}\nu(dy)\right] \\
&< \infty. \qquad (12)
\end{aligned}
$$

In order to prove our results we shall actually need to extend the domain of $\mathcal{A}$ to a larger class of functions $\mathcal{D}$ defined below.

Definition 3.1. *Let $\mathcal{D}$ be the class of functions $f : \Re \mapsto \Re$ for which there exist an integer $n \geq 0$ and a set of real numbers $S = \{x_1, x_2, \ldots, x_n\}$ (if $n = 0$ then S is the empty set) such that the following conditions hold:*
(i) $f \in C^1(\Re) \cap C^2(\Re \setminus S)$
(ii) The derivative f' is bounded on $\Re$ and the second derivative f'' is bounded on $\Re \setminus S$. We shall call the points in S the exceptional points.

From Lemma A.2 follows that (12) holds even if $f \in \mathcal{D}$. Indeed, let $(f_n, n \geq 0)$ be a sequence of functions guaranteed by Lemma A.2, then we have

$$
\int_{\Re} \left|f(x+y) - f(x) - f'(x)y\right| 1_{\{|y|<1\}}\nu(dy) = \int_{\Re} \lim_{n\to\infty} \left|f_n(x+y) - f_n(x) - f_n'(x)y\right| 1_{\{|y|<1\}}\nu(dy) \leq
$$

$$
\text{const} \times \int_{\Re} y^2 1_{\{|y|<1\}}\nu(dy) < \infty.
$$

It follows that we can and will extend the operator $\mathcal{A}$ to $\mathcal{D}$. This way $\mathcal{A}f(x)$ may be undefined if x is an exceptional point of f, but this will not cause any problems. For $f \in \mathcal{D}$ Itô's rule applied to $f(X_t)$ holds in its usual form (see the Appendix A).

We now conjecture that the optimal impulse control policy takes a double bandwidth control policy form. In particular, we assume that there exist constants $a < \alpha \leq \beta < b$ such that

$$
\tau_1^* = \inf\left\{t \geq 0 : \; Y_t \in \mathbb{R} \setminus (a, b)\right\}, \qquad (13)
$$

and for $n \geq 2$

$$
\tau_n^* = \inf\left\{t \geq \tau_{n-1}^* : \; X_{t-} + \Delta Y_t \in \mathbb{R} \setminus (a, b)\right\}. \qquad (14)
$$

The jump sizes are given by the following equations. If $x \in \mathbb{R} \setminus (a, b)$ then

$$
\xi_1^* = \begin{cases} \beta - x, & \text{if } x \geq b, \\ \alpha - x, & \text{if } x \leq a, \end{cases} \qquad (15)
$$

and for all other cases (including the case of $n = 1$ and $x \in (a, b)$)

$$
\xi_n^* = \begin{cases} \beta - \left(X(\tau_n^*-) + \Delta Y(\tau_n^*)\right), & \text{if } X(\tau_n^*-) + \Delta Y(\tau_n^*) \geq b, \\ \alpha - \left(X(\tau_n^*-) + \Delta Y(\tau_n^*)\right), & \text{if } X(\tau_n^*-) + \Delta Y(\tau_n^*) \leq a. \end{cases} \qquad (16)
$$

Note that $\tau_1^* = 0$ if and only if $x \in \Re \setminus (a, b)$. For $n \geq 1$ it is possible that $\Delta Y(\tau_n^*) = 0$ in which case $X(\tau_n^*-)$ is either equal to a or to b, ξ_n^* is $\alpha - a$ or $\beta - b$, and $X(\tau_n^*)$ equal to α or β, respectively. However, it is also possible that $\Delta Y(\tau_n^*) \neq 0$, in which case $X(\tau_n^*-) + \Delta Y(\tau_n^*)$ may be either larger than a or smaller then b, but we still have $X(\tau_n^*)$ equal to α or β, respectively. This control policy is admissible since the corresponding cash at hand process is bounded, even without discounting. Such a policy corresponding the constants a, α, β, b will be denoted by (T^*, Ξ^*).

Proposition 3.2. *Suppose that for some $a < \alpha \leq \beta < b$ the sequence of stopping times $(\tau_n^*)_{\{n \geq 1\}}$ is given in (13)- (14). If Y is not constant then $\tau_n^* < \infty$, $\tau_n^* < \tau_{n+1}^*$ for $n \geq 1$ almost surely, and $\lim_{n\to\infty} \tau_n^* = \infty$.*

Proof: We are going to show that $\tau_n^* < \infty$, P_x-a.s. by induction. We have $\tau_1^* < \infty$ because the sample paths of the Lévy process Y are unbounded functions (Sato [20], Proposition 37.10). Suppose now that $\tau_{n-1}^* < \infty$ P_x-a.s. Notice that

$$X_t - X(\tau_{n-1}^*) = Y_t - Y(\tau_{n-1}^*), \quad t \in [\tau_{n-1}^*, \tau_n^*), \tag{17}$$

and on the event $\{\tau_n^* = \infty\}$ we have $|X_t - X(\tau_{n-1}^*)| < b - a$ for $t \in [\tau_{n-1}^*, \infty)$, thus on $\{\tau_n^* = \infty\}$ every sample function $t \mapsto Y_t - Y(\tau_{n-1}^*)$ is bounded by $b - a$ on $[\tau_{n-1}^*, \infty)$. By the strong Markov property for Y, the process $\{Y(\tau_{n-1}^* + s) - Y(\tau_{n-1}^*), s \geq 0\}$ has the same law under P_x as $\{Y_s - Y_0, s \geq 0\}$, and recalling again Proposition 37.10 in Sato [20] we conclude that $P_x(\tau_n^* = \infty) = 0$.
Next we show $\tau_n^* < \tau_{n+1}^*$. Let Y^α be the process Y started at $Y_0 = \alpha$ and Y^β be the process Y started at $Y_0 = \beta$. Let $\tau_\alpha = \inf\{s \geq 0 : \ Y_s^\alpha \notin (a, b)\}$ and $\tau_\beta = \inf\left\{s \geq 0 : \ Y_s^\beta \notin (a, b)\right\}$. By the right-continuity of Y we have $\tau_\alpha > 0$ and $\tau_\beta > 0$ a.s. By the strong Markov property for Y, conditionally on $X(\tau_n^*) = \alpha$ the inter-arrival time $\tau_{n+1}^* - \tau_n^*$ has the same distribution as τ_α and conditionally on $X(\tau_n^*) = \beta$ the inter-arrival time $\tau_{n+1}^* - \tau_n^*$ has the same distribution as τ_β. Hence $\tau_n^* < \tau_{n+1}^*$.
Finally we show that $\lim_{n\to\infty} \tau_n^* = \infty$. We take limits in (17) as $t \uparrow \tau_n^*$ and derive that

$$X(\tau_n^*-) + \Delta Y(\tau_n^*) - X(\tau_{n-1}^*) = Y(\tau_n^*) - Y(\tau_{n-1}^*).$$

Notice that the absolute value of the left-hand side is bounded below by $\min\{\alpha - a, b - \beta\}$. Indeed, $X(\tau_n^*-) + \Delta Y(\tau_n^*)$ is not included in the interval (a, b), whereas $X(\tau_{n-1}^*)$ is either equal to α or to β. Then the absolute value of the right-hand side is also bounded below by $\min\{\alpha - a, b - \beta\}$. Let $\tau = \lim_{n\to\infty} \tau_n^*$. It follows that the sample paths of Y do not have a limit on the left at τ on $\{\tau < \infty\}$, which implies that $P_x(\tau < \infty) = 0$.

The following theorems contain the main results of this Section. The first theorem gives conditions under which a function is a lower bound for the value function of the optimization problem.

Theorem 3.1. *Suppose that a function $f : \Re \mapsto (0, \infty)$ such that $f \in D$ with a set of exceptional points S satisfies the following conditions:*

(i) $\mathcal{A}f(x) - \lambda f(x) + \phi(x) \geq 0 \quad$ *for* $x \in \Re \setminus D$*;*
(ii) $f(x) \leq Mf(x) \quad$ *for* $x \in \Re$*.*
Then $f(x) \leq V(x)$

The next theorem will characterize the value function and the optimal double band policy.

Theorem 3.2. *Suppose that there exist constants* $a < \alpha \leq \beta < b$ *and a function* $f : \Re \mapsto (0, \infty)$ *such that* $f \in D$ *with a set of exceptional points* $\{a, b\}$*, and the following conditions are satisfied:*
(a) $\mathcal{A}f(x) - \lambda f(x) + \phi(x) = 0 \quad$ *for* $x \in (a, b)$*;*
(b) $f(x) \leq Mf(x) \quad$ *for* $x \in (a, b)$*;*
(c) $\mathcal{A}f(x) - \lambda f(x) + \phi(x) \geq 0 \quad$ *for* $x \in \Re \setminus [a, b]$*;*
(d) $f(a) = Mf(a) = f(\alpha) + C + c(\alpha - a), \quad f(b) = Mf(b) = f(\beta) + D + d(b - \beta)$*;*
(e) f *is linear on* $(-\infty, a]$ *with slope* $-c$*, and also linear on* $[b, \infty)$ *with slope* d*.*

Then $f(x) = V(x)$*. Furthermore, the control* (T^*, Ξ^*) *given in* (**??**) - (16) *with these values for* a, α, β, b *is optimal.*

The requirements imposed on $f(\cdot)$ in Theorem 3.2 are stronger than the ones imposed in Theorem 3.1, though it is not obvious since in Theorem 3.1 we required that $f(x) \leq Mf(x)$ on the entire of $\Re$, whereas in Theorem 3.2 we only required this inequality to hold on $[a, b]$ (see conditions (b) and (d)). However, it follows from the conditions of Theorem 3.2 that actually $f(x) = Mf(x)$ for $x \in \Re \setminus [a, b]$, as shown in the lemma below.

Lemma 3.3. *Under the conditions of Theorem 3.2 we have that*

$$f(x) = Mf(x) = f(\alpha) + C + c(\alpha - x), \quad x \leq a \tag{18}$$

and

$$f(x) = Mf(x) = f(\beta) + D + d(x - \beta), \quad x \geq b. \tag{19}$$

Proof: We shall prove only (18); the proof of (19) is similar. First we note that we can calculate $Mf(a)$ by taking infimum over $(0,\infty)$ only, i.e.,

$$Mf(a) = \inf\{f(a + \eta) + g(\eta),\ \eta > 0\}, \tag{20}$$

because by condition (d) we have that $f(a) = Mf(a)$, and by condition (e) and by (8) we have that $\inf\{f(a + \eta) + g(\eta), \eta < 0\} = f(a) + D > f(a)$. Next let $x \leq a$. The function $\eta \mapsto f(x + \eta) + g(\eta)$ is constant on $(0, a - x]$. Indeed, if $\eta \in (0, a - x]$ then by condition (e) and by (8) we have that $f(x + \eta) + g(\eta) = f(x) + C + cx$. On the other hand $\inf\{f(x + \eta) + g(\eta),\ \eta < 0\} = f(x) + D$, hence using the notation $a \wedge b = \min\{a, b\}$ we have that

$$Mf(x) = \inf\{f(x + \eta) + g(\eta),\ \eta \in \Re \setminus \{0\}\} = (f(x) + D) \wedge \inf\{f(x + \eta) + g(\eta),\ \eta > a - x\} =$$

$$(f(x) + D) \wedge \inf\{f(a + \gamma) + g(a + \gamma - x),\ \gamma > 0\} =$$

$$(f(x)+D)\wedge(\inf\{f(a+\gamma)+C+c\gamma,\ \gamma>0\}+c(a-x))=$$

$$(f(x)+D)\wedge(Mf(a)+c(a-x))=(f(x)+D)\wedge(f(a)+c(a-x))=f(x).$$

The last identity follows from condition (e), since under that condition $f(a)+c(a-x)=f(x)$. Then the first identity of (18) follows. From the above calculation follows that $Mf(x)=Mf(a)+c(a-x)$. We substitute $Mf(a)=f(\alpha)+C+c(\alpha-a)$ from condition (d) into this equation, and the second identity of (18) follows. •

Proof of Theorem 3.1: Let $(T,\Xi)=\{\tau_1,\tau_2,\dots,\xi_1,\xi_2,\dots\}$ be an arbitrary admissible impulse control for which (9) and (10) hold. We are going to show that

$$f(x)\le I(x,T,\Xi). \tag{21}$$

The assumption that (9) and (10) hold does not restrict the generality of this proof, since if any of (9) or (10) does not hold, then (21) is obvious by Lemma 2.2. By a similar calculation to (12) based on Taylor's theorem one can show that

$$\int_{(0,t]\times\Re} e^{-\lambda s}\left\{f(X_{s-}+y)-f(X_{s-})-f'(X_{s-})y1_{\{|y|<1\}}\right\}J(ds,dy)\le$$

$$\text{const}\times\left[\int_{(0,t]\times\Re}|y|1_{\{|y|\ge1\}}J(ds,dy)+\int_{(0,t]\times\Re}y^2 1_{\{|y|<1\}}J(ds,dy)\right]<\infty. \tag{22}$$

Notice that by our conditions $f\in\mathcal{D}$, thus by (12) and (22) the process $\{U_t,t\in[0,\infty)\}$ given by

$$U_t=\int_{(0,t]\times\Re} e^{-\lambda s}\left\{f(X_{s-}+y)-f(X_{s-})-f'(X_{s-})y1_{\{|y|<1\}}\right\}(J(ds,dy)-ds\nu(dy)) \tag{23}$$

is well defined. This process is a local martingale by Corollary 11.10 and Theorem 11.45, part 3 in [12]. In order to apply this Corollary, we need that the process on the left-hand side of (22) has locally integrable variation; but it actually has integrable variation, since (22) holds even if we take the absolute value of the integrand on the left-hand side, and the expected value of the right-hand side is finite. Since U is a local martingale, there exists a sequence of stopping times (S_n) such that $\lim_{n\to\infty}S_n=\infty$, almost surely, and $\{U_{t\wedge S_n},\ t\in[0,\infty)\}$ is a martingale for every n. Let $t>0$ fixed, and let $T_n=\min\{S_n,t\}$. By the generalized Itô's rule (Proposition A.1) and by the integration of parts formula (Protter, [18], Corollary 2 to Theorem II.22), we have that

$$e^{-\lambda T_n}f\left(X_{T(n)}\right)-f(x)=\int_{(0,T_n]}e^{-\lambda s}f'(X_{s-})dX_s+\int_0^{T_n}e^{-\lambda s}\left\{\frac{\sigma^2}{2}f''(X_s)-\lambda f(X_s)\right\}ds$$

$$+\sum_{s\le T_n}e^{-\lambda s}\left\{f(X_x)-f(X_{s-})-f'(X_{s-})\Delta X_s\right\}. \tag{24}$$

By (2) and (4) this can be cast in the form

$$e^{-\lambda T_n} f\left(X_{T(n)}\right) - f(x) = \int_{(0,T_n]} e^{-\lambda s} f'(X_{s-})(dA_s + d\varphi_s + \sigma w_s) + \tag{25}$$

$$\int_{(0,T_n]} e^{-\lambda s} \left\{ \frac{\sigma^2}{2} f''(X_s) - \lambda f(X_s) + \mu f'(X_s) \right\} ds + \tag{26}$$

$$\sum_{s \le T_n} e^{-\lambda s} \left\{ f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s \right\} + \sum_{i:\tau_i \le T_n} e^{-\lambda \tau_i} f'\left(X_{\tau(i)-}\right) \xi_i. \tag{27}$$

The jumps of X consist of the jumps of Y plus the jumps included in the control, thus we can cast (27) in the form

$$\sum_{s \le T_n} e^{-\lambda s} \left\{ f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta Y_s \right\}, \tag{28}$$

which can be written as

$$\sum_{s \le T_n} e^{-\lambda s} \left\{ f\left(X_s\right) - f\left(X_{s-} + \Delta Y_s\right) \right\} + \sum_{s \le T_n} e^{-\lambda s} \left\{ f(X_{s-} + \Delta Y_s) - f(X_{s-}) - f'(X_{s-})\Delta Y_s \right\}. \tag{29}$$

Note that the first sum in the above expression has only finitely many terms; the number of terms is equal to the number of i's such that $\tau_i \le T_n$, which is finite by (9). We substitute (29) into (27), and get

$$e^{-\lambda T_n} f\left(X_{T(n)}\right) - f(x) = \int_{(0,T_n]} e^{-\lambda s} f'(X_{s-})(dA_s + d\varphi_s + \sigma w_s) + \tag{30}$$

$$\int_{(0,T_n]} e^{-\lambda s} \left\{ \frac{\sigma^2}{2} f''(X_s) - \lambda f(X_s) + \mu f'(X_s) \right\} ds + \tag{31}$$

$$\sum_{s \le T_n} e^{-\lambda s} \left\{ f\left(X_s\right) - f\left(X_{s-} + \Delta Y_s\right) \right\} + \sum_{s \le T_n} e^{-\lambda s} \left\{ f(X_{s-} + \Delta Y_s) - f(X_{s-}) - f'(X_{s-})\Delta Y_s \right\}. \tag{32}$$

We cancel the integral with respect to A_s in (30) with the "large" jumps in the second summation in (32), and write

$$e^{-\lambda T_n} f\left(X_{T(n)}\right) - f(x) = \int_{(0,T_n]} e^{-\lambda s} f'(X_{s-})(d\varphi_s + \sigma w_s) + \tag{33}$$

$$\int_{(0,T_n]} e^{-\lambda s} \left\{ \frac{\sigma^2}{2} f''(X_s) - \lambda f(X_s) + \mu f'(X_s) \right\} ds + \tag{34}$$

$$\sum_{s\le T_n} e^{-\lambda s}\left\{f\left(X_s\right)-f\left(X_{s-}+\Delta Y_s\right)\right\}+\sum_{s\le T_n} e^{-\lambda s}\left\{f(X_{s-}+\Delta Y_s)-f(X_{s-})-f'(X_{s-})\Delta Y_s 1_{\{|\Delta Y_s|<1\}}\right\}. \tag{35}$$

The second sum in (35) can be written as

$$U_{T(n)}+\int_{(0,T_n]\times\Re} e^{-\lambda s}\left\{f(X_{s-}+y)-f(X_{s-})-f'(X_{s-})y1_{\{|y|<1\}}\right\}\nu(dy)ds, \tag{36}$$

so after substituting this back to (35), and using (11) we get the following equality:

$$e^{-\lambda T_n} f\left(X_{T(n)}\right)-f(x)=\int_{(0,T_n]} e^{-\lambda s} f'(X_{s-})(d\varphi_s+\sigma w_s)+ \tag{37}$$

$$\int_{(0,T_n]} e^{-\lambda s}\left\{\mathcal{A}f(X_{s-})-\lambda f(Xs-)\right\}ds+ \tag{38}$$

$$\sum_{s\le T_n} e^{-\lambda s}\left\{f\left(X_s\right)-f\left(X_{s-}+\Delta Y_s\right)\right\}+U_{T(n)}. \tag{39}$$

Next we take expectations; all martingale terms will disappear. Indeed, the local martingale

$$Z_t=\int_{(0,t]} e^{-\lambda s} f'(X_{s-})d\varphi_s$$

has quadratic variation

$$[Z]_t=\int_{(0,t]} e^{-2\lambda s}\left(f'(s)\right)^2 d\left[\varphi\right]_s$$

([18], Chapter II, Theorem 29), and the boundedness of f' and (3) imply that Z is a square-integrable martingale (see also [18], Chapter II, Corollary 3 to Theorem 27). Then by the Optional sampling Theorem for bounded stopping times we have $E[Z_{T_n}]=0$. The boundedness of $f'(\cdot)$ implies that the stochastic integral with respect to the Brownian motion also has zero expected value. On the other hand, $U_{T(n)}$ has zero expectation because $(S_n, n\ge 1)$ is a localizing sequence for the local martingale U. Therefore we can write the identity

$$f(x)=E_x\left[e^{-\lambda T_n} f\left(X_{T(n)}\right)\right]-E_x\left[\int_{(0,T_n]} e^{-\lambda s}\left\{\mathcal{A}f(X_{s-})-\lambda f(Xs-)\right\}ds\right]-$$

$$E_x\left[\sum_{s\le T_n} e^{-\lambda s}\left\{f\left(X_s\right)-f\left(X_{s-}+\Delta Y_s\right)\right\}\right]. \tag{40}$$

Using conditions (i), (ii) we arrive at

$$f(x) \leq E_x\left[e^{-\lambda T_n} f\left(X_{T(n)}\right)\right] + E_x\left[\int_0^{T_n} e^{-\lambda s}\phi(X_s)ds + \sum_{\tau_j \leq T_n} e^{-\lambda\tau_j} g(\xi_j)\right]. \tag{41}$$

There is a minor problem here since (i) and implies the inequality $\mathcal{A}f(x) - \lambda f(x) + \phi(x) \geq 0$ only for $x \in \Re \setminus S$. But either $\sigma = 0$, in which case the function $x \mapsto \mathcal{A}f(x) - \lambda f(x) + \phi(x)$ is continuous, thus the inequality holds for all $x \in \Re$, or $\sigma \neq 0$, in which case (41) still holds by Lemma A.3 in Appendix A. Next we take limits of both sides in (41) as $n \to \infty$. Since f is continuously differentiable and has bounded derivative on $\Re$, it satisfies a linear growth condition $f(x) \leq K_1|x| + K_2$ for some positive constants K_1, K_2, hence by (5)

$$0 \leq e^{-\lambda t} f(X_t) \leq e^{-\lambda t}(K_1|X_t| + K_2) \leq K_3. \tag{42}$$

The first term on the right-hand side of (41) converges to $E_x\left[e^{-\lambda t} f(X_t)\right]$ by (42) and by the Bounded Convergence Theorem. For the second term on the right-hand side we apply the Monotone Convergence Theorem, and conclude that

$$f(x) \leq E_x\left[e^{-\lambda t} f(X_t)\right] + E_x\left[\int_0^{t} e^{-\lambda s}\phi(X_s)ds + \sum_{\tau_j \leq t} e^{-\lambda\tau_j} g(\xi_j)\right]. \tag{43}$$

Next we take the lim inf in (43) as $t \to \infty$. By (10) the lim inf of the first term on the right-hand side of (43) is zero. We again apply the Monotone Convergence Theorem for the second term, and conclude that

$$f(x) \leq E_x\left[\int_0^{\infty} e^{-\lambda s}\phi(X_s)ds + \sum_{j=1}^{\infty} e^{-\lambda\tau_j} g(\xi_j)\right], \tag{44}$$

which completes the proof. •

Proof of Theorem 3.2: By Lemma 3.3 the conditions of Theorem 3.2 are stronger than those of Theorem 3.1, hence all statements in the proof of Theorem 3.1 are valid. In particular, (40) is still true. From condition (a), Lemma A.3, (18), and (19) follows that we have equalities in (41), (43), and (44). •

4 Analysis of the Optimal Control (T^*, Ξ^*)

We now set out to determine the transient and steady-state behavior of the controlled cash on hand process X_t under the optimal control (T^*, Ξ^*) assuming that Y_t is a spectrally positive Lévy process. In other words,

$$\int_{\Re_-} \nu(dy) = 0$$

and Y_t is not a subordinator. The case of a spectrally negative Lévy process may be treated similarly. Our main result will be to determine the Laplace transform (with respect to time) of the transition probabilities of X_t and also to determine the limiting distribution of X_t.

We begin by determining the Laplace transform of the transition probabilities of X_t. Let $\mathcal{A} \in \mathcal{B}(\Re)$ be a Borel set of $\Re$ and let e_q be an exponential random variable with rate q independent of X. Now consider $P_x[X_{e_q} \in \mathcal{A}]$ for $x \in (a, b)$. It then follows conditioning on the value of e_q relative to the stopping time τ_1^* that

$$P_x[X_{e_q} \in \mathcal{A}] = E_x[1\{X_{e_q} \in \mathcal{A}\}1\{e_q < \tau_1^*\}] + E_x[1\{X_{e_q} \in \mathcal{A}\}1\{e_q \geq \tau_1^*\}]. \tag{45}$$

However, since $X_t = Y_t$ for $0 \leq t < \tau_1^*$, it follows by the memoryless property of the exponential distribution and the strong Markov property that $E_x[1\{X_{e_q} \in \mathcal{A}\}1\{e_q \geq \tau_1^*\}]$ is equal to

$$E_\alpha[1\{X_{e_q} \in \mathcal{A}\}]P_x[\{e_q \geq \tau_1^*\} \cap \{Y(\tau_1^*) \leq a\}] + E_\beta[1\{X_{e_q} \in \mathcal{A}\}]P_x[\{e_q \geq \tau_1^*\} \cap \{Y(\tau_1^*) \geq b\}].$$

Substituting the above into (45), we have

$$\begin{aligned} E_x[1\{X_{e_q} \in \mathcal{A}\}] &= E_x[1\{X_{e_q} \in \mathcal{A}\}1\{e_q < \tau_1^*\}] \\ &\quad + E_\alpha[1\{X_{e_q} \in \mathcal{A}\}]P_x[\{e_q \geq \tau_1^*\} \cap \{Y(\tau_1^*) \leq a\}] \\ &\quad + E_\beta[1\{X_{e_q} \in \mathcal{A}\}]P_x[\{e_q \geq \tau_1^*\} \cap \{Y(\tau_1^*) \geq b\}]. \end{aligned} \tag{46}$$

It therefore remains to determine expressions for the three quantities on the righthand side of (46). We proceed term by term.

First note that τ_1^* is equal to the fist time the Lévy process Y_t exits the open interval (a, b). We then have that in general one may write

$$\begin{aligned} E_x[1\{X_{e_q} \in \mathcal{A}\}1\{e_q < \tau_1^*\}] &= E_x[1\{Y_{e_q} \in \mathcal{A}\}1\{e_q < \tau_1^*\}] \\ &= q\int_0^\infty e^{-qt}E_x[1\{Y_t \in \mathcal{A}\}1\{\tau_1^* > t\}]dt \\ &= q\int_{\mathcal{A}}\int_0^\infty e^{-qt}P_x[Y(t) \in dy, \tau_1^* > t]dt \\ &= q\int_{\mathcal{A}} U^{(q)}(x, dy), \end{aligned} \tag{47}$$

where $U^{(q)}$ is the q-potential measure of Y_t (see (78)). This then provides an expression for the first term on the righthand side of (46). Also note, by Theorem 8.7 of [15] (quoted in this paper under

Theorem B.2), if Y_t is spectrally positive then its q-potential measure $U^{(q)}(x,dy)$ has a density $u^{(q)}(x,y)$ given by

$$u^{(q)}(x,y) \quad = \quad W^{(q)}(b-x)\frac{W^{(q)}(y-a)}{W^{(q)}(b-a)} - W^{(q)}(y-x), \tag{48}$$

where

$$\int_0^\infty e^{-sy}W^{(q)}(y)dy \quad = \quad \frac{1}{\psi(-s)-q}, \tag{49}$$

whenever $-s$ is large enough so that $\psi(-s) > q$ and $\psi(s) = \log E_0[e^{sY_1}]$ is the Laplace exponent of Y_t. Note that $\psi(s) < \infty$ for all $s \leq 0$ by the spectral positivity of Y.

Next note that setting $x = \alpha$ in (46), we obtain

$$\begin{aligned} & E_\alpha[1\{X_{e_q} \in \mathcal{A}\}](1 - P_\alpha[\{e_q \geq \tau_1^*\} \cap \{Y(\tau_1^*) \leq a\})) \\ & -E_\beta[1\{X_{e_q} \in \mathcal{A}\}]P_\alpha[\{e_q \geq \tau_1^*\} \cap \{Y(\tau_1^*) \geq b\}] \\ = \quad & E_\alpha[1\{X_{e_q} \in \mathcal{A}\}1\{e_q < \tau_1^*\}], \end{aligned} \tag{50}$$

and similarly, setting $x = \beta$, we have

$$\begin{aligned} & E_\beta[1\{X_{e_q} \in \mathcal{A}\}](1 - P_\beta[\{e_q \geq \tau_1^*\} \cap \{Y(\tau_1^*) \geq b\}]) \\ & -E_\alpha[1\{X_{e_q} \in \mathcal{A}\}]P_\beta[\{e_q \geq \tau_1^*\} \cap \{Y(\tau_1^*) \leq a\}] \\ = \quad & E_\beta[1\{X_{e_q} \in \mathcal{A}\}1\{e_q < \tau_1^*\}]. \end{aligned} \tag{51}$$

(50) and (51) constitute a set of linear equations for $E_\alpha[1\{X_{e_q} \in \mathcal{A}\}]$ and $E_\beta[1\{X_{e_q} \in \mathcal{A}\}]$. Moreover, so long as $q > 0$, we have that

$$\begin{aligned} & (1 - P_\beta[\{e_q \geq \tau_1^*\} \cap \{Y(\tau_1^*) \geq b\}])(1 - P_\alpha[\{e_q \geq \tau_1^*\} \cap \{Y_{\tau_1^*} \leq a\}]) \\ > \quad & P_\alpha[\{e_q \geq \tau_1^*\} \cap \{Y(\tau_1^*) \geq b\}]P_\beta[\{e_q \geq \tau_1^*\} \cap \{Y_{\tau_1^*} \leq a\}] \end{aligned}$$

and so the determinant associated with (50) and (51) is non-zero and hence a solution exists. Solving for $E_\alpha[1\{X_{e_q} \in \mathcal{A}\}]$ and $E_\beta[1\{X_{e_q} \in \mathcal{A}\}]$ then yields that $E_\alpha[1\{X_{e_q} \in \mathcal{A}\}]$ is given by

$$\frac{1}{C_{a,\alpha,\beta,b}} \times \begin{pmatrix} E_\alpha[1\{X_{e_q} \in \mathcal{A}\}1\{e_q < \tau_1^*\}](1 - P_\beta[\{e_q \geq \tau_1^*\} \cap \{Y(\tau_1^*) \geq b\}] \\ +E_\beta[1\{X_{e_q} \in \mathcal{A}\}1\{e_q < \tau_1^*\}]P_\alpha[\{e_q \geq \tau_1^*\} \cap \{Y(\tau_1^*) \geq b\}] \end{pmatrix} \tag{52}$$

and $E_\beta[1\{X_{e_q} \in \mathcal{A}\}]$ is given by

$$\frac{1}{C_{a,\alpha,\beta,b}} \times \begin{pmatrix} E_\alpha[1\{X_{e_q} \in \mathcal{A}\}1\{e_q < \tau_1^*\}]P_\beta[\{e_q \geq \tau_1^*\} \cap \{Y_{\tau_1^*} \leq a\}] \\ +E_\beta[1\{X_{e_q} \in \mathcal{A}\}1\{e_q < \tau_1^*\}](1 - P_\alpha[\{e_q \geq \tau_1^*\} \cap \{Y_{\tau_1^*} \leq a\}]) \end{pmatrix}, \tag{53}$$

where

$$C_{a,\alpha,\beta,b} = (1 - P_\beta[\{e_q \geq \tau_1^*\} \cap \{Y(\tau_1^*) \geq b\}])(1 - P_\alpha[\{e_q \geq \tau_1^*\} \cap \{Y_{\tau_1^*} \leq a\}]) \\ -P_\alpha[\{e_q \geq \tau_1^*\} \cap \{Y(\tau_1^*) \geq b\}]P_\beta[\{e_q \geq \tau_1^*\} \cap \{Y_{\tau_1^*} \leq a\}].$$

We now proceed to compute the terms appearing on the right-hand sides of (52) and (53).

Firs note that the expressions of the form $E_x[1\{X_{e_q} \in \mathcal{A}\}1\{e_q < \tau_1^*\}]$ appearing in (52) and (53) have already been determined by (47). It remains to determine expressions for the probabilities appearing in (52) and (53). However, note that

$$\begin{aligned} P_x[\{e_q \geq \tau_1^*\} \cap \{Y_{\tau_1^*} \leq a\}] &= \int_0^\infty qe^{-qt} P_x[\{t \geq \tau_1^*\} \cap \{Y_{\tau_1^*} \leq a\}]dt \\ &= E_x\left[\int_{\tau_1^*}^\infty qe^{-qt}dt1\{Y_{\tau_1^*} \leq a\}\right] \\ &= E_x\left[e^{-q\tau_1^*}1\{Y_{\tau_1^*} \leq a\}\right] \\ &= Z^{(q)}(b-x) - Z^{(q)}(b-a)\frac{W^{(q)}(b-x)}{W^{(q)}(b-a)}, \end{aligned}$$

where the final equality follows from Theorem 8.1 of [15] (quoted in this paper under Theorem B.1 in Appendix B) and we have the relationship

$$Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)dy.$$

In a similar fashion, using Theorem 8.1 of [15] (quoted in this paper under Theorem B.1 in Appendix B), one may compute

$$P_x[\{e_q \geq \tau_1^*\} \cap \{Y(\tau_1^*) \geq b\}] = \frac{W^{(q)}(b-x)}{W^{(q)}(b-a)}.$$

Substituting the above results into (52) and (53), and subsequently into (46), one obtains an expression for the Laplace transform of the transition probabilities of X_t.

We now proceed towards obtaining an expression for the limiting distribution of X_t as $t \to \infty$. Our main result in this regard will be the following. Recall the definition of $U^{(q)}$ as the q-potential measure associated with Y.

Proposition 4.1. *If Y_t is spectrally positive, then under a double bandwidth control policy (a, α, β, b), the limiting distribution of X_t is given by*

$$\pi(\mathcal{A}) = \frac{1}{K_{a,\alpha,\beta,b}}\left(\left(1 - \frac{W^{(0)}(b-\beta)}{W^{(0)}(b-a)}\right)\int_{\mathcal{A}} U^{(0)}(\alpha, dy) + \left(\frac{W^{(0)}(b-\alpha)}{W^{(0)}(b-a)}\right)\int_{\mathcal{A}} U^{(0)}(\beta, dy)\right),$$

for each $\mathcal{A} \in \mathcal{B}$, where $K_{a,\alpha,\beta,b}$ is an appropriate normalizing constant, as given by (57).

We prove Proposition 4.1 in a series of lemmas. First note that by the strong Markov property, X_t is a regenerative process with possible regeneration points either α or β. Let us consider the point α and define $n_\alpha^\star = \inf\{n \geq 1 : X(\tau_n^\star) = \alpha\}$. By the standard theory of regenerative processes, see for instance Theorem 1.2 of Chapter VI of [1], if we may show that $E_\alpha[\tau_{n_\alpha^*}] < \infty$ and that $\tau_{n_\alpha^*}$ is nonlattice[1], then $\lim_{t\to\infty} P_x(X_t \in \mathcal{A}) = \pi(\mathcal{A})$ exists for all $\mathcal{A} \in \mathcal{B}(\Re)$ and $x \in \Re$ and is given by

$$\pi(\mathcal{A}) \quad = \quad \frac{E_\alpha[\int_0^{\tau_{n_\alpha^*}} 1\{X_s \in \mathcal{A}\}ds]}{E_\alpha[\tau_{n_\alpha^*}]}.$$

The following lemma now shows that $E_\alpha[\tau_{n_\alpha^*}] < \infty$.

Lemma 4.2. *If the Lévy process Y_t is spectrally positive, then $E_\alpha[\tau_{n_\alpha^*}] < \infty$.*

Proof: Note first that

$$\begin{aligned}\tau_{n_\alpha^\star} &= \tau_1^\star 1\{Y(\tau_1^*) \leq a\} + \tau_{n_\alpha^\star} 1\{Y(\tau_1^*) \geq b\} \\ &= \tau_1^\star + (\tau_{n_\alpha^\star} - \tau_1^*) 1\{Y(\tau_1^*) \geq b\}.\end{aligned}$$

Hence, by the strong Markov property,

$$\begin{aligned}E_\alpha[\tau_{n_\alpha^\star}] &= E_\alpha[\tau_1^\star] + E_\alpha[(\tau_{n_\alpha^\star} - \tau_1^*) 1\{Y(\tau_1^*) \geq b\}] \\ &= E_\alpha[\tau_1^\star] + E_\beta[\tau_{n_\alpha^\star}] P_\alpha[Y(\tau_1^*) \geq b].\end{aligned}$$

Similarly, we may show

$$E_\beta[\tau_{n_\alpha^\star}] \quad = \quad E_\beta[\tau_1^\star] + E_\beta[\tau_{n_\alpha^\star}] P_\beta[Y(\tau_1^*) \geq b],$$

from which we obtain

$$E_\beta[\tau_{n_\alpha^\star}] \quad = \quad \frac{E_\beta[\tau_1^\star]}{1 - P_\beta[Y(\tau_1^*) \geq b]}.$$

Now note that since Y_t is spectrally positive, we have by Theorem 8.1 of [15] (quoted in this paper under Theorem B.1 in Appendix B) that

$$P_\beta[Y(\tau_1^*) \geq b] \quad = \quad \frac{W^{(0)}(b-\beta)}{W^{(0)}(b-a)} \quad < \quad 1$$

and so it suffices from the above to show $E_\alpha[\tau_1^\star], E_\beta[\tau_1^\star] < \infty$.

[1] A random variable is called *lattice* if its distribution is concentrated on a set of the form $\{\delta, 2\delta, \dots\}$. It is called nonlattice if it is not.

We now show that in general for $x \in (a,b)$, $E_x[\tau_1^\star] < \infty$. Recall by [15], the potential measure of Y_t upon exiting $[a,b]$ is given by

$$U(x,dy) = \int_0^\infty P_x[Y_t \in dy, \tau_1^\star > t]dt.$$

Integrating over $[a,b]$, we obtain that

$$\begin{aligned}
\int_{[a,b]} U(x,dy) &= \int_{[a,b]} \int_0^\infty P_x[Y_t \in dy, \tau_1^\star > t]dt \\
&= \int_0^\infty \int_{[a,b]} P_x[Y_t \in dy, \tau_1^\star > t]dt \\
&= \int_0^\infty P_x[\tau_1^\star > t]dt \\
&= E_x[\tau_1^\star].
\end{aligned}$$

However, by Theorem 8.7 of [15] (quoted in this paper in Appendix B, Theorem B.2), since Y_t is spectrally positive, $U(x,dy)$ has a density given by

$$u(x,y) = W^{(0)}(b-x)\frac{W^{(0)}(y-a)}{W^{(0)}(b-a)} - W^{(0)}(y-x).$$

Integrating over $[a,b]$, we therefore find that

$$\begin{aligned}
\int_{[a,b]} U(x,dy) &= \int_{[a,b]} \left(W^{(0)}(b-x)\frac{W^{(0)}(y-a)}{W^{(0)}(b-a)} - W^{(0)}(y-x) \right) dy \\
&< \infty,
\end{aligned}$$

where the inequality follows since $W^{(0)}$, due to its continuity on $[0,\infty)$ (see [15] Theorem 8.1, quoted in this paper in Appendix B, Theorem B.1), is bounded on compact sets. By the above, this completes the proof. •

The following lemma allows us to take the limit as $q \to 0$ in (52) and (53) in order to obtain the limiting distribution of X_t.

Lemma 4.3. *For each $x \in \Re$,*

$$\lim_{q\to 0} P_x[X(e_q) \in \mathcal{A}] = \pi(\mathcal{A}) \tag{54}$$

Proof: Select $T > 0$ large enough so that $|P_x[X_t \in \mathcal{A}] - \pi(\mathcal{A})| < \epsilon$. Then

$$|P_x[X(e_q) \in \mathcal{A}] - \pi(\mathcal{A})| = \left| \int_0^\infty \{P_x[X_t \in \mathcal{A}] - \pi(\mathcal{A})\} q e^{-qt} dt \right|$$

$$\leq \left| \int_0^T \{P_x[X_t \in \mathcal{A}] - \pi(\mathcal{A})\} qe^{-qt} dt \right| + \left| \int_T^\infty \{P_x[X_t \in \mathcal{A}] - \pi(\mathcal{A})\} qe^{-qt} dt \right|.$$

The second term in the above expression is bounded by ϵ and the first term converges to zero as $q \to 0$ by the Dominated Convergence Theorem, which completes the proof. •

We now provide the proof of Proposition 4.1.

Proof of Proposition 4.1: Using Lemma 4.3, we now wish to take limits $q \to 0$ in (52) in order to determine the limiting distribution δ. However, both the numerator and denominator in (52) converge to 0 as $q \to 0$ and so we must apply L'Hoptial's rule. Before doing so, however, we first must verify that both the numerator and denominator in (52) are differentiable.

By Lemma 8.3 and Corollary 8.5 in [15] we have that for each $x > 0$, both $W^q(x)$ and $Z^{(q)}(x)$ are differentiable in q. Moreover, since

$$\frac{W^{(q)}(b-x)}{W^{(q)}(b-a)} = E_x[e^{-q\tau_1^*} 1\{Y_{\tau_1^*} \geq b\}],$$

for $a \leq x < b$, it follows that

$$\begin{aligned} \frac{d}{dq} \frac{W^{(q)}(b-x)}{W^{(q)}(b-a)} &= -E_x[\tau_1^* e^{-q\tau_1^*} 1\{Y_{\tau_1^*} \geq b\}] \\ &< \infty, \end{aligned}$$

where the inequality follows as in the proof of Lemma 4.2. Finally, since for each $a \leq x \leq b$, $U^q(x)$ has a density $u^{(q)}(x, y)$ given by (48) it follows that for each $\mathcal{A} \in \mathcal{B}(\Re)$,

$$\frac{d}{dq} \int_{\mathcal{A}} U^{(q)}(x, dy) = \int_{\mathcal{A}} \frac{d}{dq} u^{(q)}(x, y) dy.$$

Thus, noting that

$$\begin{aligned} &(E_\alpha[1\{X_{e_q} \in \mathcal{A}\}1\{e_q < \tau_1^*\}](1 - P_\beta[\{e_q \geq \tau_1^*\} \cap \{Y(\tau_1^*) \geq b\}]) \\ &+E_\beta[1\{X_{e_q} \in \mathcal{A}\}1\{e_q < \tau_1^*\}]P_\alpha[\{e_q \geq \tau_1^*\} \cap \{Y(\tau_1^*) \geq b\}]) \\ = \;& q \int_{\mathcal{A}} U^{(q)}(\alpha, dy) \left(1 - \frac{W^{(q)}(b-\beta)}{W^{(q)}(b-a)}\right) + q \int_{\mathcal{A}} U^{(q)}(\beta, dy) \left(\frac{W^{(q)}(b-\alpha)}{W^{(q)}(b-a)}\right) \end{aligned} \tag{55}$$

and

$$\begin{aligned} &((1 - P_\beta[\{e_q \geq \tau_1^*\} \cap \{Y(\tau_1^*) \geq b\}])(1 - P_\alpha[\{e_q \geq \tau_1^*\} \cap \{Y_{\tau_1^*} \leq a\}]) \\ &-P_\alpha[\{e_q \geq \tau_1^*\} \cap \{Y(\tau_1^*) \geq b\}]P_\beta[\{e_q \geq \tau_1^*\} \cap \{Y_{\tau_1^*} \leq a\}]) \\ = \;& \left(1 - \frac{W^{(q)}(b-\beta)}{W^{(q)}(b-a)}\right) \left(1 - \left(Z^{(q)}(b-\alpha) - Z^{(q)}(b-a)\frac{W^q(b-\alpha)}{W^q(b-a)}\right)\right) \end{aligned} \tag{56}$$

$$-\left(Z^{(q)}(b-\beta)-Z^{(q)}(b-a)\frac{W^q(b-\beta)}{W^q(b-a)}\right)\frac{W^{(q)}(b-\alpha)}{W^{(q)}(b-a)},$$

we see that both the numerator and denominator in (52) are differentiable.

Let us now take derivatives on the righthand sides of (55) and (56).

Taking the derivative of the right hand side of (55) and evaluating at $q=0$ we obtain

$$\int_{\mathcal{A}} U^{(0)}(\alpha,dy)\left(1-\frac{W^{(0)}(b-\beta)}{W^{(0)}(b-a)}\right)+\int_{\mathcal{A}} U^{(0)}(\beta,dy)\left(\frac{W^{(0)}(b-\alpha)}{W^{(0)}(b-a)}\right).$$

Next, recalling that

$$Z^q(x) \quad = \quad 1+q\int_0^x W^{(q)}(y)dy,$$

it follows upon taking the derivative of the righthand side of (56) and evaluating at $q=0$ that we obtain

$$\begin{aligned}
&\frac{d}{dq}\left(\left(1-\frac{W^{(q)}(b-\beta)}{W^{(q)}(b-a)}\right)\left(1-\left(Z^{(q)}(b-\alpha)-Z^{(q)}(b-a)\frac{W^q(b-\alpha)}{W^q(b-a)}\right)\right)\right.\\
&\left.-\left(\left(Z^{(q)}(b-\beta)-Z^{(q)}(b-a)\frac{W^q(b-\beta)}{W^q(b-a)}\right)\frac{W^{(q)}(b-\alpha)}{W^{(q)}(b-a)}\right)\right)\\
=\;&\frac{W^{(0)}(b-\beta)}{W^{(0)}(b-a)}\int_0^{b-\alpha}W^0(x)dx+\frac{W^{(0)}(b-\alpha)}{W^{(0)}(b-a)}\int_{b-\beta}^{b-a}W^0(x)dx-\int_0^{b-\alpha}W^0(x)dx\\
=\;&K_{a,\alpha,\beta,b}. \qquad (57)
\end{aligned}$$

Thus, by (52) and Lemma 4.2 we obtain the desired result. •

5 An Example

We suppose in this section that

$$Y_t = x+\sigma w_t + N_t,$$

where N is a compound Poisson process independent of w such that the rate of jump arrivals is equal to 1 and the Lévy measure ν of N is

$$\nu(dy)=\theta e^{-\theta y}dy, \quad y\geq 0$$

for some $\theta>0$, and $\nu((-\infty,0])=0$. In this case, we can write the linear operator $\mathcal{A}$ in the form

$$\mathcal{A}f(x)=\frac{\sigma^2}{2}f''(x)+\int_0^\infty \left[f(x+y)-f(x)\right]\nu(dy). \qquad (58)$$

We also specify the opportunity cost function as

$$\phi(x) = (x-\rho)^2$$

where ρ is a fixed target value. Suppose now that $x \in (a, b)$. Using (58), the equation in (i) in Theorem 3.1 may be written as

$$\frac{1}{2}\sigma^2 f''(x) + (x-\rho)^2 - \lambda f(x) + \int_0^\infty [f(x+y) - f(x)]\, \theta e^{-\theta y} dy = 0.$$

This becomes

$$\frac{1}{2}\sigma^2 f''(x) + (x-\rho)^2 - (1+\lambda) f(x) + \theta e^{\theta x} \int_x^b f(z) e^{-\theta z} dz +$$

$$e^{\theta x} \int_b^\infty [f(b) + d(z-b)]\, \theta e^{-\theta z} dz = 0, \tag{59}$$

and also

$$\frac{1}{2}\sigma^2 e^{-\theta x} f''(x) + e^{-\theta x}(x-\rho)^2 - (1+\lambda) e^{-\theta x} f(x) + \theta \int_x^b f(z) e^{-\theta z} dz + \zeta = 0, \tag{60}$$

where

$$\zeta = \int_b^\infty [f(b) + d(z-b)]\, \theta e^{-\theta z} dz.$$

Let us now introduce $e^{-\theta x} f(x) = g(x)$. We then obtain the following equation from (60):

$$(\frac{1}{2}\sigma^2\theta^2 - \lambda - 1) g(x) + \sigma^2 \theta g'(x) + \frac{1}{2}\sigma^2 g''(x) + e^{-\theta x}(x-\rho)^2 + \theta \int_x^b g(z) dz + \zeta = 0. \tag{61}$$

Differentiating the above with respect to x we get the following inhomogeneous linear ordinary differential equation of the third order:

$$\frac{1}{2}\sigma^2 g''' + \sigma^2 \theta g'' + \left(\frac{1}{2}\sigma^2\theta^2 - \lambda - 1\right) g' - \theta g + 2e^{-\theta x}(x-\rho) - \theta e^{-\theta x}(x-\rho)^2 = 0. \tag{62}$$

A particular solution for the inhomogeneous equation, denoted by g_p, is given by

$$g_p(x) = e^{-\theta x}\left[K_1(x-\rho)^2 + K_2(x-\rho) + K_3\right],$$

where

$$K_1 = \frac{1}{\lambda} \quad , \; K_2 = \frac{2}{\theta\lambda^2} \quad , \; K_3 = \frac{1}{\lambda^3\theta^2}\left[2\lambda + 2 + \theta^2\lambda\sigma^2\right].$$

The general solution of the homogeneous equation is given by g_h, that is,

$$g_h(x) = L_1 e^{c_1 x} + L_2 e^{c_2 x} + L_3 e^{c_3 x}$$

where c_1, c_2, c_3 are the roots of the equation

$$P(x) = \frac{1}{2}\sigma^2 x^3 + \sigma^2\theta x^2 + \left(\frac{1}{2}\sigma^2\theta^2 - \lambda - 1\right)x - \theta = 0$$

and L_1, L_2, L_3 are "free parameters". Notice that $P(0) = -\theta < 0$, $P(-\theta) = \theta\lambda > 0$, and $\lim_{x\to-\infty} P(x) = -\infty$, $\lim_{x\to\infty} P(x) = \infty$ thus $P(x)$ has three roots, say $c_1 < -\theta$, $-\theta < c_2 < 0$ and $c_3 > 0$.

We have now arrived at the following family of candidate solutions:

$$g(x; L_1, L_2, L_3, b) = e^{-\theta x}\left[K_1(x-\rho)^2 + K_2(x-\rho) + K_3\right] + L_1 e^{c_1 x} + L_2 e^{c_2 x} + L_3 e^{c_3 x}.$$

This gives

$$f(x; L_1, L_2, L_3, b) = K_1(x-\rho)^2 + K_2(x-\rho) + K_3 + L_1 e^{(\theta+c_1)x} + L_2 e^{(\theta+c_2)x} + L_3 e^{(\theta+c_3)x}. \quad (63)$$

For simplicity we shall use the notation $f(x; L_1, L_2, L_3, b) = f(x)$.

We now have 7 unknown parameters $a, \alpha, \beta, b, L_1, L_2, L_3$. From the conditions of Theorem 3.1, we may derive the following 6 equations for these constants:

$$f'(a) = -c \quad (64)$$

$$f'(\alpha) = -c \quad (65)$$

$$f'(b) = d \quad (66)$$

$$f'(\beta) = d \quad (67)$$

$$f(a) = f(\alpha) + C + c(\alpha - a) \quad (68)$$

$$f(b) = f(\beta) + D + d(b - \beta). \quad (69)$$

In addition, if we trace back our derivation in the above, then we see that we must have (59) hold for at least for one particular x since in going from (61) to (62) we took a derivative. Select $x = b$. This then gives us our 7th equation

$$\frac{1}{2}\sigma^2 f''(b) + (b-\rho)^2 - (1+\lambda)f(b) + e^{\theta b}\zeta = 0. \quad (70)$$

We now have the following.

Theorem 5.1. *Suppose that there exist seven constants $L_1 \leq 0, L_2 \leq 0, L_3 \leq 0, a < \alpha \leq \beta < b$ satisfying the seven equations (64)-(70). We define h by*

$$h(x) = \begin{cases} f(a) - c(x-a), & \text{if } x \leq a, \\ f(x), & \text{if } a \leq x \leq b \\ f(b) + d(x-b), & \text{if } x \geq b. \end{cases}$$

Then $h(x) = V(x)$, i.e., $h(x)$ is the value function of the optimization problem. Furthermore, the policy (T^*, Ξ^*) described in (14) and (16) with this choice of a, α, β, b is optimal.

In order to prove this theorem, we need the following lemma.

Lemma 5.1. *Assume the conditions of Theorem 5.1. Then there exists a constant $\xi \in (\alpha, \beta)$ such that h' is convex on $[a, \xi]$, concave on $[\xi, b]$. Furthermore $h'(x) \leq -c$ if $x \in [a, \alpha]$, $h'(x) \geq d$ if $x \in [\beta, b]$, and $-c \leq h'(x) \leq d$ if $x \in [\alpha, \beta]$.*

Proof: From the condition that $L_1, L_2, L_3 \leq 0$ it follows that $h'''(x)$ is decreasing on (a, b). Therefore $h''(x)$ has at most two zero points, which implies that $h'(x)$ has at most two local extreme values in (a, b). By (64)-(67) the derivative function h' must be first decreasing then increasing then again decreasing on $[a, b]$. Since h'' is concave, it must be either increasing, or decreasing, or first increasing then decreasing on $[a, b]$. But the first two possibilities are not possible, since h' can not be neither convex nor concave on $[a, b]$. Therefore it must be first convex then concave.

Proof of Theorem 5.1: We need to prove that the conditions of Theorem 3.1 are satisfied. Condition (i) and the required smoothness of h follows from our construction. Next we prove (ii) and (iv). From conditions (65), (67) and Lemma 5.1 it follows that

$$Mh(x) = \begin{cases} h(\alpha) + C + c(\alpha - x), & \text{if } a \leq x \leq \alpha, \\ h(x) + \min\{C, D\}, & \text{if } \alpha < x < \beta, \\ h(\beta) + D + d(x - \beta), & \text{if } \beta \leq x \leq b. \end{cases}$$

Condition (ii) follows from (68) and (69). We show condition (iv) for the case of $x \in [a, \alpha]$, the case of $x \in [\beta, b]$ is similar and the case of $x \in (\alpha, \beta)$ is obvious. We need to show that $0 \leq h(\alpha) - h(x) + C + c(\alpha - x)$. For $x = a$ we have equality by (68), and the derivative of the right-hand side of the inequality with respect to x is $-h'(x) - c$, which is non-negative on $[a, \alpha]$ by Lemma 5.1. Next we show condition (iii). First we look at the case of $x > b$. Let

$$K(x) = \mathcal{A}h(x) - \lambda h(x) + (x-\rho)^2; \quad x \in \Re \setminus \{a, b\}.$$

Then

$$K(x) = \frac{\sigma^2}{2} h''(x) + \int_0^\infty [h(x+y) - h(x)]\, \nu(dy) - \lambda h(x) + (x-\rho)^2 = 0, \quad x \in (a, b) \tag{71}$$

and

$$K(x)=\int_0^\infty [h(x+y)-h(x)]\,\nu(dy)-\lambda h(x)+(x-\rho)^2,\quad x\in(-\infty,a)\cup(b,\infty). \tag{72}$$

Therefore $0=K(b-)=\frac{\sigma^2}{2}h''(b-)+K(b+)$, and $\frac{\sigma^2}{2}h''(b-)\le 0$ implies $K(b+)\ge 0$. On the other hand for $x>b$ we have $K'(x)=-\lambda d+2(x-\rho)$ and $K'(b+)=-\lambda d+2(b-\rho)$. We also have by (71)

$$0=K'(x)=\frac{\sigma^2}{2}h'''(x)+\int_0^\infty \left[h'(x+y)-h'(x)\right]\nu(dy)-\lambda h'(x)+2(x-\rho),\quad x\in(a,b) \tag{73}$$

hence

$$0=K'(b-)=\frac{\sigma^2}{2}h'''(b-)-\lambda d+2(b-\rho)=\frac{\sigma^2}{2}h'''(b-)+K'(b+).$$

Since $h'''(b-)\le 0$ so we must have $K'(b+)\ge 0$. But K' is increasing on (b,∞), thus $K'(x)\ge 0$ whenever $x\in(b,\infty)$. This in turn implies that K is increasing on (b,∞), thus $K(x)\ge 0$ for $x\in(b,\infty)$.

Next we show that $K(x)\ge 0$ for $x<a$. By (71) and (72) $0=K(a+)=K(a-)+\frac{\sigma^2}{2}h''(a+)$ which implies that $K(a-)\ge 0$. Hence all we need to show is that $K'(x)\le 0$ for $x\le a$. Differentiating (72) we get

$$K'(x)=\int_0^\infty \left[h'(x+y)-h'(x)\right]\nu(dy)-\lambda h'(x)+2(x-\rho),\quad x<a,$$

and with a change of variable in the integral one can see that

$$K'(x)=e^{\theta(x-a)}\int_0^\infty [h'(a+z)+c]\nu(dz)+\lambda c+2(x-\rho),\quad x<a \tag{74}$$

and

$$K''(x)=\theta e^{\theta(x-a)}\int_0^\infty [h'(a+z)+c]\nu(dz)+2,\quad x<a. \tag{75}$$

Thus K'' is either increasing or decreasing on $(-\infty,a)$ depending on the sign of $\int_0^\infty [h'(a+z)+c]\nu(dz)$ which makes K' either convex or concave on $(-\infty,a)$. However, the fact that $\lim_{x\to-\infty}K'(x)=-\infty$ implies that K' must be concave and K'' decreasing on $(-\infty,a)$. In addition, the integral in (75) and in (74) is negative. By (73) and (74)

$$0=K'(a+)=\frac{\sigma^2}{2}h'''(a+)+K'(a-)$$

thus $K'(a-)\le 0$ follows from $h'''(a+)\ge 0$. Formula (74) imply that K' has at most one zero-point on $(-\infty,a)$. These facts about K' imply that indeed $K'(x)\le 0$. •

We conclude this section by providing an example showing how the limiting distribution π of Proposition 4.1 may be calculated for an arbitrary double bandwidth control policy (a,α,β,b). We will assume that $Y_t = \vartheta t + N_t$, where $\vartheta < 0$ and N_t is a compound Poisson process which has jumps at rate one and jump sizes which are exponentially distributed with rate θ. Note that by (48) and Proposition 4.1, it suffices to determine the function $W^{(0)} = \lim_{q\to 0} W^{(q)}$. By (49), the Laplace transform of $W^{(0)}$ is given by $1/\psi(-s)$ where $\psi(s)$ is the Lévy exponent of Y_t. Morover, by (8.1) in [15] it then follows that

$$\psi(s) \quad = \quad \vartheta s - \int_0^\infty (1 - e^{xs})\theta e^{-\theta x} dx,$$

for $s < \theta$, which reduces to $\psi(s) = \vartheta s + s(\theta - s)^{-1}$. One may now proceed to verify that

$$\frac{1}{\psi(-s)} \quad = \quad \frac{-\theta}{s(\vartheta s + \vartheta\theta + 1)} - \frac{1}{\vartheta s + \vartheta\theta + 1}.$$

In the case in which $\theta\vartheta \neq -1$ inverting each of the terms in the above, one obtains that the function $W^{(0)}$ is given by

$$W^{(0)}(x) \quad = \quad \frac{-\theta}{\theta\vartheta + 1} - \left(\frac{1}{\vartheta} - \frac{\theta}{\theta\vartheta + 1}\right) \exp\left(-\frac{\theta\vartheta + 1}{\vartheta} x\right).$$

For the case in which $\theta\vartheta = 1$, one has that $W^{(0)}(x) = \vartheta^{-1}(\theta x + 1)$. Substituting into the formula of Proposition 4.1, one may now obtain the density of π.

6 Acknowledgements

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A Appendix

In Appendix A, we provide a proof of the fact that Itô's rule applies to test functions $f \in \mathcal{D}$. For $f \in \mathcal{D}$ the second derivative $f''(x)$ may not exist in points $S = \{x_1, \ldots, x_m\}$. We shall call S the set of exceptional points. We extend f'' to the entire of $\Re$ by assuming an arbitrary value for $f''(x_i)$. This convention will be used in the rest of this section. The following is then the main result.

Proposition A.1. *If $f \in \mathcal{D}$ and X is a controlled cash on hand process with an arbitrary impulse control $(T, \Xi) = (\tau_1, \tau_2, \ldots, \tau_n, \ldots,\ \xi_1, \xi_2, \ldots, \xi_n, \ldots,)$ then Itô's rule holds in its usual form:*

$$f(X_t) - f(X_0) = \int_{(0,t]} f'(X_{s-})dX_s + \frac{\sigma^2}{2}\int_{(0,t]} f''(X_s)ds +$$

$$\sum_{0<s\leq t} \left\{f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\right\}. \tag{76}$$

In order to prove Proposition A.1, we need the following two lemmas.

Lemma A.2. *Let $f \in \mathcal{D}$ with set of exceptional points S. Then there exists a sequence $(f_n)_{n\geq 1} \subset C^2(\Re)$ such that the following hold;*
(i) $f_n(x) \to f(x)$ and $f'_n(x) \to f'(x)$ for every $x \in \Re$ as $n \to \infty$;
(ii) $f''_n(x) \to f''(x)$ for every $x \in \Re \setminus S$ as $n \to \infty$;
(iii) f'_n and f''_n are bounded uniformly in n, i.e., $|f'_n(x)| \leq C_1$ and $|f''_n(x)| \leq C_1$ for some constant C_1 and all n and $x \in \Re$.

The proof of this lemma can be based on the proof of a similar lemma in Økesendal [16], Appendix D with some obvious modifications.

Lemma A.3. *If $\sigma \neq 0$ then for every $z \in \Re$*

$$\int_0^\infty 1_{\{z\}}(X_s)ds = 0, \quad P_x\text{–a.s.}$$

In other words, the Lebesgue measure of the time the controlled cash on hand process spends at level z is zero.

Proof:

$$\begin{aligned} & E_x\left[\int_0^\infty 1_{\{z\}}(X_s)ds\right] \\ = & \int_0^\infty P_x[X_s = z]ds \\ \leq & \int_0^\infty P_x[\tau_i = s \text{ for some } i]ds + \int_0^\infty P_x[X_s = z,\ \tau_i \neq s \text{ for all } i]ds. \end{aligned}$$

We deal with these last two integrals separately.

$$\int_0^\infty P_x[\tau_i = s \text{ for some } i]ds \leq \int_0^\infty \sum_{i=1}^\infty P_x[\tau_i = s]ds = \sum_{i=1}^\infty \int_0^\infty P_x[\tau_i = s]ds$$

and this last expression is zero because the cardinality of the set $\{s \geq 0 : \ P_x[\tau_i = s] > 0\}$ is either countably infinite or finite. For the second integral we have

$$\int_0^\infty P_x[X_s = z,\ \tau_i \neq s \text{ for all } i]ds = \sum_{i=0}^\infty \int_0^\infty P_x\left[X_s = z,\ \tau_i < s < \tau_{i+1}\right]ds.$$

Now it suffices to show that the probability in the right-hand side is zero. Indeed,

$$P_x\left[X_s = z,\ \tau_i < s < \tau_{i+1}\right]$$

$$\begin{aligned}
&= \int_{[0,s]\times\Re} P_x\left[X_s = z,\ s < \tau_{i+1} \mid \tau_i = u, X_u = y\right] P_x\left[\tau_i \in du, X_u \in dy\right] \\
&= \int_{[0,s]\times\Re} P_x\left[X_s - X_u = z - y,\ s < \tau_{i+1} \mid \tau_i = u, X_u = y\right] P_x\left[\tau_i \in du, X_u \in dy\right] \\
&= \int_{[0,s]\times\Re} P_x\left[Y_s - Y_u = z - y,\ s < \tau_{i+1} \mid \tau_i = u, X_u = y\right] P_x\left[\tau_i \in du, X_u \in dy\right] \\
&\leq \int_{[0,s]\times\Re} P_x\left[Y_s - Y_u = z - y,\ \mid \tau_i = u, X_u = y\right] P_x\left[\tau_i \in du, X_u \in dy\right] \\
&= \int_{[0,s]\times\Re} P_x\left[Y_s - Y_u = z - y\right] P_x\left[\tau_i \in du, X_u \in dy\right] \\
&= \int_{[0,s]\times\Re} P_x\left[Y_{s-u} = z - y\right] P_x\left[\tau_i \in du, X_u \in dy\right]
\end{aligned}$$

and $P_x\left[Y_{s-u} = z\right] = 0$ follows from our assumption $\sigma \neq 0$ and Sato [20], Theorem 27.4.

We now provide the proof of Proposition A.1.

Proof of Proposition A.1: Let $(f_n)_{n\geq 1}$ be the sequence approximating f in the sense of Lemma A.2. Itô's rule holds for each f_n, i.e.,

$$\begin{aligned}
f_n(X_t) - f_n(X_0) &= \int_{(0,t]} f_n'(X_{s-})dX_s + \frac{\sigma^2}{2}\int_{(0,t]} f_n''(X_s)ds \\
&\quad + \sum_{0<s\leq t}\left\{f_n(X_s) - f_n(X_{s-}) - f_n'(X_{s-})\Delta X_s\right\}.
\end{aligned} \tag{77}$$

All we need to show that all three terms in the right-hand side of (77) converge to the corresponding terms in the right-hand side of (76) as $n \to \infty$. We can write $X = X_0 + M_1(t) + A_1(t)$ where M_1 is a local martingale with bounded jumps (thus also locally square-integrable) and A_1 is a finite variation process (Jacod & Shiryaev [13], Proposition I.4.17). We then have

$$\int_{(0,t]} f_n'(X_{s-})dM_1(s) \to \int_{(0,t]} f'(X_{s-})dM_1(s) \text{ in probability as } n \to \infty$$

by Theorem I.4.40 iii' in Jacod & Shiryaev [13]. Also

$$\int_{(0,t]} f_n'(X_{s-})dA_1(s) \to \int_{(0,t]} f'(X_{s-})dA_1(s) \text{ a.s, as } n \to \infty$$

by the Dominated Convergence Theorem. Therefore, the first integral in the right-hand side of (77) indeed converges to the corresponding integral in (76). The convergence

$$\sum_{0<s\leq t}\left\{f_n(X_s) - f_n(X_{s-}) - f_n'(X_{s-})\Delta X_s\right\} \to \sum_{0<s\leq t}\left\{f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\right\},\ P_x\text{−a.s.}$$

follows from the discrete time version of the Dominated Convergence Theorem since $|f_n(X_s) - f_n(X_{s-}) - f_n'(X_{s-})\Delta X_s|$ is bounded by $\frac{C_1}{2}(\Delta X_s)^2$ and $\sum_{0<s\le t}(\Delta X_s)^2 < \infty$. Finally we need to show that

$$\frac{\sigma^2}{2}\int_{(0,t]} f_n''(X_s)ds \to \frac{\sigma^2}{2}\int_{(0,t]} f''(X_s)ds,\ P_x\text{−a.s.}$$

as $n \to \infty$. If $\sigma = 0$ then there is nothing to prove and if $\sigma \neq 0$ then this follows from Lemma A.3 and the Dominated Convergence Theorem.

B Appendix

In Appendix B we recall some results from [15] for the convenience of the reader. We minimally changed the notation in order to accommodate the present notations. It will be assumed in this section, just like in [15], that Y is a spectrally negative Lévy process, that is, $\nu(0,\infty) = 0$, and $-Y$ is not a subordinator. In Section 4 we assumed that Y is spectrally positive, but it requires only minimal modification to adapt the results below to the spectrally positive case.

As explained at the beginning of Section 8.1 in [15], rather than working with the Lévy-Khintchine exponent, it is preferable to work with the Laplace exponent

$$\psi(\lambda) := \frac{1}{t}\log E\left[e^{\lambda Y(t)}\right],$$

which is finite for all $\lambda \ge 0$. The function $\psi : [0,\infty) \mapsto \mathbb{R}$ is zero at zero and tends to infinity at infinity. Further, it is infinitely differentiable and strictly convex. Define the right inverse

$$\Phi(q) = \sup\{\lambda \ge 0 : \psi(\lambda) = q\}$$

for each $q \ge 0$. If $\psi'(0) \ge 0$ then $\lambda = 0$ is the unique solution of $\psi(\lambda) = 0$ and otherwise there are two solutions to the latter with $\lambda = \Phi(0) > 0$ being the larger of the two, the other is $\lambda = 0$.

For all $z \in \mathbb{R}$ let

$$\tau_z^+ = \inf\{t > 0 : Y_t > z\} \text{ and } \tau_z^- = \inf\{t > 0 : Y_t < z\}.$$

Next we recall Theorem 8.1 in [15].

Theorem B.1. (One- and two-sided exit formulae). *There exists a family of functions $W^{(q)} : \mathbb{R} \to [0,\infty)$ and*

$$Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)dy,\ for\ x \in \mathbb{R}$$

defined for each $q \ge 0$ such that the following hold.

(i) For any $q \geq 0$, we have $W^{(q)}(x) = 0$ for $x < 0$ and $W^{(q)}$ is characterized on $[0, \infty)$ as a strictly increasing and continuous function whose Laplace transform satisfies

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\psi(\beta) - q} \text{ for } \beta > \Phi(q).$$

(ii) For any $x \in \mathbb{R}$ and $q \geq 0$

$$E_x \left[\exp\{-q\tau_0^-\} 1\{\tau_0^- < \infty\} \right] = Z^{(q)}(x) - \frac{q}{\phi(q)} W^{(q)}(x),$$

where we understand $q/\phi(q)$ in the limiting sense for $q = 0$, so that

$$P_x(\tau_0^- < \infty) = \begin{cases} 1 - \psi'(0+) W^{(0)}(x) & \text{if } \psi'(0+) > 0 \\ 1 & \text{if } \psi'(0+) \leq 0. \end{cases}$$

(iii) For any $x \leq a$ and $q \geq 0$,

$$E_x \left[\exp\{-q\tau_a^+\} 1\{\tau_0^- > \tau_a^+\} \right] = \frac{W^{(q)}(x)}{W^{(q)}(a)},$$

and

$$E_x \left[\exp\{-q\tau_0^-\} 1\{\tau_0^- < \tau_a^+\} \right] = Z^{(q)}(x) - Z^{(q)}(a) \frac{W^{(q)}(x)}{W^{(q)}(a)}.$$

Next we recall the q-potential measure

$$U^{(q)}(x, dy) = \int_0^\infty e^{-qt} P_x(Y_t \in dy, \tau > t) dt \tag{78}$$

for $q \geq 0$, where $a > 0$ and $\tau = \tau_a^+ \wedge \tau_0^-$. We recall Theorem 8.7 in [15].

Theorem B.2. *Suppose, for $q \geq 0$, that $U^{(q)}(x, dy)$ is the q-potential measure of Y killed on exiting $[0, a]$ where $x, y \in [0, a]$. Then it has a density $u^{(q)}(x, y)$ given by*

$$u^{(q)}(x, y) = \frac{W^{(q)}(x) W^{(q)}(a - y)}{W^{(q)}(a)} - W^{(q)}(x - y).$$

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