Utility Maximization with Partial Information

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Abstract

In the present paper we address two maximization problems: the maximization of expected total utility from consumption and the maximization of expected utility from terminal wealth. The price process of the available financial assets is assumed to satisfy a system of functional stochastic differential equations. The difference between this paper and the existing papers on the same subject is that here we require the consumption and investment processes to be adapted to the natural filtration of the price processes. This requirement means that the only available information for agents in this economy at a certain time are the prices of the financial assets up to that time. The underlying Brownian Motion and the drift process in the system of equations for the asset prices are not directly observable. Particular details will be worked out for the "Bayesian" example, when the dispersion coefficient is a fixed invertible matrix and the drift vector is an F_0–measurable, unobserved random variable with known distribution.

Keywords: Security price process, stochastic differential equation, investment and consumption, utility maximization.

1. Introduction

A financial market model is formulated in this paper with $d+1$ assets, one bond and d stocks. The price of the bond is assumed to be 1 over the entire continuous time–horizon $[0,T]$, where T is a fixed, finite terminal time, and the d–dimensional price process X of the stocks is assumed to satisfy a system of functional stochastic differential equations. The price process X, the d–dimensional Brownian Motion w appearing in the system of

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stochastic differential equations for X, the drift vector, and the dispersion matrix are all adapted to a fixed filtration F.

We consider two related maximization problems here: one is the maximization of expected total utility from consumption, and the other is the maximization of expected utility from terminal wealth. The contribution of the present paper compared to the existing literature is that we require the consumption and investment processes to be adapted to the natural filtration of the stock price process X, which is usually smaller than F. In this framework, an agent in the economy can observe the prices only; the Brownian Motion and the drift coefficient are not directly observable. This is quite realistic since the drift vector and w are fictitious mathematical objects, and certainly not directly observable to agents in the economy. However, in our model the dispersion matrix will be observable since it is assumed to be adapted to the natural filtration of X.

In previous works the consumption and investment processes are adapted to F which is generated by w (complete markets), or generated by w and by some additional Brownian Motions (incomplete markets). However, in both cases the Brownian Motions generating F are assumed to be observable. We use the terminology that in this case full information is available to agents in the economy. In our problem, when only the prices are observable, we use the terminology that agents have partial information.

The solution to the utility maximization problem with full information is available in the literature. For the case of complete markets we refer to Cox, Ingersoll, & Ross (1985), Cox & Huang (1989), Karatzas, Lehoczky, & Shreve (1987), Duffie & Zame (1989), Karatzas, Lakner, Lehoczky, & Shreve (1991), and Ocone & Karatzas (1991). For the case of incomplete markets we refer to Karatzas, Lehoczky, Shreve, & Xu (1991) and He & Pearson (1991).

In the rest of this introduction we give an outline of the paper. In Section 2, we write down the exact assumptions on the price process X and give some examples. Our assumptions guarantee the existence of a probability measure $\tilde{P}$, equivalent to the original measure P, such that X is a d–dimensional $\tilde{P}$–local martingale with respect to the filtration F. In Section 3, we define the admissible consumption/investment pairs, list our conditions on the utility functions, and pose the expected total utility from consumption maximization problem. The utility function has the form $U(\omega, t, c)$, and it quantifies the utility an agent gaines by consuming at the rate c at time t when the state of the world is $\omega \in \Omega$. In

Section 4 some crucial technical results are presented.

In Section 5, we show that the solution of the optimization problem posed in Section 3 is unique, and give an explicit formula for the optimal consumption process. At this level of generality we don't have an explicit formula for the optimal investment process, but offer an equation by which it is uniquely determined.

In Section 6, we pose the expected utility from terminal wealth maximization problem, show that it has a unique solution, and present an explicit formula for the optimal level of terminal wealth. Again, at this level of generality we have no explicit formula for the optimal investment process but present an equation by which it is uniquely determined.

In the formulae for the optimal consumption process and for the optimal level of terminal wealth, the process ζ enters in a prominent fashion, where $\zeta(t)$ is the conditional expectation of the Radon–Nicodym derivative $\frac{d\tilde{P}}{dP}$ with respect to the available information to the agents at time t. In Section 7, we give several explicit representations for $\zeta(t)$ assuming that the system of stochastic differential equations for X is a special case of the one postulated in Section 2. In particular, we assume in this section that the dispersion coefficient is a fixed invertible matrix, and the drift vector is an F_0–measurable d–dimensional random variable with known distribution. This can be regarded as a Bayesian problem.

In Section 8, we keep the assumptions of the previous section, and also suppose that the utility function is given by $\log c$. In this case we can give explicit representations to the optimal investment processes of both maximization problems, and to the corresponding wealth processes. One can derive from these results the already known formulae for the case of full information by assuming that the distribution of the drift vector is degenerate. Further specialization occurs when $d = 1$ and the drift coefficient follows a one–dimensional normal distribution with known mean and variance. This is exactly the problem addressed by Browne & Whitt (1994) and we specialize our explicit formulae to this situation.

In Section 9, we give another example when the conditions of the main theorems are satisfied. Here we still assume that the system of stochastic differential equations for X is the one described in Section 7 (the Bayesian case), and the utility function is given by a discount process multiplied by c^η where $\eta \in (0, 1)$ is a fixed constant. Finally, in Section 10, we prove a purely technical lemma needed in Section 7.

2. Security price processes

Let $(\Omega, \mathcal{F}, P)$ a probability triplet and $F = \{F_t;\ 0 \leq t \leq T\}$ a filtration in $\mathcal{F}$ satisfying the usual conditions (augmented and right–continuous). The terminal time $T < \infty$ is a fixed constant. In our model of the financial market there are d risky securities (stocks) and a bond available for trading. The bond is assumed to have price unity over the entire time–horizon $[0,T]$, and the price process of the stocks is denoted by $\{X(t) = (X_1(t), \ldots, X_d(t))^*;\ 0 \leq t \leq T\}$ (* denotes transposition), where X is an F-adapted process satisfying the system of functional stochastic integral equations

$$X_i(t) = x_i + \int_0^t A_i(\omega, u, X)du + \sum_{j=1}^{d} \int_0^t D_{ij}(u, X)dw_j(t), \quad i = 1 \ldots d, \tag{2.1}$$

where $x = (x_1, \ldots, x_d)^*$ is a fixed vector. Assumptions on the ingredients of (2.1) will be spelled out below. The process w is a standard, d-dimensional Brownian motion under the probability measure P and with respect to the filtration F (we specify the filtration and the probability measure here because later in the paper another filtration and probability measure will be defined).

In order to write down our assumptions on $A = (A_1, \ldots, A_d)^*$ and $D = (D_{ij})_{i,j=1,\ldots,d}$ we need some notations. Let $C^d[0,T]$ be the class of continuous mappings from $[0,T]$ to $\Re^d$, and ξ_u be the coordinate mapping from $C^d[0,T]$ to $\Re^d$, i.e., $\xi_u(f) = f(u)$ for $f \in C^d[0,T]$ and $u \in [0,T]$. Let B_t be the σ-field generated by the class of mappings $\{\xi_u;\ 0 \leq u \leq t\}$. Let $A : \Omega \times [0,T] \times C^d[0,T] \mapsto \Re^d$ and $D : [0,T] \times C^d[0,T] \mapsto \Re^{d \times d}$ be measurable mappings such that $A(\cdot, t, \cdot)$ is $F_t \otimes B_t$–measurable and $D(t, \cdot)$ is B_t–measurable for every $t \in [0,T]$, and they satisfy the forthcoming Assumptions 2.1, 2.2 and 2.3.

2.1 Assumption: For every $f \in C^d[0,T]$ the mapping $t \mapsto D(t,f)$ is LCRL (left–continuous with finite right limits) on $[0,T]$ and the following Lipschitz–condition holds: there exists a constant $K > 0$ such that for every $f, g \in C^d[0,T]$

$$\|D(t,f) - D(t,g)\| \leq K \sup_{0 \leq u \leq t} \|f(u) - g(u)\|\ , \quad t \in [0,T]. \tag{2.2}$$

The norm in the left and right–hand side of (2.2) is the Euclidean norm in $\Re^{d \times d}$ and $\Re^d$, respectively.

We do not assume that X is the unique solution of (2.1). The system may have more than one solutions and X may be one of them. We denote by α the measurable, adapted d–dimensional process

$$\alpha(\omega, t) = \alpha(t) = A(\omega, t, X(\omega)),$$

and by δ the measurable, adapted, $\Re^{d\times d}$–valued process

$$\delta(t,\omega) = \delta(t) = D(t, X(\omega)).$$

The process δ has LCRL paths. The assumption that X satisfies (2.1) means, in a more rigorous form, that

$$\int_0^T |\alpha_i(t)|dt < \infty \text{ ,a.s.,} \quad i = 1, \ldots, d, \tag{2.3}$$

$$\int_0^T \|\delta(t)\|^2 dt < \infty, \ a.s., \tag{2.4}$$

and

$$X_i(t) = x_i + \int_0^t \alpha_i(u)du + \sum_{j=1}^d \int_0^t \delta_{ij}(u)dw_j(u) \ . \tag{2.5}$$

2.2 Assumption: For every $(\omega, t) \in \Omega \times [0, T]$, $\text{rank}(\delta(\omega, t)) = \text{rank}(\delta(\omega, t+)) = d$.

We denote by $\delta^{-1}(t)$ the inverse of $\delta(t)$. We did *not* assume that $\text{rank}(D(t, f)) = \text{rank}(D(t+, f)) = d$ for every $f \in C^d[0, T]$. This would be too restrictive for Examples 2.4 and 2.5. Of course this condition would be sufficient to guarantee that Assumption 2.2 holds.

2.3 Assumption: We suppose that

$$\int_0^T \|\delta^{-1}(t)\alpha(t)\|^2 dt < \infty \text{ ,a.s.,} \tag{2.6}$$

and the positive local martingale Z defined by

$$Z(t) = \exp\{-\sum_{i=1}^d \int_0^t \left(\delta^{-1}(u)\alpha(u)\right)_i dw_i(u) - \frac{1}{2}\int_0^t \|\delta^{-1}(u)\alpha(u)\|^2 du\} \tag{2.7}$$

is actually a martingale (with respect to the filtration F and the probability measure P).

In (2.7) the expression $\left(\delta^{-1}(u)\alpha(u)\right)_i$ is a notation for the i–th entry of the vector $\delta^{-1}(u)\alpha(u)$. If the process $\delta^{-1}\alpha$ is bounded then Assumption 2.3 certainly holds. However, assuming boundedness of $\delta^{-1}\alpha$ would be too restrictive to include Examples 2.5 and 2.7.

2.4 Example: Let X be the unique solution of the system of stochastic integral equations

$$X_i(t) = x_i + \int_0^t m_i(u)X_i(u)du + \sum_{j=1}^d \int_0^t \sigma_{ij}(u)X_i(u)dw_j(u) \ , \ i = 1, \ldots, d, \tag{2.8}$$

where $m = (m_1, \ldots, m_d)^*$ is a measurable, F–adapted, bounded process, and $\sigma = (\sigma_{ij})_{i,j=1,\ldots,d}$ is a (deterministic) LCRL mapping from $[0,T]$ to $\Re^{d\times d}$ such that $\mathrm{rank}(\sigma(t)) = \mathrm{rank}(\sigma(t+)) = d$. By Karatzas & Shreve, (1988), Problem 5.6.15, equation 2.8 indeed has a unique solution with positive paths. It is easy to check that Assumptions 2.1 and 2.2 hold, and Assumption 2.3 follows since σ^{-1} is LCRL thus bounded.

2.5 Example: Let σ be a mapping from $[0,T]$ to $\Re^{d\times d}$ satisfying the same conditions as in Example 2.4, $\mu = (\mu_1, \ldots, \mu_d)^*$ be an F_0–measurable, d–dimensional random variable, and $X = (X_1, \ldots, X_d)^*$ be the unique solution of the stochastic differential equation

$$X_i(t) = x_i + \int_0^t \mu_i X_i(u)du + \sum_{j=1}^d \int_0^t \sigma_{ij}(u)X_i(u)dw_j(u)\ ,\ i = 1\ldots, d. \tag{2.9}$$

Assumption 2.1 holds because of the boundedness of σ, and Protter (1990), Theorem II.8.36 guarantees that (2.9) has indeed a unique solution and the components of the solution have positive paths. Assumption 2.2 follows from the positivity of X_i. Finally, to show that Assumption 2.3 holds it is sufficient to verify that

$$E\big[Z(T)\big] = E\big[\exp\{-\sum_{j=1}^d \mu_j \sum_{i=1}^d \int_0^T (\sigma^{-1}(u))_{ij}dw_i(u) - \frac{1}{2}\int_0^T \|\sigma^{-1}(u)\mu\|^2 du\}\big] = 1\ . \tag{2.10}$$

(Z, F) is a positive local martingale, thus by Fatou's lemma a supermartingale, so $E[Z(T)] \leq 1$. The random variable μ is F_0–measurable thus independent of the Brownian Motion w, therefore

$$E\big[Z(T)|\mu = \lambda\big] = E\big[\exp\{-\sum_{j=1}^d \lambda_j \sum_{i=1}^d \int_0^T (\sigma^{-1}(u))_{ij}dw_i(u) - \frac{1}{2}\int_0^T \|\sigma^{-1}(u)\lambda\|^2 du\}\big] = 1 \tag{2.11}$$

for $P \circ \mu^{-1}$–almost every $\lambda \in \Re^d$, and (2.10) now follows. The second identity of (2.11) follows from Novikov's condition (Karatzas & Shreve, (1988), Corollary 3.5.13). A special case of this example has been studied by Browne & Whitt (1994), assuming that $d = 1$, μ follows a normal distribution and the utility function is logarithmic.

2.6 Example: Let X be given by

$$X_i(t) = x_i + \int_0^t m_i(u)du + \sum_{j=1}^d \int_0^t \sigma_{ij}(u)dw_j(u)\ ,\ \ i = 1, \ldots, d, \tag{2.12}$$

where m and σ are the same as in Example 2.4.

2.7 Example: Let X be given by

$$X_i(t) = x_i + \mu_i t + \sum_{j=1}^{d} \int_0^t \sigma_{ij}(u) dw_j(u) \ , i = 1 \ldots, d, \tag{2.13}$$

where μ and σ are the same as in Example 2.5 and 2.4, resp. It can be shown that this example satisfies Assumption 2.3 similarly to Example 2.5.

3. Consumption and investment processes

Let $\{G_t;\ 0 \le t \le T\}$ be the augmented natural filtration of X. In the present paper we require the consumption and investment processes to be adapted to G. This means that the only observation at time t is the price of the stocks up to that time. Neither $\{w(u);\ 0 \le u \le t\}$ nor $\{\alpha(u);\ 0 \le 0 \le t\}$ can be directly observed. However, the process δ and its inverse δ^{-1} are adapted to G (Lemma 10.3), so these are observable.

3.1 Definition: A $[0, \infty)$–valued, measurable, G–adapted process c is called a consumption process if

$$\int_0^T c(u) du < \infty \ , \text{a.s.} \tag{3.1}$$

3.2 Definition: A d-dimensional, measurable, G–adapted process π is called an investment process if

$$\int_0^T \|\pi^*(u)\delta(u)\|^2 du < \infty \ , \text{a.s.} \tag{3.2}$$

The quantity $c(t)$ represents the rate of consumption at time t, and $\pi_i(t)$ represents the number of shares of the i–th stock held by an agent at time t.

If π is an investment process then (2.6), (3.2), and the Cauchy–Schwarz inequality imply that

$$\int_0^T |\pi^*(u)\alpha(u)| du < \infty \ . \tag{3.3}$$

The wealth process corresponding to the consumption/investment pair (c, π) is

$$V^{c,\pi}(t) = v + \int_t^t \pi^*(u)\alpha(u) du + \sum_{j=0}^{d} \int_0^t \left(\pi^*(u)\delta(u)\right)_j dw_j(u) - \int_0^t c(u) du \ , \tag{3.4}$$

where $v \geq 0$ is the fixed initial capital. We refer to Karatzas, Lehoczky, & Shreve (1987) for a derivation of (3.4). By (3.3) and (3.2) the wealth process is well–defined.

3.3 Definition: A consumption/investment pair (c, π) is called admissible (for the initial capital $v \in [0, \infty)$) if $V^{c,\pi}(t) \geq 0$, a.s., for every $t \in [0, T]$. We denote the class of admissible consumption/investment pairs by $\mathcal{A}_1(v)$.

In the next definition we introduce the concept of utility functions.

3.4 Definition: An $\mathcal{F} \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}([0,\infty))/\mathcal{B}(\Re \cup \{-\infty\})$ measurable mapping

$$U : \Omega \times [0,T] \times [0,\infty) \mapsto \Re \cup \{-\infty\} \tag{3.5}$$

is called a utility function if the following three conditions hold:

(i) For every $(\omega, t) \in \Omega \times [0, T]$ the function $U(\omega, t, \cdot)$ is finite and continuously differentiable on $(0, \infty)$, and the function $c \mapsto \frac{\partial}{\partial c} U(\omega, t, c)$ is strictly decreasing on $(0, \infty)$, and

$$\lim_{c \to \infty} \frac{\partial}{\partial c} U(\omega, t, c) = 0 \ . \tag{3.6}$$

(ii) For every $(t, c) \in [0, T] \times [0, \infty)$ the random variable $U(\cdot, t, c)$ is G_t–measurable.

(iii) There exists a constant $\tilde{c} > 0$ such that

$$E\left[\int_0^T U^-(\omega, t, \tilde{c})dt\right] < \infty \tag{3.7}$$

and

$$E[U^-(\omega, T, \tilde{c})] < \infty \ , \tag{3.8}$$

where $U^-(\omega, t, c) = -U(\omega, t, c)$ if $U(\omega, t, c) < 0$ and $U^-(\omega, t, c) = 0$ otherwise.

The above definition implies that if U is a utility function then for every $(\omega, t) \in \Omega \times [0, T]$ the function $U(\omega, t, \cdot)$ is strictly increasing and strictly concave, and $\frac{\partial}{\partial c} U(\cdot, t, c)$ is G_t–measurable for every $(t, c) \in [0, T] \times (0, \infty)$. The quantity $U(\omega, t, c)$ expresses the amount of utility an agent gains by consuming at the rate c at time t. Condition (ii) guarantees that the agent expresses his/her utility at time t based on the available information at that time. Examples include

$$U_1(\omega, t, c) = \gamma(\omega, t) \log c \tag{3.9}$$

and

$$U_2(\omega,t,c) = \gamma(\omega,t)c^{\eta} \ , \tag{3.10}$$

where γ is a measurable, G–adapted, strictly positive discount process, and $\eta \in (0,1)$ a constant.

The expected total utility from consumption maximization problem is the following:

$$\text{maximize} \quad E\Big[\int_0^T U(\omega,t,c(t))dt\Big] \quad \text{over} \quad (c,\pi) \in \mathcal{A}_1(v) \tag{3.11}$$

under the additional constraint

$$E\Big[\int_0^T U^-(\omega,t,c(t))dt\Big] < \infty. \tag{3.12}$$

We call a consumption process $\bar{c}$ optimal for the problem (3.11)–(3.12) if there exists an investment process $\bar{\pi}$ such that $(\bar{c},\bar{\pi})$ is optimal. Similarly, we call an investment process $\bar{\pi}$ optimal for the problem if there exists a consumption process $\bar{c}$ such that $(\bar{c},\bar{\pi})$ is optimal for (3.11)–(3.12). Finally, if $(\bar{c},\bar{\pi})$ is optimal then we call $V^{\bar{c},\bar{\pi}}$ the optimal wealth process.

4. The filtration generated by the stock prices

We introduce the auxiliary probability measure $\tilde{P}$ by

$$\frac{d\tilde{P}}{dP} = Z(T), \tag{4.1}$$

where Z is defined in (2.7). In the solution of the optimal consumption/investment problem with full information the process Z and the probability measure $\tilde{P}$ was essential (Cox & Huang (1989), Karatzas, Lehoczky & Shreve (1987), or Karatzas, Lakner, Lehoczky & Shreve (1991)). In our case of restricted information we have to project $Z(t)$ to the available information, so we define

$$\zeta(t) = E[Z(t)|G_t] = E\Big[E\big[Z(T)|F_t\big] \mid G_t\Big] = E\Big[Z(T) \mid G_t\Big] \ . \tag{4.2}$$

After taking the appropriate modification, the process (ζ, G) becomes a P–martingale with RCLL (right–continuous with finite left–hand limits) paths. We denote by $\tilde{E}$ the expectation corresponding to the probability measure $\tilde{P}$ and note that for any G_t–measurable random variable L

$$\tilde{E}[L] = E[\zeta(t)L] \ , \quad t \in [0,T],$$

in the sense that if one of the above expectations exists then the other also exists, and they are equal.

We also introduce the d–dimensional process $\tilde{w}$ by

$$\tilde{w}(t) = w(t) + \int_0^t \delta^{-1}(u)\alpha(u)du \ , \tag{4.3}$$

which is a Brownian Motion under the probability measure $\tilde{P}$ with respect to the filtration F (Girsanov's Theorem, Karatzas & Shreve, (1988)). We can write (2.5) in the form

$$X_i(t) = x_i + \sum_{j=1}^d \int_0^t \delta_{ij}(u)d\tilde{w}_j(u) \quad i = 1, \ldots, d, \tag{4.4}$$

thus X is a d–dimensional $\tilde{P}$–local martingale with respect to both filtrations F and G.

The following theorem will be essential in solving the optimization problem. In the proof we shall use the fact that stochastic integrals with continuous integrator are invariant if computed under a smaller filtration, as long as both the integrand and the integrator are adapted to the smaller filtration which satisfies the usual conditions.

4.1 Proposition: The filtration G is the augmented natural filtration of $\tilde{w}$, and $(\tilde{w}, G)$ is a $\tilde{P}$–Brownian Motion. The filtration G is continuous.

Proof: Let $\tilde{F}$ be the augmented natural filtration of $\tilde{w}$ and $\tilde{G}_t = G_{t+}$, $\tilde{G} = \{\tilde{G}_t;\ 0 \leq t \leq T\}$. These notations are tentative in this proof since we are showing that these two filtrations are identical to G. We note that $\tilde{F}$ is continuous (Karatzas & Shreve (1988), Corollary 2.7.8). First we shall show that $\tilde{F} = \tilde{G}$.
By its definition $\tilde{G}$ satisfies the usual conditions. Since the process δ^{-1} has LCRL paths and is adapted to $\tilde{G}$, one can easily derive from (4.4) the identity

$$\sum_{j=1}^d \int_0^t \left(\delta^{-1}(u)\right)_{ij} dX_j(u) = \tilde{w}_i(u) \ , \quad , i = 1, \ldots, d, \tag{4.5}$$

which implies that $\tilde{F}_t \subseteq \tilde{G}_t$ since the stochastic integral on the left–hand side of (4.5) is $\tilde{G}$–adapted. In order to show the reverse inclusion we consider the system of stochastic differential equations

$$Y_i(t) = x_i + \sum_{j=1}^d \int_0^t D_{ij}(u, Y)d\tilde{w}_j(u) \ , \quad i = 1 \ldots d. \tag{4.6}$$

By Protter (1990), Theorem V.3.7, (4.6) has a unique solution which is $\tilde{F}$–adapted. By (4.4) X is a solution of (4.6), so X must be $\tilde{F}$–adapted thus $G_t \subseteq \tilde{F}_t$. The continuity of $\tilde{F}$ implies that $\tilde{G}_t \subseteq \tilde{F}_t$ also holds, thus $\tilde{G}_t = \tilde{F}_t$ and $\tilde{G}$ is continuous. From the definition of $\tilde{G}$ follows that $G_{t-} = \tilde{G}_{t-}$, which, together with the the continuity of $\tilde{G}$, imply that

$$G_{t-} = \tilde{G}_{t-} = \tilde{G}_t = G_{t+},$$

hence the proof is complete.

A consequence of the above proposition is the following

4.2 Proposition: If (M, G) is a real–valued $\tilde{P}$–local martingale with RCLL paths then there exists an investment process π such that

$$M_t = M_0 + \sum_{j=1}^{d} \int_0^t \big(\pi^*(u)\delta(u)\big)_j d\tilde{w}_j(u) \ . \tag{4.7}$$

Proof: By the representation theorem for Brownian Motion (Karatzas & Shreve, (1988), Problem 3.4.17) there exists a G–adapted d–dimensional process θ such that

$$\int_0^T \|\theta(u)\|^2 du < \infty \tag{4.8}$$

and M has the representation

$$M_t = M_0 + \sum_{j=1}^{d} \int_0^t \theta_j(u) d\tilde{w}_j(u) \ . \tag{4.9}$$

We define

$$\pi(u) = \big(\delta^{-1}(u)\big)^* \theta(u) \ , \tag{4.10}$$

which is indeed an investment process by (4.8), and (4.7) follows.

We can write the wealth–equation (3.4) in the form

$$V^{c,\pi}(t) = v + \sum_{j=1}^{d} \int_0^t \big(\pi^*(u)\delta(u)\big)_j d\tilde{w}_j(u) - \int_0^t c(u) du \ , \tag{4.11}$$

thus the wealth–process $V^{c,\pi}$ is G–adapted.

5. Optimal consumption, investment and wealth processes

The two propositions in the previous section allow us to transfer the known results for the case when full information is available to agents in the economy to our case with only partial information. In this and the next section we present results similar to the ones in Karatzas, Lehoczky, & Shreve (1987), or Karatzas, Lakner, Lehoczky, & Shreve (1991). The basic difference here is that instead of Z we have to use the G–measurable ζ, and apply Proposition 4.2.

5.1 Proposition: For a consumption process c there exists an investment process π such that $(c,\pi) \in \mathcal{A}_1(v)$ if and only if

$$\tilde{E}\Big[\int_0^T c(u)du\Big] \leq v \ . \tag{5.1}$$

If the stronger condition

$$\tilde{E}\Big[\int_0^T c(u)du\Big] = v \tag{5.2}$$

holds then the above investment process π is unique up to equivalence, and satisfies the equation

$$\tilde{E}\Big[\int_0^T c(u)du|G_t\Big] = v + \sum_{j=1}^d \int_0^t \big(\pi^*(u)\delta(u)\big)_j d\tilde{w}_j(u) \ , \tag{5.3}$$

and the corresponding wealth process is given by

$$V^{c,\pi}(t) = \tilde{E}\Big[\int_t^T c(u)du|G_t\Big] \ . \tag{5.4}$$

In particular, in this case

$$V^{c,\pi}(T) = 0, \quad \text{a.s.} \tag{5.5}$$

Proof: Let us suppose first that $(c,\pi) \in \mathcal{A}_1(v)$. The process N is a $\tilde{P}$–local martingale with respect to both filtrations F and G where N is defined by

$$N(t) = \sum_{j=1}^d \int_0^t \big(\pi^*(u)\delta(u)\big)_j d\tilde{w}_j(u) \ . \tag{5.6}$$

Equation (4.11) implies that N is bounded below by $-v$, thus by Fatou's lemma (N,F) is a $\tilde{P}$–supermartingale, hence (5.1) follows.

Next we suppose that c is a consumption process satisfying (5.1) and define the process M by

$$M(t) = \tilde{E}\Big[\int_0^T c(u)du|G_t\Big] \ . \tag{5.7}$$

After taking an appropriate modification (M, G) becomes a $\tilde{P}$–martingale with RCLL paths, and Proposition 4.2 guarantees the existence of an investment process π such that (4.7) holds. By (4.7), (5.7) and (4.11)

$$V^{c,\pi}(t) = v - \tilde{E}\left[\int_0^T c(u)du\right] + \tilde{E}\left[\int_t^T c(u)du|G_t\right] \geq 0 \ , \tag{5.8}$$

hence indeed $(c, \pi) \in \mathcal{A}_1(v)$.
Suppose now that (5.2) is true. The investment process π created above satisfies (5.3) and (5.4) by the construction. We need to establish the uniqueness of π. Suppose that π^i are investment processes such that $(c, \pi^i) \in \mathcal{A}_1(v)$ for $i = 1, 2$. Let N^i be the $\tilde{P}$–supermartingale defined in (5.6) with π substituted by π^i for $i = 1, 2$. Equations (4.11), (5.2) and the supermartingale property for N^i implies that $\tilde{E}[N^i(T)] = 0$ thus (N^i, F) is a $\tilde{P}$–martingale and $V^{c,\pi^i}(T) = 0$ for $i = 1, 2$. Now we conclude that $N^1(T) - N^2(T) = 0$ thus the martingale $N^1 - N^2$ is zero, hence its quadratic variation is also zero. Therefore

$$\int_0^T \|(\pi^1(u) - \pi^2(u))^* \delta(u)\|^2 du = 0$$

and the equivalence of π^1 and π^2 follows.

Remark: Inequality (5.1) implies that if $v = 0$ then $(c, \pi) \in \mathcal{A}_1(0)$ implies $c \equiv 0$, thus our optimization problem is trivial. Therefore, when dealing with problem (3.11)–(3.12), we lose no generality by assuming that $v > 0$. Indeed, this will be a standing assumption for the rest of this section.

5.2 Proposition: The optimal consumption/investment pair for problem (3.11)–(3.12) is unique up to equivalence.

Proof: Let us suppose that $(c^i, \pi^i) \in \mathcal{A}_1(v)$ is optimal for problem (3.11)–(3.12) for $i = 1, 2$. We consider the consumption and investment processes

$$c^3 = \frac{1}{2}(c^1 + c^2) \quad \pi^3 = \frac{1}{2}(\pi^1 + \pi^2) \ . \tag{5.9}$$

It is clear that (c^3, π^3) is in the class $\mathcal{A}_1(v)$. The function $U^-(\omega, t, \cdot)$ is convex for every $(\omega, t) \in \Omega \times [0, T]$ thus c^3 satisfies (3.12) because c^1 and c^2 satisfy (3.12). The strict concavity of $U(\omega, t, \cdot)$ now imply that c^1 and c^2 are equivalent. In order to complete the proof, by Proposition 5.2 we need to show only that $c = c^1 = c^2$ satisfies (5.2). Suppose the opposite, i.e.,

$$a = \tilde{E}\left[\int_0^T c(u)du\right] < v \ . \tag{5.10}$$

There are two cases now. If $a = 0$ then we consider the constant investment process $c^4 = \frac{v}{T}$ which is positive by (5.10), and satisfies (5.2). Proposition 5.1 guarantees the existence an investment process π^4 such that $(c^4, \pi^4) \in \mathcal{A}_1(v)$. The consumption process c^4 satisfies (3.12) since $c = 0$ satisfies it and $U^-(\omega, t, \cdot)$ is non–increasing. But this contradicts the optimality of (c^i, π^i). If $a > 0$ then we consider the consumption/investment pair

$$c^5 = \frac{v}{a} c \quad \pi^5 = \frac{v}{a} \pi^1 \tag{5.11}$$

Now $(c^5, \pi^5) \in \mathcal{A}_1(v)$, $c^5 > c^1$ and c^5 satisfies (3.12) which is a contradiction.

We introduce the $\mathcal{F} \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}((0,\infty))/\mathcal{B}([0,\infty))$ measurable mapping $I : \Omega \times [0,T] \times (0,\infty) \mapsto [0,\infty)$ by

$$I(\omega, t, y) = \inf\{c \geq 0;\ \frac{\partial}{\partial c} U(\omega, t, c) \leq y\}\ .$$

By condition (ii) of Definition 3.4, for every $t \in [0,T]$ the mapping $I(\cdot, t, \cdot)$ is $G_t \otimes \mathcal{B}((0,\infty))/\mathcal{B}([0,\infty))$ measurable. Furthermore, for every $(\omega, t) \in \Omega \times [0,T]$ the function $I(\omega, t, \cdot)$ is continuous on $(0,\infty)$, strictly decreasing on $(0, \frac{\partial}{\partial c} U(\omega, t, 0))$, and if $\frac{\partial}{\partial c} U(\omega, t, 0) < \infty$ then it is zero on $[\frac{\partial}{\partial c} U(\omega, t, 0), \infty)$. We also have the relations

$$\lim_{y \to 0+} I(\omega, t, y) = \infty \qquad \lim_{y \to \infty} I(\omega, t, y) = 0.$$

We also define the function $\mathcal{X}_1 : (0,\infty) \mapsto (0,\infty]$ by

$$\mathcal{X}_1(y) = E\left[\int_0^T \zeta(t) I(\omega, t, y\zeta(t)) dt\right]\ . \tag{5.12}$$

5.3 Lemma: Suppose that $\mathcal{X}_1(y) < \infty$ for every $y \in (0,\infty)$. Then there exists a unique number $\bar{y} \in (0,\infty)$ such that

$$\mathcal{X}_1(\bar{y}) = v\ . \tag{5.13}$$

Proof: One can see easily that $\mathcal{X}_1$ is strictly decreasing on an interval $(0, \kappa_1)$ for some $\kappa_1 \in (0,\infty]$. If $\kappa_1 < \infty$ then $\mathcal{X}_1$ is zero on $[\kappa_1, \infty)$. The Monotone Convergence Theorem and the Dominated Convergence Theorem imply that $\mathcal{X}_1$ inherits other properties of $I(\omega, t, \cdot)$ as well, i.e., it is continuous and

$$\lim_{y \to \infty} \mathcal{X}_1(y) = 0 \qquad \lim_{y \to 0+} \mathcal{X}_1(y) = \infty\ , \tag{5.14}$$

hence the statement of the lemma follows from our assumption that $v > 0$.

The following inequality will play a major role in the proof of Theorem 5.4. By the concavity of $U(\omega, t, \cdot)$, for every $\omega \in \Omega$, $t \in [0, T]$, $c \in [0, \infty)$ and $y \in (0, \infty)$ the following inequality holds:

$$U(\omega, t, I(\omega, t, y)) \geq U(\omega, t, c) + yI(\omega, t, y) - yc \; . \tag{5.15}$$

This inequality was used already in Karatzas, Lehoczky & Shreve (1987) and in Karatzas, Lakner, Lehoczky, & Shreve (1991) in the case of full information.

5.4 Theorem: Suppose that $\mathcal{X}_1(y) < \infty$ for every $y \in (0, \infty)$. Then there exists a unique optimal consumption/investment pair $(\bar{c}, \bar{\pi}) \in \mathcal{A}_1(v)$ for problem (3.11)–(3.12). The optimal consumption process is given by

$$\bar{c}(t) = I(\omega, t, \bar{y}\zeta(t)) \; , \tag{5.16}$$

where $\bar{y}$ is the constant satisfying (5.13). The optimal investment process is uniquely determined by the equation

$$\tilde{E}\left[\int_0^T \bar{c}(u)du|G_t\right] = v + \sum_{j=1}^{d} \int_0^t \left(\bar{\pi}^*(u)\delta(u)\right)_j d\tilde{w}_j(u) \; , \tag{5.17}$$

and the optimal wealth process is given by

$$V^{\bar{c},\bar{\pi}}(t) = \tilde{E}\left[\int_t^T \bar{c}(u)du|G_t\right] \; . \tag{5.18}$$

Proof: Since $I(\cdot, t, \cdot)$ is $\mathcal{F} \otimes \mathcal{B}((0, \infty))$ measurable and ζ is adapted to G, the process $\bar{c}$ is adapted to G. It satisfies (5.2) which implies (3.1) thus $\bar{c}$ is indeed a consumption process. Proposition 5.1 guarantees the existence of an investment process $\bar{\pi}$ such that $(\bar{c}, \bar{\pi}) \in \mathcal{A}_1(v)$, and (5.17), (5.18) follow from (5.3) and (5.4). Next we are going to show that $\bar{c}$ satisfies (3.12). Recall $\tilde{c}$ from (3.7). Inequality (5.15) implies that

$$U(\omega, t, \bar{c}(t)) \geq U(\omega, t, \tilde{c}) - \bar{y}\zeta(t)\tilde{c} \tag{5.19}$$

thus

$$U^-(\omega, t, \bar{c}(t)) \leq U^-(\omega, t, \tilde{c}) + \bar{y}\zeta(t)\tilde{c} \tag{5.20}$$

and (3.12) for $\bar{c}$ follows from (3.7) and the identity

$$E\left[\int_0^T \bar{y}\zeta(t)\tilde{c}dt\right] = T\bar{y}\tilde{c} < \infty \; . \tag{5.21}$$

Finally we are going to show that $(\bar{c},\bar{\pi})$ is optimal for (3.11)–(3.12). Let $(c,\pi)\in\mathcal{A}_1(v)$ an arbitrary admissible consumption/ investment pair satisfying (3.12). Inequality (5.15) implies that

$$U(\omega,t,\bar{c}(t))\geq U(\omega,t,c(t))+\bar{y}\zeta(t)\bar{c}(t)-\bar{y}\zeta(t)c(t) \tag{5.22}$$

and the optimality follows from (5.1) and from the fact that $\bar{c}$ satisfies (5.2).

6. Maximization of expected utility from terminal wealth

In this section we suppose that the agent in our economy is investing only without any consumption, and wants to maximize the expected utility of his/her wealth at time T. If π is an investment process then the corresponding wealth process is

$$V^{\pi}(t)=v+\sum_{j=1}^{d}\int_0^t\big(\pi^*(u)\delta(u)\big)_j d\tilde{w}_j(u)\ , \tag{6.1}$$

which one can derive from (4.11) by taking $c\equiv 0$.

Definition: An investment process is called admissible (for the maximization of expected utility of terminal wealth with initial capital $v\geq 0$) if the corresponding wealth process given by (6.1) is non–negative. We denote the class of admissible investment processes by $\mathcal{A}_2(v)$.

The maximization of expected utility from terminal wealth problem is the following:

$$\text{Maximize}\quad E\Big[U(\omega,T,V^{\pi}(T))\Big]\quad\text{over}\quad\pi\in\mathcal{A}_2(v) \tag{6.2}$$

subject to the additional constraint

$$E\Big[U^{-}(\omega,T,V^{\pi}(T))\Big]<\infty\ . \tag{6.3}$$

A random variable $\hat{R}$ will be called an optimal terminal wealth if there exists an investment process $\hat{\pi}\in\mathcal{A}_2(v)$ optimal for problem (6.2)–(6.3) such that $V^{\hat{\pi}}=\hat{R}$.

The steps of solving this problem are quite similar to steps of the previous section so we shall suppress some details of some proofs in this section.

6.2 Proposition: For every $\pi\in\mathcal{A}_2(v)$

$$\tilde{E}\Big[V^{\pi}(T)\Big]\leq v\ . \tag{6.4}$$

Proof: The $\tilde{P}$ local martingale N (with respect to both filtrations F and G) of (5.6) is bounded below thus it is a supermartingale, and (6.4) follows from (6.1).

Remark: This inequality implies that if $v = 0$ then the terminal wealth corresponding to any admissible investment process is zero, thus problem (6.2)–(6.3) is trivial in this case. Since both optimization problems of this paper are trivial in the case of $v = 0$, we can assume without loss of generality that v is positive. This will be a standing assumption for the rest of the paper.

6.3 Proposition: Suppose that R is a non–negative, G_T–measurable random variable such that

$$\tilde{E}[R] = v \ . \tag{6.5}$$

Then there exists a unique (up to equivalence) investment process $\pi \in \mathcal{A}_2(v)$ such that

$$V^{\pi}(T) = R \tag{6.6}$$

and the corresponding wealth process is

$$V^{\pi}(t) = \tilde{E}\big[R|G_t\big] \ . \tag{6.7}$$

Proof: (sketch) The existence follows from Proposition 4.2 applied to the $(\tilde{P}, G)$–martingale $M(t) = \tilde{E}[R|G_t]$. The investment process π created this way satisfies (6.7). Uniqueness can be shown similarly to the uniqueness part of the proof of Proposition 5.1 and we skip the details.

6.4 Proposition: The optimal investment process for problem (6.2)–(6.3) is unique up to equivalence.

Proof: Let $\pi^i \in \mathcal{A}_2(v)$ optimal for problem (6.2)–(6.3) for $i = 1, 2$. Then $\pi^3 = \frac{1}{2}(\pi^1 + \pi^2)$is also in $\mathcal{A}_2(v)$, and satisfies

$$V^{\pi^3}(T) = \frac{1}{2}\Big(V^{\pi^1}(T) + V^{\pi^2}(T)\Big) \ . \tag{6.8}$$

By the convexity of $U^-(\omega, t, \cdot)$, the investment process π^3 satisfies (6.3) since π^1 and π^2 satisfy it. Strict concavity of $U(\omega, t, \cdot)$ implies that

$$V^{\pi^1}(T) = V^{\pi^2}(T), \quad \text{a.s.} \tag{6.9}$$

By Proposition 6.3 all we have to show that $\tilde{E}\big[v^{\pi^1}(T)\big] = v$. This can be done similarly to the last part of the proof of Proposition 5.2 and we suppress the details.

We introduce the function $\mathcal{X}_2 : (0,\infty) \mapsto (0,\infty]$

$$\mathcal{X}_2(y) = E\big[\zeta(T) I(\omega, T, y\zeta(T)\big) \ . \tag{6.10}$$

6.5 Lemma: If $\mathcal{X}_2(y) < \infty$ for every $y \in (0,\infty)$ then there exists a unique constant $\hat{y} \in (0,\infty)$ such that

$$\mathcal{X}_2(\hat{y}) = v \ . \tag{6.11}$$

Proof: Just like in the proof of Lemma 5.3 one can see that there exists a constant $\kappa_2 \in (0,\infty]$ such that $\mathcal{X}_2$ is strictly decreasing on $(0,\kappa_2)$, and vanishes on (κ_2,∞) if $\kappa_2 < \infty$. Furthermore, $\mathcal{X}$ is continuous on its domain and

$$\lim_{y\to 0} \mathcal{X}_2(y) = \infty \qquad \lim_{y\to\infty} \mathcal{X}_2(y) = 0,$$

and the statement of the lemma follows from our assumption that $v > 0$.

6.6 Theorem: Suppose that $\mathcal{X}_2(y) < \infty$ for every $y \in (0,\infty)$. Then the unique optimal terminal wealth is given by

$$\hat{R} = I(\omega, T, \hat{y}\zeta(T)) \ , \tag{6.12}$$

where $\hat{y}$ is the constant from (6.11). The unique optimal investment process $\hat{\pi}$ and the corresponding wealth process $V^{\hat{\pi}}$ satisfy the equation

$$\tilde{E}\big[\hat{R}|G_t\big] = V^{\hat{\pi}}(t) = v + \sum_{j=1}^{d} \int_0^t \big(\hat{\pi}^*(u)\delta(u)\big)_j d\tilde{w}_j(u) \ . \tag{6.13}$$

Proof: Let $\hat{R}$ be defined by (6.12). We have the identity

$$\tilde{E}\big[\hat{R}\big] = \mathcal{X}_2(\hat{y}) = v \tag{6.14}$$

thus Proposition 6.3 and the G_T–measurability of $\hat{R}$ guarantees the existence of an investment process $\hat{\pi} \in \mathcal{A}_2(v)$ such that

$$V^{\hat{\pi}}(T) = \hat{R} \tag{6.15}$$

and (6.13) holds. Similarly to (5.20) one can show that

$$U^-(\omega, T, \hat{R}) \leq U^-(\omega, T, \tilde{c}) + \hat{y}\zeta(T)\tilde{c} \tag{6.16}$$

where $\tilde{c}$ is the constant from (3.8),and (6.16) implies that $\hat{\pi}$ satisfies (6.3).
Let $\pi \in \mathcal{A}_2(v)$ an arbitrary admissible investment process satisfying (6.3). Inequality (5.15) implies that

$$U(\omega, t, \hat{R}) \geq U(\omega, t, V^\pi(T)) + \hat{y}\zeta(T)\hat{R} - \hat{y}\zeta(T)V^\pi(T) \ , \tag{6.17}$$

and the optimality of $\hat{\pi}$ follows from (6.14) and (6.4).

7. A Bayesian problem; explicit computation of ζ.

In this section we shall suppose that the stock price process X is the unique solution of the system of stochastic differential equations

$$X_i(t) = x_i + \int_0^t \mu_i X_i(u)du + \sum_{j=1}^d \int_0^t \sigma_{ij} X_i(u)dw_j(u) \ , \tag{7.1}$$

where $\mu = (\mu_1, \dots, \mu_d)^*$ is a d–dimensional, F_0–measurable random variable with a known distribution $\check{P}_\mu$, and $\sigma = (\sigma_{ij})_{i,j=1\dots,d}$ is a fixed $d \times d$ matrix with rank d. This is a special case of example 2.5 and it can be regarded as a Bayesian problem. Our objective in this section is to give an explicit representation for ζ which plays an important role in formula (5.16) for the optimal consumption process and in (6.12) for the optimal terminal wealth.

Under the above conditions $\tilde{w}$ and Z are given by

$$\tilde{w}(t) = w(t) + \sigma^{-1}\mu t \tag{7.2}$$

$$Z(t) = \exp\{-(\sigma^{-1}\mu)^* w(t) - \frac{1}{2}\|\sigma^{-1}\mu\|^2 t\} \ . \tag{7.3}$$

In order to give an explicit formula for $\zeta(t) = E[Z(t)|G_t]$ we need the conditional distribution of μ given G_t under the probability measure P (the posterior distribution). We introduce the mapping $H : [0,T] \times \Re^d \times \Re^d \mapsto (0, \infty)$

$$H(t, x, b) = \exp\{(\sigma^{-1}b)^* x - \frac{1}{2}\|\sigma^{-1}b\|^2 t\} \ . \tag{7.4}$$

We will show in the Appendix (Lemma 10.1) that

$$\check{P}_\mu(B|G_t) \triangleq \frac{\int_B H(t, \tilde{w}(t), b)\check{P}_\mu(db)}{\int_{\Re^d} H(t, \tilde{w}(t), b)\check{P}_\mu(db)} \ , \quad B \in \mathcal{B}(\Re^d), \tag{7.5}$$

determines a regular conditional distribution for μ given G_t under P. This implies that

$$E\big[\mu_i|G_t\big] = \frac{\int_{\Re^d} H(t,\tilde{w}(t),b)b_i\check{P}_\mu(db)}{\int_{\Re^d} H(t,\tilde{w}(t),b)\check{P}_\mu(db)} \quad , i=1,\dots,d, \tag{7.6}$$

whenever $E|\mu_i| < \infty$ for every $i = 1,\dots,d$.

7.1 Proposition: The process ζ is given by the following explicit representation:

$$\zeta(t) = \left(\int_{\Re^d} H(t,\tilde{w}(t),b)\check{P}_\mu(db)\right)^{-1} , \tag{7.7}$$

where $\tilde{w}$ is given by

$$\tilde{w}_j(t) = \sum_{i=1}^{d}\int_0^t (\sigma^{-1})_{ij} X_j^{-1}(u) dX_j(u) \ , \quad j=1,\dots,d.$$

Proof: We write $Z(t)$ in the form

$$Z(t) = \big(H(t,\tilde{w}(t),\mu)\big)^{-1} . \tag{7.8}$$

It is easy to see that for $Q, B \in \mathcal{B}(\Re^d)$

$$P(\tilde{w}(t)\in Q, \mu\in B|G_t) = 1_{\{\tilde{w}(t)\in Q\}}\check{P}(B|G_t) , \tag{7.9}$$

which determines a regular conditional distribution for $(\tilde{w}(t),\mu)$ given G_t , by extending the right–hand side of (7.9) from the class of measurable rectangles of $\mathcal{B}(\Re^d\times\Re^d)$ to the entire $\mathcal{B}(\Re^d\times\Re^d)$. The conditional expectation of $Z(t)$ given G_t can be computed by integrating $\big(H(t,\cdot,\cdot)\big)^{-1}$ with respect to the conditional distribution for $(\tilde{w}(t),\mu)$ given G_t. This gives exactly the right–hand side of (7.7), using (7.9) and (7.5).

Remark: In order to avoid further unnecessary notations we agree that in the rest of the paper the notation $E[\mu|G_t]$ stands for the measurable version of this conditional expectation represented by the right–hand side of (7.6). Therefore, $E[\mu|G_t]$ is a measurable, G–adapted, d–dimensional process, provided that $E|\mu_i| < \infty$ for every $i \le d$.

The next proposition gives two additional representations for ζ, and a characterization of ζ^{-1} as a unique solution of a linear stochastic differential equation.

7.2 Proposition: Suppose that for every constant $K_1 > 0$

$$E\Big[\|\mu\|^2\exp\{K_1\|\mu\|\}\Big] < \infty . \tag{7.10}$$

Then we have the following representation:

$$\zeta^{-1}(t) = 1 + \sum_{j=1}^{d} \int_0^t \int_{\Re^d} H(t, \tilde{w}(u), b)(\sigma^{-1}b)_j \check{P}_\mu(db) d\tilde{w}_j(u) \ . \tag{7.11}$$

Furthermore, ζ^{-1} satisfies the following linear stochastic differential equation

$$\zeta^{-1}(t) = 1 + \sum_{j=1}^{d} \int_0^t \zeta^{-1}(u)\big(\sigma^{-1}E[\mu|G_u]\big)_j d\tilde{w}_j(u) \tag{7.12}$$

and, in addition to (7.11), ζ also has the following representation:

$$\zeta(t) = \exp\{-\sum_{j=1}^{d} \int_0^t \big(\sigma^{-1}E[\mu|G_u]\big)_j d\tilde{w}_j(u) + \frac{1}{2}\int_0^t \|\sigma^{-1}E[\mu|G_u]\|^2 du\} \ . \tag{7.13}$$

Proof: (7.12) is an immediate consequence of (7.11) using (7.6),(7.7). Representation (7.13) follows from (7.12), thus we have to show (7.11) only. We introduce the mapping $J : [0,T] \times \Re^d \mapsto \Re$

$$J(t,x) = E[H(t,x,\mu)] \ . \tag{7.14}$$

$J(t,x)$ is indeed finite because

$$E[H(t,x,\mu)] \le E\Big[\exp\big\{(\sigma^{-1}\mu)^* x\big\}\Big] \le$$

$$\exp\big\{\|\sigma^{-1}\|\|x\|\big\} + E[\|\mu\|^2 \exp\big\{\|\sigma^{-1}\|\|\mu\|\|x\|\big\}] < \infty \ . \tag{7.15}$$

By Lemma 10.2 of the Appendix J is in $C^{1,2}$, (meaning that the partial derivatives $\frac{\partial}{\partial t}J$ and $(\frac{\partial^2}{\partial x_i \partial x_j}J;\ 1 \le i,j \le d)$ exist and are continuous on $(0,T) \times \Re^d$, and J is continuous on $[0,T] \times \Re^d$), and the expectation and differentiation are exchangeable, i.e.,

$$\frac{\partial}{\partial x_i}J(t,x) = E\Big[\frac{\partial}{\partial x_i}H(t,x,\mu)\Big] \quad i = 1,\dots,d \tag{7.16}$$

$$\frac{\partial^2}{\partial x_i \partial x_j}J(t,x) = E\Big[\frac{\partial^2}{\partial x_i \partial x_j}H(t,x,\mu)\Big] \quad i,j = 1,\dots,d \tag{7.17}$$

and

$$\frac{\partial}{\partial t}J(t,x) = E\Big[\frac{\partial}{\partial t}H(t,x,\mu)\Big] \quad t \in [0,T]. \tag{7.18}$$

By (7.7) we can write $\zeta^{-1}(t)$ in the following form:

$$\zeta^{-1}(t) = J(t, \tilde{w}(t)) \ . \tag{7.19}$$

One can apply Ito's rule to (7.19), use (7.17), (7.18) and the identity

$$\frac{\partial}{\partial t} H + \frac{1}{2} \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} H = 0 \ , \tag{7.20}$$

and (7.11) follows. Now our proof is complete.

We have the explicit formula (7.7) for $\zeta(t)$ so we can substitute it to (5.16) and (6.12) and arrive at the following formulae for the optimal consumption process and the optimal terminal wealth under the assumption that $\mathcal{X}_i(y) < \infty$ for $i = 1, 2$ and $y \in (0, \infty)$:

$$\bar{c}(t) = I\Big(\omega, t, \bar{y}\big(\int_{\Re^d} H(t, \tilde{w}(t), b) \check{P}_\mu(db)\big)^{-1}\Big) \tag{7.21}$$

and

$$\hat{R} = I\Big(\omega, T, \hat{y}\big(\int_{\Re^d} H(T, \tilde{w}(T), b) \check{P}_\mu(db)\big)^{-1}\Big) \ , \tag{7.22}$$

where $\bar{y}$ and $\hat{y}$ are the constants from (5.13) and (6.14).

We note that condition (7.10) is satisfied if, for example, μ follows a d–dimensional normal distribution.

7.3 Remark: It appears from (7.12) and (7.13) that ζ depends on μ only through the conditional expectation $E[\mu|G_u]$. Therefore, we can *formally* derive (7.11) and (7.12) in the following way: write down the corresponding equations for deterministic μ, and then substitute μ by the conditional expectation of μ given G_u.

8. The logarithmic utility function

In this section we shall still assume that X is the (unique) solution of the system (7.1) where μ is a d–dimensional, F_0–measurable random variable with distribution $\check{P}_\mu$ and σ is a fixed, $d \times d$ matrix with full rank.Additionally, we assume (7.10) and that the utility function is given by

$$U(\omega, t, c) = \log c \ , \tag{8.1}$$

in which case

$$I(t, \omega, y) = \frac{1}{y} \ . \tag{8.2}$$

Under these conditions we shall give explicit representations for the optimal investment processes $\bar{\pi}$ and for $\hat{\pi}$ and the corresponding wealth processes $V^{\bar{c},\bar{\pi}}$ and $V^{\hat{\pi}}$. It is easy to see that $\mathcal{X}_1(y) < \infty$ and $\mathcal{X}_2(y) < \infty$ for every $y \in (0,\infty)$. First we shall consider the optimization problem (3.11)–(3.12). By (7.20) the optimal consumption process is

$$\bar{c}(t) = \frac{v}{T}\zeta^{-1}(t) = \frac{v}{T}\int_{\Re^d} H(t,\tilde{w}(t),b)\check{P}_\mu(db) \; . \tag{8.3}$$

8.1 Proposition: The optimal investment process for the problem (3.11)– (3.12) is given by

$$\bar{\pi}_i(t) = \frac{v}{T}(T-t)X_i^{-1}(t)\sum_{j=1}^d ((\sigma\sigma^*)^{-1})_{ij}\int_{\Re^d} H(\omega,\tilde{w}(t),b)b_j\check{P}_\mu(db) \; , \tag{8.4}$$

and the corresponding wealth process is

$$V^{\bar{c},\bar{\pi}}(t) = \frac{v}{T}(T-t)\zeta^{-1}(t) = \frac{v}{T}(T-t)\int_{\Re^d} H(t,\tilde{w}(t),b)\check{P}_\mu(db) \; . \tag{8.5}$$

Furthermore, the optimal consumption and investment processes can be written in the following form:

$$\bar{c}(t) = \frac{1}{T-t}V^{\bar{c},\bar{\pi}}(t) \tag{8.6}$$

$$\bar{\pi}_i(t) = V^{\bar{c},\bar{\pi}}(t)X_i^{-1}(t)\sum_{j=1}^d ((\sigma\sigma^*)^{-1})_{ij}E[\mu_j|G_t] \; . \tag{8.7}$$

Proof: By (8.3) and (5.17) we have the identity

$$v + \sum_{j=1}^d \int_0^t \Big(\sum_{i=1}^d \bar{\pi}_i(u)\sigma_{ij}X_i(u)\Big)d\tilde{w}_j(u) = \frac{v}{T}\tilde{E}\Big[\int_0^T \zeta^{-1}(u)du|G_t\Big] \; . \tag{8.8}$$

The process ζ^{-1} is a martingale with respect to the filtration G and probability measure $\tilde{P}$ hence the right–hand side of (8.8) can be written as

$$\frac{v}{T}\Big(\int_0^t \zeta^{-1}(u)du + (T-t)\zeta^{-1}(t)\Big) = v + \frac{v}{T}\int_0^t (T-u)d\zeta^{-1}(u) \; , \tag{8.9}$$

where the last identity follows from Ito's rule applied to the process $(T-t)\zeta^{-1}(t)$. Using (7.12) one can write the right–hand side of (8.9) in the form

$$v + \frac{v}{T}\sum_{j=1}^d \int_0^t (T-u)\zeta^{-1}(u)\big(\sigma^{-1}E[\mu|G_u]\big)_j d\tilde{w}_j(u) \; . \tag{8.10}$$

The left–hand side of (8.8) and (8.10) are equal, therefore

$$\sum_{i=1}^{d} \bar{\pi}_i(u)\sigma_{ij}X_i(u) = (T-u)\frac{v}{T}\zeta^{-1}(u)\big(\sigma^{-1}E[\mu|G_u]\big)_j \quad j=1,\dots,d. \tag{8.11}$$

We can write this in the form

$$\sigma^*\nu(u)\bar{\pi}(u) = (T-u)\frac{v}{T}\zeta^{-1}(u)\sigma^{-1}E[\mu|G_u]\ , \tag{8.12}$$

where the $d\times d$ matrix $\nu(u)=\nu(\omega,u)$ is defined as $\nu_{ii}(u)=X_i(u)$ for $i=1,\dots,d$ and $\nu_{ij}(u)=0$ for $i,j=1,\dots,d, i\neq j$. Substituting (7.6) to (8.12) yields (8.4). Identity (8.5) follows from (5.18) and (8.3). Equations (8.6) and (8.7) follow from (8.5), (8.3) and (8.12).

Next we are going to give an explicit formula for the optimal investment process and the corresponding wealth process for problem (6.2)–(6.3). The optimal terminal wealth of (6.12) has the simple form

$$\hat{R} = v\zeta^{-1}(T)\ , \tag{8.13}$$

and (6.13) implies that the wealth process corresponding to the optimal investment process has the form

$$V^{\hat{\pi}}(t) = v\zeta^{-1}(t)\ . \tag{8.14}$$

We still have to compute the optimal investment process.

8.2 Proposition: The optimal investment process for problem (6.2)–(6.3) is

$$\hat{\pi}_i(t) = vX_i^{-1}(t)\sum_{j=1}^{d}((\sigma\sigma^*)^{-1})_{ij}\int_{\Re^d} H(t,\tilde{w}(t),b)b_j\check{P}_\mu(db)\ , \tag{8.15}$$

which can be written in the form

$$\hat{\pi}_i(t) = V^{\hat{\pi}}(t)X_i^{-1}(t)\sum_{j=1}^{d}((\sigma\sigma^*)^{-1})_{ij}E[\mu_j|G_t]\ . \tag{8.16}$$

Proof: (6.13),(8.14) and (7.12) imply (8.16). Identity (8.15) follows from (8.16) and (7.6).

We can make a similar remark to 7.3. In (8.7) and (8.16) μ enters only through its conditional expectation given G_t. Therefore, one can *formally* derive (8.7) and (8.16) by writing down the corresponding formulae with deterministic μ, i.e., when $\check{P}$ is degenerate, and then substitute μ by its conditional expectation given G_t.

Finally, we shall compute $\bar{c}$, $\bar{\pi}$,$\hat{\pi}$, and the corresponding wealth processes for the even more special case when $d = 1$ and μ follows a one–dimensional normal distribution with mean f and variance $l^2 > 0$. This is exactly the model studied by Browne and Whitt (1994). Condition (7.10) holds in this case and from (7.7) one can derive the formula (dropping the indices since $d = 1$)

$$\zeta^{-1}(t) = \frac{|\sigma|}{\sqrt{l^2 t + \sigma^2}} \exp\left\{ -\frac{f^2}{2l^2} + \frac{(l^2 \tilde{w}(t) + f\sigma)^2}{2l^2(l^2 t + \sigma^2)} \right\} \tag{8.17}$$

The optimal wealth process for problem (3.11)–(3.12), $V^{\bar{c},\bar{\pi}}$ is determined by (8.17) and (8.5), and the optimal consumption process is given by (8.3) or (8.6). The posterior expectation of μ is

$$E[\mu | G_t] = \frac{\sigma l^2 \tilde{w}(t) + f\sigma^2}{l^2 t + \sigma^2} \ . \tag{8.18}$$

One can substitute this to (8.7) and (8.16), and derive

$$\bar{\pi}(t) = X^{-1}(t) V^{\bar{c},\bar{\pi}}(t) \frac{l^2 \tilde{w}(t) + f\sigma}{\sigma(l^2 t + \sigma^2)} \tag{8.19}$$

$$\hat{\pi}(t) = X^{-1}(t) V^{\hat{\pi}}(t) \frac{l^2 \tilde{w}(t) + f\sigma}{\sigma(l^2 t + \sigma^2)} \ , \tag{8.20}$$

where $V^{\hat{\pi}}$ is determined by (8.14) and (8.17).

9. Power utility function

Next we shall give another example when our assumptions for Theorems 5.4 and 6.6 hold, i.e., $\mathcal{X}_i(y) < \infty$ for $i = 1, 2$ and $y \in (0, \infty)$. In particular, in this section we shall suppose that X satisfies (7.1) with the same assumptions on μ and σ, and the utility function is given by

$$U(\omega, t, c) = \gamma(\omega, t) c^{\eta} \tag{9.1}$$

(as in (3.10)) with some fixed $\eta \in (0, 1)$, where the discount process γ is G–adapted and for some constant $K_2 > 0$

$$0 < \gamma(\omega, t) < K_2 \ , \quad (\omega, t) \in \Omega \times [0, T]. \tag{9.2}$$

Furthermore, in this section we suppose that μ is bounded, i.e., there exists a constant $K_3 > 0$ such that

$$\|\mu\|^2 \le K_3 \quad \text{a.s.} \tag{9.3}$$

If the utility function is given by (9.1) then

$$I(\omega, t, y) = \left(\frac{\eta\gamma(\omega, t)}{y}\right)^{\frac{1}{1-\eta}} . \tag{9.4}$$

9.1 Proposition: Under the above assumptions for every $y \in (0, \infty)$

$$\mathcal{X}_1(y) < \infty \quad \text{and} \quad \mathcal{X}_2(y) < \infty . \tag{9.5}$$

Proof: Using (7.7) and Jensen's inequality, with

$$K_4 = \left(\frac{\eta K_2}{y}\right)^{\frac{1}{1-\eta}}$$

we can see the following:

$$\mathcal{X}_1(y) \le K_4 \int_0^T \tilde{E}\Big[\zeta^{\frac{1}{\eta-1}}(t)\Big] dt = K_4 \int_0^T \tilde{E}\Big[\Big(\int_{\Re^d} H(t, \tilde{w}(t), b)\check{P}_\mu(db)\Big)^{\frac{1}{1-\eta}}\Big] dt \le$$

$$K_4 \int_0^T \tilde{E}\Big[\int_{\Re^d} \exp\{\frac{1}{1-\eta}\}(\sigma^{-1}b)^*\tilde{w}(t) - \frac{1}{2(1-\eta)}\|\sigma^{-1}b\|^2 t\}\check{P}_\mu(db)\Big] dt$$

$$= K_4 \int_0^T \int_{\Re^d} \exp\{\frac{\eta}{2(1-\eta)^2}\|\sigma^{-1}b\|^2 t\}\check{P}_\mu(db) dt \le$$

$$K_4 \int_0^T \exp\{\frac{\eta}{2(1-\eta)^2}\|\sigma^{-1}\|^2 K_3 t\} dt < \infty.$$

Going from the second to the third line we used the identity

$$\tilde{E}\Big[\exp\{\frac{1}{1-\eta}(\sigma^{-1}b)^*\tilde{w}(t) - \frac{1}{2(1-\eta)^2}\|\sigma^{-1}b\|^2 t\}\Big] = 1 .$$

One can show similarly that

$$\mathcal{X}_2(y) \le K_4 \exp\{\frac{\eta}{2(1-\eta)^2}\|\sigma^{-1}\|^2 K_3 T\} < \infty.$$

According to Theorems 5.4 and 6.6 the optimal consumption process for problem (3.11)–(3.12) and the optimal terminal wealth for problem (6.2)–6.3) is

$$\bar{c}(t) = K_5 \left(\frac{\zeta(t)}{\gamma(t)} \right)^{\frac{1}{\eta-1}}$$

and

$$\hat{R} = K_6 \left(\frac{\zeta(T)}{\gamma(T)} \right)^{\frac{1}{\eta-1}},$$

where K_5 and K_6 are constants selected to satisfy $\tilde{E}\big[\int_0^T \bar{c}(t)dt\big] = v$ and $\tilde{E}[\hat{R}] = v$.

10. Appendix

10.1 Lemma: Formula (7.5) determines a regular conditional distribution for μ given G_t under P.

Proof: First we are going to compute the conditional distribution of $(\tilde{w}(t_1), \ldots, \tilde{w}(t_n))$ given G_t (under P) for $0 = t_0 \le t_1 < t_2 < \ldots < t_n = t$. Let $C \in \mathcal{B}(\Re^{d\times n})$. With the notations

$$K_7 = \left[(2\pi)^n \prod_{i=1}^{n} (t_i - t_{i-1}) \right]^{-\frac{d}{2}} \quad \text{and} \quad x_0 = 0 \in \Re^d, \tag{10.1}$$

using (7.2) and the independence of μ and w under P, we can compute:

$$P((\tilde{w}(t_1), \ldots \tilde{w}(t_n)) \in C|\mu) =$$

$$K_7 \int_C \exp\{-\sum_{i=1}^{n} \frac{1}{2(t_i - t_{i-1})} \|x_i - x_{i-1} - \sigma^{-1}\mu(t_i - t_{i-1})\|^2\} d(x_1 \times \ldots \times x_n) =$$

$$K_7 \int_C \exp\{-\sum_{i=1}^{n} \frac{1}{2(t_i - t_{i-1})} \|x_i - x_{i-1}\|^2\} H(t, x_n, \mu) d(x_1 \times \ldots \times x_n)\ . \tag{10.2}$$

To show that (7.5) indeed determines a conditional distribution for μ given G_t we need to prove that for every $L \in G_t$ and $B \in \mathcal{B}(\Re^d)$

$$P(\{\mu \in B\} \cap L) = E\left[\frac{\int_B H(t, \tilde{w}(t), b)\check{P}_\mu(db)}{\int_{\Re^d} H(t, \tilde{w}(t), b)\check{P}_\mu(db)} 1_L \right]\ . \tag{10.3}$$

However, by Proposition 4.1, it suffices to show (10.3) for events L of the form

$$L = \{(\tilde{w}(t_1), \ldots, \tilde{w}(t_n)) \in C\}\ , \tag{10.4}$$

where $t_1, \ldots, t_n$ and C are the same as above. Putting (10.4) and (10.2) together we get

$$P(\{\mu \in B\} \cap L) =$$

$$\int_C K_7 \exp\{-\sum_{i=1}^{n} \frac{1}{2(t_i - t_{i-1})} \|x_i - x_{i-1}\|^2\} \int_B H(t, x_n, b)\check{P}_\mu(db) d(x_1 \times \ldots \times x_n) =$$

$$\int_C K_7 \exp\{-\sum_{i=1}^{n} \frac{1}{2(t_i - t_{i-1})} \|x_i - x_{i-1}\|^2\} \int_{\Re^d} H(t, x_n, b)\check{P}_\mu(db) \times$$

$$\frac{\int_B H(t, x_n, b)\check{P}_\mu(db)}{\int_{\Re^d} H(t, x_n, b)\check{P}_\mu(db)} d(x_1 \times \ldots \times x_n) = E\left[\frac{\int_B H(t, \tilde{w}(t), b)\check{P}_\mu(db)}{\int_{\Re^d} H(t, \tilde{w}(t), b)\check{P}_\mu(db)} 1_{\{(\tilde{w}(t_1), \ldots, \tilde{w}(t_n)) \in C\}}\right],$$

so (10.3) is indeed true. In order to show that (7.5) gives a regular version of the conditional probabilities, it suffices to prove that the denominator is finite for every $\omega \in \Omega$. This can be seen from the following computation:

$$H(t, \tilde{w}(t), b) = \exp\left\{-\frac{t}{2}\big(\|\sigma^{-1}b - \frac{\tilde{w}(t)}{t}\|^2 - \frac{\|\tilde{w}(t)\|^2}{t^2}\big)\right\} \le \exp\left\{\frac{\|\tilde{w}(t)\|)^2}{2t}\right\}, \qquad (10.5)$$

which completes the proof of the lemma.

10.2 Lemma: Let $H : [0,T] \times \Re^d \times \Re^d \mapsto \Re$ and $J : [0,T] \times \Re^d \mapsto \Re$ be defined as in (7.4) and (7.14), and suppose that (7.10) holds for every $K_1 \in (0, \infty)$. Then the function J is continuous on $[0,T] \times \Re^d$, the partial derivatives $\frac{\partial}{\partial t} J$, and $(\frac{\partial^2}{\partial x_i \partial x_j} J,\ 1 \le i, j \le d)$ exist and are continuous on $(0,T) \times \Re^d$ and (7.16), (7.17), and (7.18) hold.

Proof: First we are going to show (7.16). Let $(t, x) \in (0,T) \times \Re^d$ be arbitrary and e_i be the i–th basic unit vector of $\Re^d$. The Intermediate Value Theorem guarantees that for every $h \in \Re$, $0 < |h| < 1$ there exists a vector $\theta(\omega, h) \in \Re^d$ between x and $x + he_i$ (the comparison of vectors is understood componentwise) such that

$$\frac{1}{h}\Big(H(t, x + he_i, \mu) - H(t, x, \mu)\Big) = \frac{\partial}{\partial x_i} H(t, \theta(\omega, h), \mu).$$

The mapping $\theta : \Omega \times \big((-1,1) \setminus \{0\}\big) \mapsto \Re^d$ may not be measurable, but this does not bother us. However, the expression on the right–hand side of the above equation is measurable as a function of (ω, h) because the left–hand side is measurable, and we also have

$$\frac{1}{h}\Big(J(t, x + he_i) - J(t, x)\Big) = E\left[\frac{\partial}{\partial x_i} H(t, \theta(\omega, h), \mu)\right]. \qquad (10.6)$$

We would like to apply the Dominated Convergence Theorem, so we bound the quantity after the expectation sign on the right–hand side of (10.6) in the following way:

$$|\frac{\partial}{\partial x_i}H(t,\theta(\omega,h),\mu)| = |(\sigma^{-1}\mu)_i|H(t,\theta(\omega,h),\mu) \le |(\sigma^{-1}\mu)_i|\exp\{\|\sigma^{-1}\|(\|x\|+1)\|\mu\|\} \le$$

$$\|\sigma^{-1}\|\|\mu\|\exp\{\|\sigma^{-1}\|(\|x\|+1)\|\mu\|\} \le$$

$$\|\sigma^{-1}\|\|\mu\|^2\exp\{\|\sigma^{-1}\|(\|x\|+1)\|\mu\|\} + \|\sigma^{-1}\|\exp\{\|\sigma^{-1}\|(\|x\|+1)\}$$

which has a finite expectation (under E) by (7.10). Formula (7.16) follows from (10.6) and the Dominated Convergence Theorem by taking limits as $h \to 0$.

To show (7.17) we proceed similarly. (7.16) and the Intermediate Value Theorem implies that for $0 < |h| < 1$ and $1 \le i,j \le d$ there exists a vector $\theta(\omega,h)$ between x and $x + he_j$ such that

$$\frac{1}{h}\left(\frac{\partial}{\partial x_i}J(t,x+he_j) - \frac{\partial}{\partial x_i}J(t,x)\right) = E\left[\frac{\partial^2}{\partial x_j \partial x_i}H(t,\theta(\omega,h),\mu)\right]. \qquad (10.7)$$

We need a bound for the expression after the expectation sign in the right–hand side of (10.7).

$$|\frac{\partial^2}{\partial x_j \partial x_i}H(t,\theta(\omega,h),\mu)| \le |(\sigma^{-1}\mu)_i(\sigma^{-1}\mu)_j|\exp\{\|\sigma^{-1}\|(\|x\|+1)\|\mu\|\} \le$$

$$\|(\sigma^{-1}\|^2\|\mu\|^2\exp\{\|\sigma^{-1}\|(\|x\|+1)\|\mu\|\} \qquad (10.8)$$

which has finite expectation by (7.10). Formula (7.17) follows from the Dominated Convergence Theorem and (10.7) by taking limits as $h \to 0$.

Next we are going to show (7.18). By the Intermediate Value Theorem for every $h \in (-1,1)$, $h \neq 0$ there exists a number $\theta(\omega,h)$ between t and $t+h$ such that

$$\frac{1}{h}\left(J(t+h,x) - J(t,x)\right) = E\left[\frac{\partial}{\partial t}H(\theta(\omega,h),x,\mu)\right]. \qquad (10.9)$$

One can see easily that

$$|\frac{\partial}{\partial t}H(\theta(h),x,\mu)| \le \frac{1}{2}\|(\sigma^{-1}\|^2\|\mu\|^2\exp\{\|\sigma^{-1}\|\|x\|\|\mu\|\} \qquad (10.10)$$

which has finite expectation, so the Dominated Convergence Theorem and (10.9) yield (7.17).

The continuity of the partial derivatives $\frac{\partial}{\partial t}J$, and $(\frac{\partial^2}{\partial x_i \partial x_j}J,\ 1 \le i,j \le d)$ on $(0,T) \times \Re^d$

follow from the continuity of the corresponding partial derivatives of H, (10.8), (10.10), and the Dominated Convergence Theorem. Finally, the continuity of J on $[0,T] \times \Re^d$ follows from (7.15) and the Dominated Convergence Theorem.

10.3 Lemma: The process δ is adapted to the filtration G.

Proof: Let C be an arbitrary, $d \times d$–dimensional Borel–set. We need to show that

$$\{\omega \in \Omega : \; D(t, X(\omega)) \in C\} \in G_t; \quad t \in [0,T] \tag{10.11}$$

(recall that $\delta(t,\omega) = D(t, X(\omega))$). Since $D(t,\cdot)$ is B_t–measurable (please recall the definition of B_t from the paragraph preceding Assumption 2.1), the event on the left–hand side of (10.11) can be written as $X^{-1}(\vartheta)$ for some $\vartheta \in B_t$, where $X^{-1}(\vartheta) = \{\omega \in \Omega : \; X(\omega) \in \vartheta\}$. Thus it suffices to show that

$$\{X^{-1}(\vartheta); \; \vartheta \in B_t\} = G_t \; . \tag{10.12}$$

First we are going to show that (10.12) holds with the equality sign replaced by the inclusion $\supseteq$. The σ–field G_t is generated by events of the form

$$L = \{\omega \in \Omega : \; X(u_1) \in C_1, \ldots, X(u_n) \in C_n\}, \tag{10.13}$$

where n runs through the set of natural numbers, $0 \le u_1 < u_2 < \ldots < u_n \le t$, and C_i is a d–dimensional Borel–set for every $i = 1, 2, \ldots, n$. However, L is included in the event on the left–hand side of (10.12) since

$$L = X^{-1}(\phi), \tag{10.14}$$

where $\phi \in B_t$ is the cylinder set

$$\phi = \{f \in C^d[0,T] : \; f(u_1) \in C_1, \ldots, f(u_n) \in C_n\}. \tag{10.15}$$

Next we show that (10.12) also holds with the equality sign replaced by the inclusion $\subseteq$. We define the σ–field

$$\mathcal{K}_t = \{\varphi \subseteq C^d[0,T] : \; X^{-1}(\varphi) \in G_t\}. \tag{10.16}$$

Since

$$\{X^{-1}(\varphi) : \; \varphi \in \mathcal{K}_t\} \subseteq G_t, \tag{10.17}$$

it suffices to show that $B_t \subseteq \mathcal{K}_t$. But this is clear since B_t is generated by sets of the form ϕ of (10.15), and all these sets are in $\mathcal{K}_t$ by (10.14) and (10.16).

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