

# Perpetual Call Options With Non-Tradability

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## Abstract

We explicitly solve an optimal stopping problem related to the exercise of a perpetual American call option when the option holder cannot trade the underlying asset. We prove the verification theorem for the solution proposed. We derive the moment generating function of the optimal exercise time and also the elasticity of the option value with respect to stock price. The class of admissible utility functions that we solve for contains the CRRA family with some parametric restrictions. This theoretical framework provides the exact exercise boundary and the value of perpetual real options for a self interested manager whose incentives are not aligned with those of the shareholders. It can also serve as an approximation to the valuation of executive stock options.

# 1 Introduction

In the absence of trading restrictions, an American call option on a non-dividend paying stock is just as valuable as its European counterpart.<sup>1</sup> Therefore the (no-arbitrage) price of such an American option can be exactly computed using the ubiquitous Black-Scholes formula for European options. The argument just made hinges strongly on the option holder's ability to trade in the underlying asset (and a risk-free asset, on a continuous basis).

However, if the option holder cannot trade in the underlying asset, a no-arbitrage price cannot be determined. Typically for options on non-tradable assets, the option itself cannot be traded and has no market valuation.

Under such trading restrictions, the decision of optimal exercise and the valuation of the option are dictated by the option holder's beliefs (such as subjective probability estimates of future prices) and preferences (such as attitudes towards risk and time). For instance, consider the American call option embedded in a new product launch decision. The new product does not yet exist and hence cannot be traded. Further, the manager whose human capital is tied to the project, cannot hedge away the idiosyncratic risk of the project. Her reward depends on success of the project, but she cannot hold a large portfolio of projects and diversify away the non-systematic risk of the project she manages. In such cases a utility maximization framework would re-

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<sup>1</sup>In other words, when there are no trading restrictions and no dividends, the early exercise feature of American call options is worthless. See Hull (2003) for an elaborate explanation.

flect managerial decision making on such projects. Similar arguments can be made for executive stock options granted to executives of firms as part of their compensation package.<sup>2</sup>

Our framework analyzes the valuation and optimal exercise decision of such an option subject to trading restrictions. We model the optimality of option exercise in this context as a (discounted) utility maximization problem. This enables us to consider the manager's risk aversion and time preference as inputs to the option exercise decision. Henceforth we assume the option holder to be a risk averse manager whose compensation directly depends on the American option payoff, but who cannot hedge by trading in the underlying, nor can she sell the option. In general, with finite maturity, this utility maximization problem is a difficult free-boundary problem. No explicit solution is known for either the optimal exercise boundary or the value of the option. Extant literature has resorted to numerical methods.<sup>3</sup> To simplify the problem setting and to provide an explicit solution we focus solely on the infinite maturity case. This is in fact a very realistic assumption for several real option scenarios. To facilitate implementation we enrich the model with a fractional recovery on the option payoff, primarily as a proxy for taxes and transaction costs.

In spite of the infinite maturity assumption, this framework may be widely applicable. It provides the exact exercise boundary and the

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<sup>2</sup> Executives cannot sell the options they are granted. They can sell the stock they own, only during certain time periods, subject to company and Securities and Exchange Commission regulations. Moreover, they cannot short the stock.

<sup>3</sup> See for instance Huddart (1994), Kulatilaka and Marcus (1994), Hall and Murphy (2002).

value of perpetual real options for a self interested manager whose incentives are not aligned with those of the shareholders. Thus it may be useful in a variety of situations with embedded real options in irreversible investments such as in new-product development, technology change, market entry etc. It also provides an approximation to the valuation and optimal exercise of long maturity American options under non-tradability restrictions, as is the case with executive stock options.

Apart from the perpetuity assumption, an important simplifying assumption we make is that the option exercise decision is independent of investment in other assets and exercise of other options. In other words, we assume that the sole control for optimization is the timing of option exercise. We model this as a stopping time.

In sum, we solve the optimal stopping problem related to the optimal exercise of a perpetual American call option, in the absence of dividends, while incorporating the option holder's utility preferences. Primarily, the contribution is in admitting an explicit solution that incorporates individual risk and time preferences. This facilitates an analytical derivation of comparative statics. Additionally, we provide a more rigorous treatment of the problem by proving of the verification theorem for the solution proposed, while considering a general class of admissible utility functions. We also derive the moment generating function of the optimal exercise time, and the elasticity of the option value with respect to the stock price.

The remainder of the article is organized as follows. In section 2 we position this paper in the relevant literature. In sections 3 and 4 we define and solve the optimal stopping problem implicit in the optimal exercise decision. In sections 5 and 6 we illustrate the model using log and power utility functions. In section 7 we check how realistic the model is and in section 8 we conclude.

## 2 Literature review

An excellent summary of results on the pricing of American options under the assumption that the option holder can trade freely is given in Myneni (1992). We do not make the tradability assumption and use a utility maximization approach. But to make the problem tractable, we do assume infinite maturity. Our derivation of the (stationary) continuation region, the (stationary) optimal exercise barrier and the value function is akin to the solution for a much simpler problem illustrated in Oksendal (1998). The problem he discusses is deciding the optimal timing of an asset sale for a risk-neutral investor in the presence of a fixed transaction cost.<sup>4</sup> The difference here is that our framework incorporates the option holder's risk preferences.

Our work is similar in spirit to Detemple and Sundaresan (1999) and to Henderson (2002). Both provide general frameworks for valuing contingent claims on non-traded assets, while mentioning executive stock options as an application. Detemple and Sundaresan (1999) assume absolute non-tradability and inability to hedge (as we do). In

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<sup>4</sup> Please see problem 5 in Section 1.4 and Solved Example 10.2.2 in the book for details.

their framework the no-short-sales constraints on the underlying asset manifest themselves in the form of an implicit dividend yield in the risk-neutralized process for the underlying asset.<sup>5</sup> They achieve the difficult task of simultaneously optimizing the option exercise and wealth investment decisions. They obtain values numerically, using a trinomial model. While our framework is much less general, we contribute in providing an explicit solution and the verification theorem for that solution.

Henderson (2002) assumes partial hedging and introduces a second non-traded log Brownian asset in the Merton problem. She focuses on the CRRA and CARA utilities. She obtains an approximation for the CRRA utility case and an explicit price for the CARA utility case. In contrast, by assuming no hedging and using a simpler model, we explicitly derive the optimal exercise boundary and the option value for the CRRA utility.

Within the context of non-traded assets, executive stock options (ESOs) have their own peculiarities, Rubinstein (1995) lists these in detail. A key distinguishing feature of ESOs is that they have much longer maturities, (thereby mitigating our perpetuity assumption to some extent).

In one stream of ESO literature, Jennergren and Naslund (1993), Cuny and Jorion (1995) and similar extensions of the Black-Scholes framework account for executive's preferences in reduced-form, by di-

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<sup>5</sup>A natural consequence of this implicit dividend yield is that early exercise of an option may be optimal even when the underlying stock does not pay dividends.

rectly modelling the executive's early departure from the firm. In contrast, we model the executive's utility preferences explicitly. Lambert, Larcker, and Verrechia (1991) were among the first to propose a utility-based model in the context of ESOs. In this utility-based stream of ESO literature, Huddart (1994), Kulatilaka and Marcus (1994), Hall and Murphy (2002) and others model utility preferences explicitly but they resort to numerical procedures. We provide an explicit solution as well as the proof of the verification theorem for the solution we propose. This proof is along the lines of an illustration in Karatzas and Wang (2000).

A significant departure from the ESO literature was made by Subramanian and Jain (2004). They value ESOs allowing for the possibility that a risk-averse employee strategically exercises her options over time rather than at a single date. We effectively assume that all options are exercised at one point in time rather than in separate chunks across time, although the latter is a more realistic scenario.

A simpler version of our framework as applied to ESOs can be seen in Kadam, Lakner, and Srinivasan (2003). While they also provide an approximation to the ESO value by assuming infinite maturity, their focus is exclusively on the negative exponential utility function. This specific utility function does not satisfy the admissibility conditions that characterize the general class of utility functions which we solve for. We do not restrict ourselves to one family of utility functions as they do. As a consequence, in implementing the model we require strong parametric restrictions whereas they do not.



The risk neutral special case of our framework yields results identical to those for the perpetual real option framework stated in Dixit and Pindyck (1994) (under the ‘contingent claim’ approach). In this seminal work on real options, the authors advocated the ‘contingent claim’ approach as a more appropriate alternative to NPV based valuation in many contexts. The recent explosion in real options literature shows that this ‘contingent claims approach’ is an extremely appealing choice for a wide range of scenarios such as oil exploration, utilities, biotechnology etc. and to address a wide variety of issues such as project valuation, supplier switching, market entry and so on. Whenever invoking an explicit solution for the option value, these applications directly or indirectly assume the existence of a portfolio of tradable assets that can replicate the uncertainty in the underlying asset price accurately. As Dixit and Pindyck (1994) themselves and (more recently) Lautier (2003), Borison (2003) caution their readers, the assumption of a replicating portfolio can be quite hard to satisfy in the context of real options. We do not assume the existence of a replicating portfolio.

If one applies our results to valuing real options, there would be an implicit assumption that the managerial incentives are not perfectly aligned with the shareholder incentives. Carpenter (2000) examines such a misalignment in the context of a manager compensated with a European call option on an asset that he controls. She finds that the manager optimally chooses to increase the underlying asset variance when the asset value is low, and decrease the underlying asset variance when the asset value is high. We obtain a similar result while deriving

the comparative statics of the optimal exercise barrier chosen by the manager.

### 3 An optimal stopping problem

In this section we define an optimal stopping problem linked to the exercise of a perpetual call option on a non-dividend paying stock by a risk averse manager in the presence of trading restrictions.

#### 3.1 The price process

We assume that the price process of the asset that determines the option payoff can be observed/estimated accurately. Let  $X_t$  denote the price process of an asset. Let  $\mathcal{S} = \{\tau: \Omega \mapsto [0, \infty)\}$  be the class of  $\mathbb{P}$ -a.s. finite stopping times where  $(\Omega, \mathcal{F}, \mathbb{P})$  is a complete probability space. For an arbitrary stopping time in this class let the dynamics of the stopped price process be defined as follows :

$$X(0) = x > 0$$

$$dX_t = \mu X_t dt + \sigma X_t dB_t \quad \forall t \in [0, \tau]$$

$$X_t = \delta(X_\tau - K) \quad \forall t \in (\tau, \infty)$$

where  $B_t$  is a standard Brownian motion on  $\mathbb{P}$  and where

$$0 < \delta \leq 1, \sigma^2 > 0, \mu \geq \frac{1}{2}\sigma^2, \text{ and } K \geq 0$$

are known constants. Thus  $\mu$  and  $\sigma$  are the drift and volatility of the price process, and  $K$  is the exercise price of the option.  $\delta$  represents the proportion of recovery after variable losses. This fractional recovery is intended to incorporate taxes and transaction costs incurred by option exercise. The condition  $\mu \geq \frac{1}{2}\sigma^2$  ensures that  $\mathbb{P}$ -a.s. the price process attains all levels larger than  $x$ .

### 3.2 Manager's preferences

We now define the time and risk preferences that drive the consumption at the time of option exercise. The manager's preference to consume earlier than later is captured by a constant positive discount rate  $\rho$  that acts on the utility of consumption. We now define  $p^*$  to be the only positive root of the quadratic equation  $\frac{1}{2}\sigma^2 p^2 + (\mu - \frac{1}{2}\sigma^2)p - \rho = 0$  and use it to characterize an admissible utility function as follows:<sup>6</sup>

$\mathbb{U}(\cdot): (0, \infty) \mapsto \mathbb{R}$  is assumed to be a continuous, strictly increasing, strongly concave and twice continuously differentiable function on  $(0, \infty)$ . Note that we allow  $\lim_{x \rightarrow 0} \mathbb{U}(x)$  to be finite or  $-\infty$ .

We extend  $\mathbb{U}(\cdot)$  to the entire  $\mathbb{R}$  by defining

$$\mathbb{U}(w) = \begin{cases} \lim_{\nu \rightarrow 0} \mathbb{U}(\nu) & \text{if } w = 0 \\ -\infty & \text{if } w < 0 \end{cases}$$

To guarantee the existence and uniqueness of an optimal level of

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<sup>6</sup>The existence and uniqueness of the positive root follows from the fact that volatility and discount rate are both strictly positive. As will become apparent from the solution to the problem, this quantity  $p^*$  can be interpreted as the elasticity of the value of the option with respect to stock price. Furthermore, Appendix B shows that  $p^*$  is above or below 1 according as  $\mu$  is less than or greater than  $\rho$ .

terminal wealth we impose some admissibility conditions on  $\mathbb{U}(\cdot)$  :

1.  $\exists \omega \geq 0$  such that  $\mathbb{U}(\omega) \geq 0$
2.  $\lim_{\omega \rightarrow \bar{\omega}} \left\{ \frac{\mathbb{U}'(\omega)}{\mathbb{U}(\omega)} \right\} = \infty$  where  $\bar{\omega} = \inf\{\omega \geq 0 : \mathbb{U}(\omega) \geq 0\}$
3.  $\lim_{\omega \rightarrow \infty} \left\{ \omega \frac{\mathbb{U}'(\omega)}{\mathbb{U}(\omega)} \right\} < p^*$
4.  $\forall \omega \quad \mathcal{R}\mathcal{R}(\omega) = \frac{-\omega \mathbb{U}''(\omega)}{\mathbb{U}'(\omega)} \geq 1 - p^*$

These conditions are sufficient, not necessary, for our solution procedure to be valid. Power utility (with some parametric restrictions) and log utility satisfy these conditions, we present specific solutions in each case. Negative exponential utility does not satisfy these conditions, however this framework can be extended to that case.<sup>7</sup>

### 3.3 The problem

Define the reward for stopping at time  $t$  as

$$g(t, X_t) = e^{-\rho t} \mathbb{U}(\delta(X_t - K))$$

where  $\rho$  is a positive discount rate. In this expression the random variable  $X_t$  captures the state of the process at time  $t$ . The value function in state  $x$  is given by

$$V(x) = \sup_{\tau \in \mathcal{S}} E^x [g(\tau, X_\tau)] = E^x [g(\tau^*, X_{\tau^*})]$$

The problem we solve (in the next section) is to determine the optimal stopping time  $\tau^*$  (if it exists) that achieves this supremum and to derive the corresponding value function.

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<sup>7</sup>See Kadam, Lakner, and Srinivasan (2003) for details.

## 4 A solution

A solution methodology using variational inequalities technique has the following three steps:

1. We first make whatever assumptions necessary to arrive at a candidate solution pair <optimal stopping time, value function> that *might* solve the original optimal stopping problem posed in section 3.3.
2. We then prove that if certain variational inequalities are satisfied by an arbitrary candidate solution then it is indeed a solution to the original optimal stopping problem posed in section 3.3.
3. Finally we verify that the candidate in Step 1. satisfies the variational inequalities in Step 2.

### 4.1 Step 1 : A candidate solution pair

Following the solution of a simpler version of this problem in Oksendal (1998) we arrive at a candidate solution.<sup>8</sup> We surmise the stationary<sup>9</sup> continuation region for the optimal stopping problem to be of the form  $[0, b)$  for some  $0 < b < \infty$ . By the dynamic programming principle, the discounted value function is a  $\mathbb{P}$ -martingale so by Ito's lemma, the stationary<sup>10</sup> value function must satisfy the PDE

$$\mu x \frac{\partial V}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} - \rho V = 0$$

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<sup>8</sup>See footnote 4 and the sentence preceding it for a description of this simpler problem.

<sup>9</sup>The stationary nature of the optimal exercise barrier as well as of the value function is obvious from the fact that the option has infinite maturity.

<sup>10</sup>See footnote 9.

with boundary condition  $V(b) = \mathbb{U}(\delta(b - K))$ . A solution to the PDE yields the expression  $V(x) = \mathbb{U}(\delta(b - K))(x/b)^{p^*} \quad \forall x \in [0, b)$  as a function of a barrier  $b$ . Maximizing this expression over  $b > K$  yields a candidate optimal solution in terms of an optimal exercise barrier  $b^*$ .

$$V(x) = \mathbb{U}(\delta(b^* - K))(x/b^*)^{p^*} \quad \forall x \in [0, b^*)$$

where  $p^*$  is the only positive root<sup>11</sup> of the quadratic equation (1).

#### 4.1.1 The final candidate

Let  $p^*$  denote the only positive root<sup>12</sup> of the quadratic equation

$$\frac{1}{2}\sigma^2 p^2 + (\mu - \frac{1}{2}\sigma^2)p - \rho = 0 \quad (1)$$

and  $b^* \in (K, \infty)$  be the unique root<sup>13</sup> of the equation

$$b\delta\mathbb{U}'(\delta(b - K)) - p^* \mathbb{U}(\delta(b - K)) = 0 \quad (2)$$

Then a candidate solution to the problem defined in section 3.3 is

$$\tau^* = \inf\{t : t \geq 0; X_t \geq b^*\}$$

$$V(x) = \mathbb{U}(\delta(b^* - K))(x/b^*)^{p^*} \quad \forall x \in [0, b^*)$$

For  $x > b^*$  we guess<sup>14</sup> that  $V(\cdot) \equiv g(\cdot)$

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<sup>11</sup>Recall from section 3.1 that  $\rho > 0$ .

<sup>12</sup>See footnote 11.

<sup>13</sup>The admissibility conditions enumerated in section 3.2 guarantee both existence and uniqueness.

<sup>14</sup>Recall that in this step 1 any assumptions necessary can be made without any rigorous proof. The limited goal here is to arrive at a candidate solution which will be verified later.

$$\therefore V(x) = \mathbb{U}(\delta(x - K)) \quad \forall x \in [b^*, \infty)$$

## 4.2 Step 2 : Verification Theorem

### 4.2.1 Variational Inequalities

Find a number  $b^* \in (K, \infty)$  and a strictly increasing function

$$\phi(\cdot) \in C([0, \infty)) \cap C^1((0, \infty)) \cap C^2((0, \infty) \setminus \{b^*\}) \quad (3)$$

such that

$$\phi(x) > \mathbb{U}(\delta(x - K)) \quad \forall x \in [0, b^*) \quad (4)$$

$$\phi(x) = \mathbb{U}(\delta(x - K)) \quad \forall x \in [b^*, \infty) \quad (5)$$

$$\mathcal{L}\phi(x) = 0 \quad \forall x \in (0, b^*) \quad (6)$$

$$\mathcal{L}\phi(x) < 0 \quad \forall x \in (b^*, \infty) \quad (7)$$

where  $\mathcal{L}$  is a differential operator acting on any twice continuously differentiable function  $h(\cdot): \mathbb{R} \mapsto \mathbb{R}$  such that

$$\mathcal{L}h(x) = -\rho h(x) + \mu x h'(x) + \frac{1}{2} \sigma^2 x^2 h''(x)$$

### 4.2.2 Theorem

Suppose  $X_t$  is the price process as defined in section 3.1. Suppose a pair  $\langle b^*, \phi(\cdot) \rangle$  can satisfy the conditions and variational inequalities

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The intuition behind this guess is that outside of the continuation region the value from continuing can be at most as much as the reward from stopping immediately.

given above. Then

$$\begin{aligned}\tau^* &= \inf\{t : t \geq 0; X_t \geq b^*\} \\ V(x) &= \phi(x) \quad \forall x \geq 0\end{aligned}$$

solves the optimal stopping problem defined in section 3.3.

### 4.2.3 Proof

We first prove that the value function is bounded above by  $\phi(\cdot)$  and then we show that using  $\tau = \tau^*$  defined above attains that upper bound.

Applying Ito's lemma<sup>15</sup> to  $f(t) = e^{-\rho t}\phi(X_t)$  we get

$$e^{-\rho t}\phi(X_t) - \phi(X_0) - \int_0^t \{e^{-\rho u} \sigma \eta \phi'(\eta)\}|_{\eta=X_u} dB_u = \int_0^t \{e^{-\rho u} \mathcal{L}\phi(\eta)\}|_{\eta=X_u} du \quad (8)$$

Now R.H.S. of equation (8) is  $\leq 0$  by variational inequalities (6) and (7) hence  $f(t) : t \in [0, \infty)$  is a local supermartingale. In fact  $f(t) \geq (\phi(0) \wedge 0)$  and variational inequality (3) implies  $\phi(\cdot) \in C([0, \infty))$  which implies that  $\phi(0) > -\infty$ . Together these imply  $f(t) > -\infty$  hence  $f(t) : t \in [0, \infty)$  is not merely a local supermartingale but also a true supermartingale<sup>16</sup>.

Then by the Optional Sampling Theorem 1.3.22 in Karatzas and Shreve (1991) (recalling from section 2.1 that  $\mathcal{S}$  is the class of  $\mathbb{P}$ - a.s.

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<sup>15</sup>Strictly speaking Ito's lemma is not applicable because the second derivative does not exist at  $b^*$ . However, the first derivatives match at  $b^*$  and the second derivatives approaching  $b^*$  from both sides exist and are finite. From Karatzas and Shreve (1991), problem 3.6.24 it follows that Ito's lemma can be extended to include this special case.

<sup>16</sup>Please see Karatzas and Shreve (1991), problem 1.3.16



finite stopping times)

$$E[f(\tau)] \leq f(0) = \phi(x) \quad \forall \tau \in \mathcal{S} \quad (9)$$

Now  $V(x) = \sup_{\tau \in \mathcal{S}} E^x [e^{-\rho\tau} \mathbb{U}(\delta(X_\tau - K))]$  so by inequalities (4) and (5) we get  $V(x) \leq \sup_{\tau \in \mathcal{S}} E^x [f(\tau)] \quad \forall x \in [0, \infty)$ . Combining this with inequality (9) yields

$$\forall x \in [0, \infty) \quad V(x) \leq \sup_{\tau \in \mathcal{S}} E^x [f(\tau)] \leq \phi(x)$$

Hence  $V(\cdot)$  is bounded above by  $\phi(\cdot)$ . It remains to be proved that this upper bound is attained when  $\tau^*$  is chosen to be  $\tau^* = \inf\{t : t \geq 0; X_t \geq b^*\}$ . There arise two cases:

**The case of  $x \geq b^*$**

In this case  $\tau^* = 0$  and  $X_{\tau^*} = x$  so that

$$V(x) = \mathbb{U}(\delta(x - K)) = \phi(x)$$

**The case of  $0 \leq x < b^* < \infty$**

In this case  $\forall t \in [0, \tau^*] \quad X_t \in [0, b^*]$  and since  $\phi(\cdot)$  was chosen to be a continuous, strictly increasing function  $\phi(X_t)$  is bounded by  $\phi(0)$  and  $\phi(b^*)$  both of which are finite. Hence

$$-\infty < \phi(0) \leq \phi(X_t) \leq \phi(b^*) < \infty \quad (10)$$

From equation (8) and equality (6) it follows that  $f(t) : t \in [0, \tau^*]$  is a local martingale. Inequality (10) further implies that it is a bounded local martingale and hence also a true martingale. By the Optional

Sampling Theorem 1.3.22 in Karatzas and Shreve (1991) it follows that

$$V(x) = E^x[f(\tau^*)] = \phi(x)$$

### 4.3 Step 3 : Verification of the inequalities

On the basis of the candidate solution obtained in Step 1 the pair  $\langle b^*, \phi(\cdot) \rangle$  to be verified can be chosen as follows:

Let  $p^*$  be the unique positive root of equation (1), let  $b^* \in (K, \infty)$  be the unique root of equation (2) and let  $\phi(\cdot)$  be defined as

$$\phi(x) = \begin{cases} \mathbb{U}(\delta(b^* - K))(x/b^*)^{p^*} & \forall x \in [0, b^*) \\ \mathbb{U}(\delta(x - K)) & \forall x \in [b^*, \infty] \end{cases}$$

It trivially follows from this definition that the condition (3) is satisfied<sup>17</sup>. Now  $\phi(\cdot)$  is strictly increasing iff the optimal reward  $\mathbb{U}(\delta(b^* - K))$  is positive<sup>18</sup>. But the existence of a positive reward follows from the fact that a solution to equation (2) exists<sup>19</sup> for a strictly increasing  $\mathbb{U}(\cdot)$ . The variational inequality (5) holds by definition of  $\phi(\cdot)$ . Verifying variational inequality (6) is straightforward upon invoking equation (1).

#### 4.3.1 Verifying variational inequality (4)

By the properties of  $\mathbb{U}(\cdot)$  listed in section 3.2  $\forall w < 0 \mathbb{U}(w) = -\infty$ . It follows that  $\forall x \in [0, K) \mathbb{U}(\delta(x - K)) = -\infty < 0 \leq \phi(x)$  and the

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<sup>17</sup>Equation (2) can be invoked to show that the first derivatives coincide at  $b^*$ .

<sup>18</sup>Recall from section 3.2 that  $\mathbb{U}(\cdot)$  was assumed to be strictly increasing.

<sup>19</sup>The admissibility conditions on  $\mathbb{U}(\cdot)$  enumerated in section 3.2 guarantee both existence and uniqueness.

inequality holds on the domain  $x \in [0, K)$ .

It remains to be verified that

$$\mathbb{U}(\delta(b^* - K))(x/b^*)^{p^*} > \mathbb{U}(\delta(x - K)) \quad \forall x \in [K, b^*)$$

i.e. to verify that

$$\mathbb{U}(\delta(b^* - K))(b^*)^{-p^*} > \mathbb{U}(\delta(x - K))x^{-p^*} \quad \forall x \in [K, b^*)$$

But this follows directly from the fact that assigning  $x = b^*$  maximizes the expression on the R.H.S. over all  $x \in [K, \infty)$ . That such a barrier  $b^* \in [K, \infty)$  exists follows from the first three admissibility conditions placed on  $\mathbb{U}(\cdot)$  in section 3.2. By the fourth admissibility condition the relative risk aversion of  $\mathbb{U}(\cdot)$  is at least  $1-p^*$ . Therefore the second order condition for this maximization problem is satisfied.

### 4.3.2 Verifying variational inequality (7)

Recall from the definition of  $\phi(\cdot)$  that  $\phi(x) = \mathbb{U}(\delta(x - K)) \quad \forall x > b^*$

$$\therefore \mathcal{L}\phi(x) = -\rho\mathbb{U}(\delta(x - K)) + \mu\delta x\mathbb{U}'(\delta(x - K)) + \frac{1}{2}\delta^2\sigma^2x^2\mathbb{U}''(\delta(x - K))$$

By the admissibility conditions placed on  $\mathbb{U}(\cdot)$  in section 3.2 it is necessary for  $\mathbb{U}(\cdot)$  to have a relative risk aversion of at least  $1-p^*$ . This implies

$$\mathcal{L}\phi(x) < -\rho\mathbb{U}(\delta(x - K)) + \mu\delta x\mathbb{U}'(\delta(x - K)) - \frac{1}{2}\sigma^2\delta x\mathbb{U}'(\delta(x - K))$$

Now  $\mathbb{U}(\cdot)$  is strictly increasing and  $p^*$  is positive

$$\therefore \mathcal{L}\phi(x) < -\rho\mathbb{U}(\delta(x - K)) + \delta x\mathbb{U}'(\delta(x - K))\left\{\mu - \frac{1}{2}\sigma^2(1 - p^*)\right\}$$

But  $p^*$  satisfies equation (1)

$$\therefore \mathcal{L}\phi(x) < -\rho\mathbb{U}(\delta(x - K)) + \frac{\rho}{p^*}\delta x\mathbb{U}'(\delta(x - K))$$

But  $\mathbb{U}(\cdot)$  is strongly concave and  $x > b^*$

$$\therefore \mathcal{L}\phi(x) < -\rho\mathbb{U}(\delta(x - K)) + \frac{\rho}{p^*}\delta x\mathbb{U}'(\delta(b^* - K))$$

But  $b^*$  satisfies equation (2)

$$\therefore \mathcal{L}\phi(x) < -\rho\mathbb{U}(\delta(x - K)) + \rho\mathbb{U}(\delta(b^* - K))$$

But  $\mathbb{U}(\cdot)$  is strictly increasing and  $x > b^*$

$$\therefore \mathcal{L}\phi(x) < 0$$

#### 4.4 The final solution

If  $p^*$  is the unique positive root of equation (1) and  $b^* > K$  is the unique root of equation (2) where  $\mathbb{U}(\cdot)$  satisfies the admissibility conditions listed in section 3.2 then the solution to the problem defined in section 3.3 is given by

$$\tau^* = \inf\{t : t \geq 0; X_t \geq b^*\}$$

$$V(x) = \begin{cases} \mathbb{U}(\delta(b^* - K))(x/b^*)^{p^*} & \forall x \in [0, b^*) \\ \mathbb{U}(\delta(x - K)) & \forall x \in [b^*, \infty] \end{cases}$$

## 4.5 Sensitivity analysis

The first order sensitivity analysis of the above solution to some of the problem parameters is summarized in Table 1. The results in the first row of the table follow from the signs of the first partial derivatives evaluated in Appendix B. The results in second and third rows are facilitated by the fact that  $\rho, \mu, \sigma$  affect the solution exclusively and entirely via  $p^*$ . In fact, to obtain results from the second and third rows one merely needs to invert the results in the first row because the first partial derivatives of  $V$  and  $b^*$  w.r.t.  $p^*$  are both negative, as shown in Appendix C. In interpreting the results it helps to recall from equation (11) that if the diffusion parameters do not change then a smaller barrier will, on an expected basis, make the process stop faster and lead to an earlier option exercise.

### 4.5.1 Sensitivity to $\mu$ and $\rho$

It turns out (as expected) that all else remaining constant the barrier  $b^*$  decreases as the drift parameter  $\mu$  decreases or as the discount rate  $\rho$  increases. Thus the manager targets for a lower exercise price for the option if the underlying is expected to grow at a slower rate or if the cost of waiting increases. The sensitivity analysis also shows (as expected) that the option value  $V$  increases with the drift of the price process and decreases with impatience.

Table 1: Summary of first order sensitivity analysis

	as $\rho$ increases	as $\mu$ increases	as $\sigma$ increases
$p^*$	increases	decreases	decreases if $\mu < \rho$ increases if $\mu > \rho$
$b^*$	decreases	increases	increases if $\mu < \rho$ decreases if $\mu > \rho$
$V(\cdot)$	decreases	increases	increases if $\mu < \rho$ decreases if $\mu > \rho$

This table summarizes the results of the analytical sensitivity analysis.  $V(\cdot)$  is the value of the option.  $b^*$  is the optimal exercise barrier.  $p^*$  is the elasticity of the option value with respect to the price of the underlying stock.  $\rho$  is the manager's impatience parameter i.e. the time discount rate.  $\mu$  and  $\sigma$  are the drift and volatility of the underlying stock price process.

#### 4.5.2 Sensitivity to $\sigma^2$

Interestingly, the sensitivity of  $b^*$  w.r.t.  $\sigma^2$  depends on the relative magnitude of the drift parameter  $\mu$  and the discount rate  $\rho$ . An increase in the volatility of the underlying leads to a lower valuation if  $\mu > \rho$  and a higher valuation otherwise. Thus if the underlying asset is expected to grow faster than the manager's time preference discount rate then the manager is hurt by an increase in volatility. If, on the other hand, the cost of waiting dominates then an increase in volatility is beneficial to the manager<sup>20</sup>. It is worth mentioning that Carpenter (2000), who examines misalignment in the context of

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<sup>20</sup> This insight is also related to the elasticity of the option value  $V$  with respect to stock price  $x$ . Recall from footnote 6 that  $p^*$  acts as the elasticity of  $V$  w.r.t.  $x$ , and that  $p^*$  is above or below 1 according as  $\mu$  is less or greater than  $\rho$ .

a manager compensated with a European call option on an asset that he controls, also obtains a similar result. She finds that the manager optimally chooses to increase the underlying asset variance when the asset value is low, and decrease the underlying asset variance when the asset value is high.

## 4.6 The m.g.f. of $\tau^*$

The moment generating function of the optimal stopping time  $\tau^*$  can be obtained using the relation

$$V(x) = \sup_{\tau \in \mathcal{S}} E^x [g(\tau, X_\tau)] = E^x [g(\tau^*, X_{\tau^*})]$$

and substituting into it the solution to the value function

$$V(x) = \mathbb{U}(\delta(b^* - K))(x/b^*)^{p^*} \quad \forall x \in [0, b^*]$$

This yields the moment generating function of  $\tau^*$  as

$$E[e^{-\rho\tau^*}] = \left(\frac{x}{b^*}\right)^{p^*}$$

Karlin and Taylor (1975) derive an identical result.<sup>21</sup> It is easy to verify that this gives the correct first order moment.

$$E[\tau^*] = - \left. \frac{\partial}{\partial \rho} \left(\frac{x}{b^*}\right)^{p^*} \right|_{\rho=0} = \frac{\ln\left(\frac{b^*}{x}\right)}{\mu - \frac{1}{2}\sigma^2} \quad (11)$$

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<sup>21</sup> For details, please refer to the solved example on Geometric Brownian Motion, immediately after Theorem 5.3 of Chapter 7 in Karlin and Taylor (1975). It builds on Laplace transform of the first passage time of a Brownian motion process to a single barrier, as given in Equation 5.5 of the book.

The computation of higher order moments is more tedious.

## 5 The case of log utility

Log utility is defined as  $U(w) = \log(w) \forall w > 0$ ,  $-\infty$  otherwise. In this section we use log utility to illustrate the theory developed in sections 3 and 4. We solve for the barrier  $b^*$  and perform sensitivity analysis of  $b^*$  with respect to the remaining problem parameters that do not feature in Table 1.

### 5.1 The optimal exercise barrier $b^*$

For log utility the transformation  $z = b - K$  reduces equation (2) to

$$\log(\delta z) = \left(1 + \frac{K}{z}\right) \frac{1}{p^*} \quad (12)$$

Since as  $z$  increases from 0 to  $\infty$  the L.H.S. of equation (12) strictly increases from  $-\infty$  to  $\infty$  while the R.H.S. strictly decreases from  $\infty$  to  $\frac{1}{p^*}$ , the root  $z^*$  must exist, must be positive and must be unique.  $b^* = z^* + K > K$  gives the unique root of equation (2), furthermore equation (12) provides the means to do some sensitivity analysis of how the barrier depends on problem parameters.

### 5.2 Sensitivity of $b^*$ to $K, \delta$

From equation (12) it can be shown that

$$\frac{\partial}{\partial K} b^* = 1 + \frac{1}{p^* + \frac{K}{(b^* - K)}} > 1$$



Hence all else remaining constant the barrier  $b^*$  increases with exercise price  $K$  and the rate of increase is bounded below by 1. Thus as the exercise price increases the manager is expected to wait longer. The increase in the manager's optimal exercise barrier is at least as much as the increase in exercise price.

Recall from section 3.1 that  $0 < \delta \leq 1$ . From equation (12) it can be shown that

$$\frac{\partial}{\partial \delta} b^* = \left( \frac{-1}{\delta} \right) \left\{ \frac{(b^* - K)^2}{(b^* - K) + \left( \frac{K}{p^*} \right)} \right\} < 0$$

Thus all else remaining constant the barrier  $b^*$  decreases with recovery rate  $\delta$  and the rate of decrease is inversely proportional to  $\delta$ . Thus as the recovery rate increases the manager is expected to exercise the option sooner and at a lower price. The absolute decrease in the manager's optimal exercise barrier is directly proportional to the proportional increase in the recovery rate.

## 6 The case of power utility

Let us define power utility for wealth  $w$  in terms of relative risk aversion  $\gamma > 0$  as<sup>22</sup>

$$\mathbb{U}(w) = \frac{w^{1-\gamma}}{1-\gamma} \quad \forall w \in (0, \infty)$$

The restrictions imposed on admissible utility functions when expressed for power utility reduce to a parametric restriction  $\gamma > 1 - p^*$ .

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<sup>22</sup>A more general definition could be  $\mathbb{U}(w) = \frac{(w-w_0)^{1-\gamma}}{1-\gamma}$  but we ignore  $w_0$  since it can be absorbed by the fixed cost parameter  $K$  and normalized to zero.

In Appendix A we show that when  $\gamma$  is at or below the lower boundary we get a trivial solution that the option holder waits forever to exercise the option. In fact, in the next section we require that  $\gamma$  should be bounded above as well, by 1, to ensure a meaningful solution. Hence effectively we will need  $\gamma \in (1 - p^*, 1)$ .

## 6.1 The optimal exercise barrier $b^*$

Under the above parametric restrictions equation (2) yields the root  $b^* \in (K, \infty)$  as

$$b^* = \frac{K}{1 - \frac{1-\gamma}{p^*}} \quad (13)$$

First we note the convergence to the risk-neutral case. As  $\gamma$  approaches zero, the optimal exercise barrier approaches that given in Oksendal (1998). Next we note that in order to ensure  $b^* \in (K, \infty)$  it is necessary to have  $\gamma \in (1 - p^*, 1)$ . In Appendix A we examine cases when the parameter value lies outside this range. In Appendix A.1 and A.2 we conclude that for a parameter value at or below the boundary i.e. for  $\gamma \leq 1 - p^*$  it is optimal to hold the option forever i.e.  $b^* = \infty$  and hence  $\mathbb{P} - a.s. \tau^* = \infty$ . This result is consistent with the limit of equation (13) as  $\gamma \downarrow 1 - p^*$ . It indicates that if the manager's risk aversion is too low then we get a trivial solution in which she waits forever to exercise the option. On the other hand Appendix A.3 demonstrates that if  $\gamma > 1$  then no exercise policy can be optimal.

## 6.2 Sensitivity of $b^*$ to $K, \delta$

From equation (13) it follows that  $b^*$  is directly proportional to the exercise price and the constant of proportionality exceeds 1. Thus as the exercise price increases the manager is expected to wait longer. The increase in the manager's optimal exercise barrier is at least as much as the increase in exercise price. Furthermore, the height of the barrier  $b^*$  does not depend on  $\delta$  so the manager's target exercise price for the option is insensitive to the recovery rate. This implies that changes in variable transaction costs (or taxes) will not affect the optimal exercise behavior of the manager.<sup>23</sup>

## 6.3 Comparison with the risk neutral case

The optimal exercise barrier for a risk neutral manager is

$$b_0^* = \frac{K}{1 - \frac{1}{p^*}}$$

Thus ignoring risk aversion gives a relative error of  $\frac{b^* - b_0^*}{b^*} = -\frac{\gamma}{p^* - 1}$ . Thus ignoring risk aversion overestimates the optimal exercise barrier. The magnitude of percentage error strictly increases with the drift of the stock price and with the risk aversion of the manager. Thus, following the 'contingent claims approach' described in Dixit and Pindyck (1994), a highly risk averse manager will overestimate the barrier with a large error and, on average, wait too long before

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<sup>23</sup>This is not the case with log utility as shown in Section 5.2. Furthermore, it is noteworthy that these changes in transaction costs will result in different option values even if the exercise barrier remains unaffected. In particular, a higher recovery rate gives the same optimal exercise barrier but a higher option value.

exercising the option.

## 6.4 Comparison with results for log utility

Log utility is a special case of power utility. If we modify the power utility function definition to

$$\mathbb{U}(w) = \frac{w^{1-\gamma} - 1}{1 - \gamma} \quad \forall w \in (1, \infty)$$

then adding a constant does not change the utility function as such, and the first as well as higher order derivatives remain unaffected. With this modified definition the power utility converges to log utility as the relative risk aversion  $\gamma$  tends to 1. Both log and power are admissible utility functions and it is interesting to check that our results for power utility converge to those for log utility.<sup>24</sup> In fact the barrier equation (2) for this modified power utility can be written as

$$(\delta z)^{1-\gamma} \left(1 + \frac{K}{z}\right) = p^* \left[ \frac{(\delta z)^{1-\gamma} - 1}{1 - \gamma} \right]$$

where  $z = b - K$ . As  $\gamma \rightarrow 1$  the barrier equation converges to

$$\left(1 + \frac{K}{z}\right) = p^* \log(\delta z)$$

which is identical to the barrier equation (12) for the log utility. The convergence of barriers indicates convergence of exercise policies. Since the utility functions themselves converge, the convergence of exercise barriers also indicates that the value functions converge.

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<sup>24</sup>We thank an anonymous referee for suggesting this verification.

To summarize, the optimal exercise behavior and option valuation for the power utility converge to those for log utility as the relative risk aversion tends to 1.

One may wonder why this modified power utility function definition was not used in the first place. While the modified form is extremely helpful to relate power utility to log utility it is not so convenient in illustrating the model. In particular the optimal exercise barrier can be explicitly solved for power utility using the original definition we adopted. With the modified definition the barrier equation becomes a non-linear equation with no explicit simple form for the barrier.

## 7 Model Implementation

In this section first we examine how realistic the model is and then illustrate the model numerically. We have imposed two parametric conditions to facilitate proofs, one on the price process and another on individual risk aversion. We examine both of them before illustrating the model numerically.

### 7.1 Check for $\mu \geq \frac{1}{2}\sigma^2$ condition

We first test on empirical grounds how often the condition  $\mu \geq \frac{1}{2}\sigma^2$  is violated in practice. This condition is used in most of the proofs presented herein, but it is not shown to be a necessary condition for the solutions obtained.<sup>25</sup>

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<sup>25</sup>In fact Kadam, Lakner, and Srinivasan (2003) prove similar results without imposing this condition but they focus only on negative exponential utility.

We use the Center for Research in Security Prices (CRSP) database for stock returns and volatility estimates. CRSP is one of the most widely used financial databases. CRSP has daily return data available from July 1962 onwards. The current version of the database includes returns up to Dec 31, 2003.<sup>26</sup> Using this data, we compute the average daily return for each security for which information is available for each year. In order for a security to be included, we require that it must have been traded for at least 100 days in the given calendar year. Once this criterion is met, we calculate the average daily return and the standard deviation of the return of this security for this year. We repeat this process for each security for each year where it has sufficient trading data available. Through this process, we are able to identify a total of 234967 firm-year pairs that have sufficient trading data for analysis. For each of these firm-year pairs, we calculate the value of  $\mu$  as the average realized return in the given year, and the value of  $\sigma$  as the average standard deviation of return over this year. We then compare the values of  $\mu$  and  $\sigma$  so obtained to see if the condition  $\mu \geq \frac{1}{2}\sigma^2$  is satisfied or not. We find that out of the total of 234,967 firm-year observations, a total of 130,331 firm-year observations satisfy the above condition. Therefore, it appears that this restriction is violated for about 44.5% of the data.

We also conducted a similar test on the data using each security's entire trading history. Out of a total of 24489 securities, we found

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<sup>26</sup>Since the period of analysis includes the stock market crash of 2000 and 2001, the data is likely to give estimates of  $\mu$  that are lower than their historical averages. On the other hand, not incorporating dividends will make the estimates higher than they should be.

that the restriction was satisfied for a total of 12682. Therefore, it appears that this restriction is violated for about 48.21% of the data.

Lastly, we repeated the above test for the subperiod starting 1990, the reason being that firm level volatility has been especially high (and increasing over time) in recent years. Out of a total of 16538 firms with valid return data, the restriction was satisfied for a total of 8618. Therefore, it appears that this restriction is violated for about 47.89% of the data.

Thus our estimate of the proportion of cases in which the condition is violated is just less than half. While it is no doubt a large fraction, we can still conclude that the parametric restriction is not so serious so as to invalidate the applicability of the model completely.

## 7.2 Check for violation of $\gamma < 1$ condition

The restriction of gamma less than 1 may appear to be restrictive for this model. Several studies such as those cited in Cochrane (2001) require a coefficient of relative risk aversion to be much greater than 1. In fact, the equity premium puzzle, namely the excess return of equity over the risk free rate Mehra and Prescott (1985) requires a risk aversion of 50. These are called puzzles precisely because of the implausibility of high values of relative risk aversion. Even in literature, there is considerable disagreement as to range of relative risk aversion values. For example, the classic study by Hansen and Singleton (1983) found estimates of relative risk aversion from .68 to .97. Jackwerth (2000) finds risk aversion estimates close to 0 using options market

data. Further, Epstein and Zin (1991) also find that the estimates are sensitive to choice of utility function. An example of this is found in Campbell and Cochrane (1999) who find that for utility functions with habit formation, a relative risk aversion of around 2.0 is sufficient to explain the predictability of stock market returns.

However, caution must be interpreted in relating these values literally to our work. The reason for this is that all of the above cited studies use aggregate stock market and consumption data to estimate the relative risk aversion of a representative consumer. The studies more relevant for the option pricing context are studies on risk aversion of an individual consumer. Unfortunately, not much work is done in this area. First, there is considerable disagreement even with regard to the appropriate functional form for utility function. For example, Friend and Blume (1975) find that utility functions must either have constant or increasing relative risk aversion depending on the treatment of household assets. For the CRRA case, they find a value of about 2.0. On the other hand, Cohn, Lewellen, Lease, and Schlarbaum (1975) find evidence decreasing relative risk aversion. Siegel and Hohan (1982) replicate their study and find that least wealthy people have increasing relative risk aversion and the most wealthy people have decreasing relative risk aversion. Szpiro (1986) finds evidence in favor of the CRRA utility function with values of relative risk aversions in the range of 1.2 - 1.8 using actual insurance purchases. Other studies attempt to measure risk aversion without specifying the functional form of the utility function. Barsky, Juster, Kimball, and Shapiro (1997) categorize individuals into four categories. The mean



relative risk aversions for the different categories range from 0.7 to 15.8. Halek and Eisenhauer (2001) find median relative risk aversion of 0.88 with individual values ranging from 0.02 to 680.

Given the lack of agreement in the literature, we believe that our results are potentially applicable to pricing of ESO's as well as real options. Having said this, managers of a corporation are likely to behave closer to risk neutrality as they are not using their own money for project selection. To that extent, the restriction of gamma less than 1 is more likely to be applicable to the real options scenarios.

### **7.3 Numerical illustration**

To illustrate the model we choose a risk averse manager with power utility (hence she exhibits constant relative risk aversion). We illustrate in the context of executive stock options where maturity is almost always finite, so that we can get a feel for how the perpetuity assumption fares in practice. Ideal benchmarks would have been Huddart (1994) and Kulatilaka and Marcus (1994), both of which are binomial models. However direct comparisons with these papers is difficult as they assume that proceeds can be invested in a risk free asset. Therefore, for valuation comparison we treat the Black-Scholes model as the benchmark.

We assume that the parameters governing the evolution of the price process for the underlying common stock are  $\mu=12.3\%$  and  $\sigma=20.5\%$ . These numbers are based on the annualized summary statistics for historical data on returns for common stocks as quoted in Table 6.1 of

Bodie, Kane, and Marcus (2002). For the manager's time preference parameter, we do not have conclusive evidence from prior literature on discounting rates for money for individuals. We refer to Warner and Pleeter (2001) who find evidence suggesting that over half of sample of military officers and over 90% of personnel had discount rates of over 18%. We assume a discount rate of 15% which would probably cover most of the sample. Given that most of the personal discount rates appear to be greater than  $\mu$ , we focus on this sub-case alone in this analysis. Since our choice of  $\rho > \mu$  implies that the option is elastic, gamma has to lie in  $(0, 1)$  and we choose it to be the midpoint i.e. 0.50. We set  $x = K = 1$  since most executive stock options are granted at the money<sup>27</sup>. Finally, to enable a comparison with the Black-Scholes model, we assume the risk free interest rate to be  $r = 3.7\%$ . This number is based on the historical average of returns on one-month bills as quoted in Table 6.1 of Bodie, Kane, and Marcus (2002).

The choice of above values yields an option elasticity  $p^* = 1.39$  and an optimal exercise barrier of 1.56 which is also the ratio  $b^*/K$  since  $K = 1$ . Had we assumed the manager to be risk neutral, as in applying the contingency claims approach of Dixit and Pindyck (1994), the ratio would have been 2.78. In comparison, Huddart and Lang (1996) report the first three quartiles of  $b^*/K$  to be 1.283, 1.626 and 2.494 respectively.

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<sup>27</sup>Carpenter (1998) makes a similar standardization and gives a more elaborate justification.

The expected time to exercise as predicted by our model would be 5.39 years. In comparison, Black-Scholes model implementations often assume a reduced maturity of 6 or 6.5 years as the expected time to exercise.

The certainty equivalent price of the option (obtained by inverting the option value from our model) is 20 cents. In comparison, the option prices computed using the Black Scholes formula are approximately 30, 40 and 100 cents respectively for reduced maturity (6 years), full maturity (10 years) and infinite maturity<sup>28</sup>.

## 8 Conclusion

We explicitly solved the optimal stopping problem related to the optimal exercise of a perpetual call option, for contexts where individual preferences matter. We also proved the verification theorem for the solution proposed, and derived the moment generating function of optimal exercise time as well as the elasticity of the option value w.r.t. stock price. Sensitivity analysis was facilitated by explicit solutions.

The option elasticity depends on the difference between the stock price drift and manager's discount rate. The manager's response to changes in volatility depends on option elasticity.

For power utility, ignoring risk aversion results in over-estimating the exercise barrier and hence longer waiting times (on average) before

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<sup>28</sup>In practice, for executive stock options the Black-Scholes formula is often used with an adjusted (reduced) maturity. This is done because the option is non-portable so it makes sense to replace the option life with its expected time to exercise.

option exercise. In this case the magnitude of the percentage error strictly increases with the manager's risk aversion and with the stock price drift.

# Appendix

## A Power utility with inadmissible $\gamma$ values

### A.1 When $\gamma < 1 - p^*$

The expected reward upon hitting barrier  $b$  is

$$E [e^{-\rho\tau_b}\mathbb{U}(\delta(X_{\tau_b} - K))] = \frac{(\delta(b - K))^{1-\gamma}}{1 - \gamma} E [e^{-\rho\tau_b}] = \frac{(\delta(b - K))^{1-\gamma}}{1 - \gamma} \left(\frac{x}{b}\right)^{p^*}$$

where the last equality follows from (Karlin and Taylor 1975).<sup>29</sup> It is also consistent with the moment generating function obtained in section 4.6. Thus the limit of the value function as  $b$  tends to infinity is proportional to  $(b - K)^{1-\gamma}b^{-p^*}$  which tends to infinity whenever  $p^* < 1 - \gamma$ . This implies that no a.s. finite stopping time can be optimal. Effectively  $b^* = \tau^* = \infty$ .

### A.2 When $\gamma = 1 - p^*$

The expected reward upon hitting barrier  $b$  is

$$E [e^{-\rho\tau_b}\mathbb{U}(\delta(X_{\tau_b} - K))] = \frac{(\delta(b - K))^{1-\gamma}}{1 - \gamma} \left(\frac{x}{b}\right)^{p^*}$$

which reduces to  $\frac{(\delta x)^{p^*}}{p^*}(1 - \frac{K}{b})^{p^*}$  when  $p^* = 1 - \gamma$ . This is strictly increasing in  $b$  so a finite barrier cannot be optimal. Effectively  $b^* = \tau^* = \infty$ .

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<sup>29</sup>See footnote 21.

### A.3 When $\gamma > 1$

Suppose  $G_t = e^{-\rho t} \mathbb{U}(\delta(X_t - K)) = \frac{e^{-\rho t} (\delta(X_t - K))^{(1-\gamma)}}{1-\gamma}$  for  $X_t > K$  and  $G_t = 0$  otherwise. From  $\mu \geq \frac{1}{2}\sigma^2$  it follows that  $\limsup_{t \rightarrow \infty} G_t = 0$ . Then we can define for any  $\epsilon > 0$  the stopping time  $\tau_\epsilon = \min\{t : G_t \geq -\epsilon\}$ . We have  $E[G(\tau_\epsilon)] \geq -\epsilon$ , so  $\tau_\epsilon$  is a ‘pseudo-optimal’ stopping time, although there will be no optimal stopping time in this case.

## B First partial derivatives of $p^*$ w.r.t $\rho, \mu, \sigma$

$p^*$  the only positive root of equation (1) is given by

$$\frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 - 4(\frac{1}{2}\sigma^2)(-\rho)}}{2(\frac{1}{2}\sigma^2)}$$

To simplify notation denote  $\mu - \frac{1}{2}\sigma^2$  by  $m$  and  $\sigma^2$  by  $n$  so that

$$p^* = \frac{-m + \sqrt{m^2 + 2n\rho}}{n}$$

$$\frac{\partial p^*}{\partial \rho} = \frac{1}{n} \left( 0 + \frac{n}{\sqrt{m^2 + 2n\rho}} \right) > 0$$

$$\frac{\partial p^*}{\partial \mu} = \frac{1}{n} \left( -1 + \frac{m}{\sqrt{m^2 + 2n\rho}} \right) < 0$$

$$\frac{\partial p^*}{\partial \sigma} = \frac{n \left( \sigma + \frac{-2m\sigma + 4\rho\sigma}{2\sqrt{m^2 + 2n\rho}} \right) - (-m + \sqrt{m^2 + 2n\rho}) 2\sigma}{n^2}$$

$$= \frac{\sigma}{n} \left( 1 + \frac{2\rho - m}{np^* + m} - 2p^* \right)$$

Substituting  $2\rho = 2mp^* + np^{*2}$  from equation (1) it follows that

$$\frac{\partial p^*}{\partial \sigma} = \frac{\sigma}{n} (np^* - np^{*2}) = \sigma p^*(1 - p^*) \leq 0 \text{ according as } p^* \geq 1$$

The values taken by the convex parabola represented by the L.H.S. of equation (1) are  $-\rho < 0$  and  $\mu - \rho$  at  $p = 0, 1$  respectively. This implies that  $p^* \geq 1$  according as  $\mu \leq \rho$ . Thus  $\frac{\partial p^*}{\partial \sigma} \leq 0$  according as  $\mu \leq \rho$ .

### C First partial derivatives of $V(\cdot)$ , $b^*$ w.r.t $p^*$

Taking logarithms and then differentiating  $V(x) = \mathbb{U}(\delta(b^* - K))(x/b^*)^{p^*}$  w.r.t.  $p^*$  and then simplifying using Equation 2 gives

$$\frac{\partial V}{\partial p^*} = V(x) \log\left(\frac{x}{b^*}\right) < 0$$

inside the continuation region since  $x < b^*$ .

Similarly, taking logarithms and then differentiating Equation 2 w.r.t.  $p^*$  and then simplifying using Equation 2 as well as using the definition  $\mathcal{R}_{\mathcal{R}}(\omega) = \frac{-\omega \mathbb{U}''(\omega)}{\mathbb{U}'(\omega)} = \gamma$  gives

$$\frac{\partial b^*}{\partial p^*} = \frac{1}{p^*} \left[ \frac{1}{b^*} + \delta \left( -\frac{\gamma}{\delta(b^* - K)} - \frac{p^*}{\delta b^*} \right) \right]^{-1} < 0$$

by the last admissibility condition in Section 3.2.

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## Response to the Editor's Comments

The authors sincerely thank the editor and the four anonymous reviewers for providing valuable advice on strengthening the paper. Most of these suggestions have been incorporated to enhance the paper. The details of editions and additions made are given in the responses to each reviewer. The section numbers in these responses refer to those in the revised version and not to those in the old version.

In the revised version the authors have removed two errors that were discovered thanks to further examination of suggestions made by the reviewers. The authors apologize for both these errors, first in the proof for the case of a highly risk averse manager, and second in the sensitivity analysis of the solution w.r.t. volatility of the underlying asset.

In the revised version the authors have enriched the paper in three ways thanks to further examination of reviewer's comments. First is the characterization of option's elasticity w.r.t. stock price. Second is the generalization of some sensitivity analysis results to all admissible utility functions. Third is showing in the context of power utility that ignoring risk aversion, as is often done in the context of real options, leads to overambitious exercise barriers (and hence longer waiting times on average).

## Response to Reviewer #1 Comments

### Major Comments

1. The reviewer makes two suggestions side by side viz. to show convergence of power to log and to compact the text by eliminating the section on log utility.

A separate section 6.4 has been added to show that the results for power utility converge to those for log utility. The authors are grateful for the modified definition of power utility suggested by the reviewer and they use this modified form to show convergence.

However as mentioned in section 6.4, the modified definition does not permit explicit solution to the optimal exercise barrier as does the original definition adopted by the authors. That explicit solution is important to illustrate the model. The sensitivity analysis is also more complicated with the exercise barrier from the modified power utility. Hence the original definition for power utility had to be retained in the remainder of the article. With that original setup, the connection between log utility and power utility is not obvious. Hence the independent section on log utility and the corresponding results on sensitivity analysis had to be retained. It is worth mentioning that the individual sections for log and power utility are now shorter in the revised version because some of the sensitivity analysis results have been generalized and moved to Section 4.5.

2. The reviewer raises two issues viz. the result in section 6.1 requiring more investigation and the condition  $\mu \geq \frac{1}{2}\sigma^2$  being too restrictive.

The authors apologize for the error in statement and proof of the result for the case of  $\gamma > \frac{\mu}{\frac{1}{2}\sigma^2} > 1$ . The revised version has a corrected proof in Appendix A.3 showing that no stopping time can be optimal for the case of  $\gamma > 1$ . The main text has been modified accordingly in section 6.1 of the revised version.

To address the second issue that the condition  $\mu \geq \frac{1}{2}\sigma^2$  is too restrictive the authors have added section 7.1. It emphasizes that the condition has been imposed to facilitate proofs and is not a necessary condition for the solution to exist. It also shows on empirical grounds that the condition is quite often not violated in reality, so the model may still be useful in reality.

### **Minor Comments**

1. Footnote 15 in the revised version does not have this repetition.
2. The phrase ‘Sensitivity to’ has been added in the title of Section 4.5.2 of the revised version.
3. The reference has been corrected to ‘Appendix A. It appears in the main text of Section 6.1 of the revised version.
4. For reasons mentioned in the response to the second Major Comment, this part of the Appendix was dropped so the typo does not exist anymore.

## Response to Reviewer #2 Comments

### Major Comments

- The definition of power utility has been modified so that the parametric restriction is stated separately, not as part of the definition. This change is made in first paragraph of Section 6 of the revised version.
- That  $p^*$  lies above or below 1 according as  $\mu$  is less or more than  $\rho$  has now been mentioned in footnote 6 and proved in Appendix B of the revised version.
- The authors apologize for the error in statement and proof of the result for the case of  $\gamma > \frac{\mu}{\frac{1}{2}\sigma^2} > 1$ . The revised version has a corrected proof in Appendix A.3 showing that no stopping time can be optimal for the case of  $\gamma > 1$ . The main text has been modified accordingly in section 6.1 of the revised version. Based on the incorrect result in the earlier version of the paper, the reviewer had commented that for realistic risk aversion values our model suggests following the NPV rule, and that this needs to be discussed. Given the lack of any optimal solution for the case of  $\gamma > 1$  in the revised version, this comment or a related discussion cannot be incorporated.
- To assuage the reviewer's concern about the parametric restriction  $\gamma < 1$ , Section 7.2 has been added in the revised version.

## Minor Comments

- The phrase ‘gulp consumption’ has been replaced with the word ‘consumption’ in the first line of section 3.2 in the revised version.
- The notation for value function in the PDE as well as in the boundary condition given in the first paragraph of Section 4.1 in the revised version have been rectified to  $V$  and made consistent with the rest of the article.
- The repetition of ‘as the’ has been rectified. This text now appears in the second line of Section 4.5.1 in the revised version.



## Response to Reviewer #3 Comments

### Major Comments

1. The main contributions are as highlighted in the second last paragraph of section 1 in the revised version. The revised version highlights one additional contribution not mentioned in the earlier version. This contribution is to characterize the elasticity of the option value with respect to the price of the underlying. As an aside, this elasticity is also related to the sensitivity of option exercise behavior w.r.t volatility; please see footnote 20.

Another contribution, not highlighted in the earlier version is to point out (in Section 6.3) that ignoring risk aversion, as is often done in the context of real options, leads to overambitious exercise barriers (and hence longer waiting times on average). The section precisely quantifies the error in setting the target exercise price.

2. Section 7.3 has been added to illustrate the model numerically. This section also compares model outputs for the illustration with values of  $b^*/K$  from literature, with Black-Scholes prices assuming full, reduced and infinite maturities, and with expected exercise times assumed in practice.
3. To address the second issue that the condition  $\mu \geq \frac{1}{2}\sigma^2$  is too restrictive the authors have added section 7.1. It emphasizes that the condition has been imposed to facilitate proofs and is not a necessary condition for the solution to exist. It also shows on empirical grounds that the condition is quite often not violated in

reality, so the model may still be useful in reality. In particular, it indicates that the increase in volatility as mentioned in reviewer comments has not materially affected the applicability of the model.

It is noteworthy that CRSP, the database used in this empirical analysis, was also the one used in Campbell, Lettau, Malkiel, and Xu (2001). Their data also started in July 1962, but their ending date was December 1997 whereas this analysis includes more recent data, ending date being Dec 31, 2003. Thus, this period of analysis included the crash in the stock market that occurred in 2000 and 2001. As such, using this data is likely to result in estimates of  $\mu$  that are lower than their historical averages. On the other hand incorporating dividends would indeed make the condition harder to satisfy in practice.

4. In the revised version, footnote 3 gives examples from extant literature. The comment on ‘other market imperfections at play’ has been dropped from Section 1.
5. In the revised version, footnote 4 clarifies the location of the problem inside the book. Furthermore, the sentence in the main text prior to the footnote describes the gist of the problem. ‘They’ and ‘DeTemple’ have been changed to ‘He’ and ‘Detemple’ in Section 2 of the revised version.
6. In the revised version, footnote 21 clarifies the location of the relevant discussion inside the book. Their result is in fact identical, but in different notation. It is an intermediate step for the

problem they are solving and has no particular equation number to refer to.

### **Minor Comments**

- Corrected to ‘The recent explosion’ as advised. This now appears on the 6th line of second last paragraph of section 2 in the revised version.
- Corrected to ‘exist and are finite’ as advised. This now appears in footnote 15 in the revised version.
- Replaced ‘performing’ by ‘performed’ as advised. This now appears in the first paragraph of section 5 of the revised version.
- The repetition of ‘as the’ has been rectified. This text now appears in the second line of Section 4.5.1 in the revised version.
- The word ‘costs’ dropped as advised. This text now appears in the second line of Section 6.2 in the revised version.

## Response to Reviewer #4 Comments

### Major Comments

- The revised version has motivated the problem with emphasis on real options, using the project management example. Section 1 of the revised version ( the section that introduces and motivates the paper) now mentions executive stock options as an aside in only three occasions viz. in the sentence preceding footnote 2, the footnote 2 itself and when defending the perpetuity assumption. Wherever possible, the word ‘executive’ has been replaced by the word ‘manager’ in the main text of the entire paper.

However, the literature review still has studies cited from ESO literature. The reasons for not completely omitting ESOs from this article are twofold

1. It widens the applicability of the model.
  2. It helps the reader obtain a link to a large body of literature that is relevant to option valuation under non-tradability restrictions.
- There was an error in the result on sensitivity to volatility, this has been corrected in the revised version. In fact, in Section 4.5 of the revised version, the sensitivity analysis results for the option value as well as for the optimal exercise barrier w.r.t. parameters  $\rho, \mu$  and  $\sigma^2$  have been generalized to all admissible utility functions. A summary of these results has been added in Table 1 of the revised version. The first paragraph in Section 4.5 explains how the results can be obtained from the proofs in

Appendix B and Appendix C that have been added in the revised version.

- Section 6.2 in the revised version now includes a short comment on the intuition behind each result.
- Section 2 in the revised version includes two of these references. Literature on executive stock options is vast, and in trying to keep the paper's real options emphasis, the authors selected only two more published papers to be added to the ESO literature already summarized in the paper. These are Lambert, Larcker, and Verrechia (1991) and Subramanian and Jain (2004). Both papers merit their mentioning because they are significant departures from prior literature, the first in advocating utility-based models, the second in recognizing the multiple-date nature of employee stock option exercises.