

An Overview of Pricing Models for Revenue Management

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Abstract

In this paper we examine the research and results of dynamic pricing policies and their relation to Revenue Management. The survey is based on a generic Revenue Management problem in which a perishable and non-renewable set of resources satisfy stochastic price-sensitive demand processes over a finite period of time. In this class of problems, the owner (or the seller) of these resources uses them to produce and offer a menu of final products to the end customers. Within this context, we formulate the stochastic control problem of capacity that the seller faces: how to dynamically set the menu and the quantity of products and their corresponding prices in order to maximize the total revenue over the selling horizon.

1 Introduction

The aim of this paper is to review the growing literature on Dynamic Pricing Policies and its connection to Revenue Management. In general terms, the Revenue Management model that we investigate considers the problem faced by a seller who owns a fixed and perishable set of resources that are sold to a price sensitive population of buyers. In this framework where capacity is fixed, the seller is mainly interested in finding an optimal pricing strategy that maximizes the revenue collected over the selling horizon.

The motivation for this work is our strong belief that pricing policies are today, more than ever before, a fundamental component of the daily operations of manufacturing and service companies. The reason is probably because price is one of the most effective variables that managers can manipulate to encourage or discourage demand in the short run. Price is not only important from a financial point of view but also from an operational standpoint. It is a tool that helps to regulate inventory and production pressures. Airline companies and retail chains are good examples of industries where dynamic pricing policies are becoming key drivers of the companies' performance. Not surprisingly, pricing models have become increasingly popular within the Management Science community. Researchers have realized that classical operational problems such as optimal capacity and inventory management or controlling congestion in a queueing network, among many others, cannot be decoupled from marketing activities and especially pricing decisions. This broad range of applications has generated an important volume of work. We believe it is time to survey the field and to present the main results and their practical implications. We do not attempt, however, an exhaustive review of the vast literature on pricing. Instead we focus on the work that has been done in the context of Revenue Management.

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The rapid evolution of information technologies and the corresponding growth of the Internet and E-commerce are sources of inspiration for a survey on dynamic pricing models for two main reasons. First, in this electronic world, it is possible to collect valuable information (about demand, inventory levels, competitors strategies, etc.) and process it in real time. This new reality allows –and forces– managers to act and react dynamically to changes in the marketplace by adjusting any variable under control, especially prices. Furthermore, Internet based selling systems make the logistics of dynamic pricing much easier. The costs associated with relabelling the prices of the products and informing customers about these changes have dropped significantly in the electronic environment when compared to traditional brick and mortar businesses (*e.g.* Brynjolfsson and Smith [18]). On the customer side, Internet price search intermediaries or web aggregators offer customers easy access to better information about product variety and price lists (*e.g.*, priceline.com). As a result, new potential applications for Revenue Management techniques are emerging in connection to the Internet. We consider it important to present the fundamental aspects of dynamic pricing models to an audience that is currently working and developing E-commerce.

From a historical perspective, the interest in Revenue Management practices started with the pioneering research of Rothstein [69]-[70] and Littlewood [52] on airline and hotel overbooking. However, it was probably after the work of Belobaba [6]-[8] and the American Airlines success [72] that the field really took off. The airline industry provided researchers with a concrete example of the tremendous impact that Revenue Management tools can have on the operations of a company (*e.g.*, Smith *et al.* [72]). The publication of a survey paper by Weatherford and Bodily [83], where a taxonomy of the field and an agenda for future work were proposed, was another symptom of this revival. At this stage, however, much of the work was done on capacity management and overbooking with little discussion of dynamic pricing policies. In essence, prices (fares) in these original models were assumed to be fixed and managers were in charge of opening and closing different fare classes as demand evolved. During the 90’s, the increasing interest in Revenue Management became evident in the different applications that were considered. Models became industry specific (*e.g.* airlines, hotels, or retail stores) with a higher degree of complexity (*e.g.* multi-class and multi-period stochastic formulations). Furthermore, it was in the last decade that pricing policies really became an active component of the Revenue Management literature (*e.g.*, Gallego and van Ryzin [36], Bitran and Mondschein [15], Feng and Gallego [27]-[28]). Today, dynamic pricing policies in a Revenue Management context is an active field of research that has reached a certain level of maturity.

In terms of applications, dynamic pricing practices are particularly useful for those industries having high start-up costs, perishable capacity, short selling horizons, and a demand that is both stochastic and price sensitive. Succinctly, the Revenue Management problem has been phrased as “selling the right product to right customer at the right time”. On one hand, the sellers would like to sell their products to those customers having a high valuation so that high margins can be achieved. On the other hand, if they wait too long for those high valuation customers to appear, they might end the selling period with unsold units that could have been sold to low valuation customers. Clearly, for this trade off to be non trivial, both perishable capacity and stochastic demand are needed. As we will discuss in this paper, it is precisely in this environment that dynamic pricing strategies are especially useful to balance utilization and profitability of the available capacity.

As we already mentioned, the airline industry pioneered the use of Revenue Management techniques in terms of capacity/seat control and dynamic pricing. Today, Revenue Management has spread out naturally to other industries such as retailers (*e.g.* Bitran and Mondschein [15], Subrahmanyam and Shoemaker [75]), car rental agencies (*e.g.* Carol and Grimes [19], Geraghty and Johnson

[39]), hotels (*e.g.*, Bitran and Mondschein [12]), Bitran and Gilbert [14]), bandwidth and Internet providers (*e.g.*, Nair and Bapna [62]), passenger railways (*e.g.*, Ciancimino *et al.* [21]), cruise lines (*e.g.*, Ladany and Arbel [48]), electric power supply (*e.g.*, Schweppe [71], Smith [73], Oren and Smith [63]). Although different in many respects, these industries all share the basic properties of the Revenue Management problems that we consider in this work, namely, perishable products, finite selling horizons, and price sensitive and stochastic demand.

We conclude this introduction by positioning this paper with respect to other similar works that have been published. In terms of goals, our objective is to present the main results that have been reported in the literature during the last decades on dynamic pricing models. We concentrate our efforts on understanding the main drivers and properties of optimal pricing strategies. In this respect, we do not discuss in detail the somewhat related research that has been done in the area of inventory and capacity control, although some related results for network Revenue Management problems are presented in section §3.2.2. Our work differs from other survey papers such as Weatherford and Bodily [83] or McGill and van Ryzin [56] since we do not attempt to provide a taxonomy or an exhaustive enumeration of all the publications in the field. Similar in many aspects to our work is the survey on dynamic pricing models by Elmaghraby and Keskinocak [26] where a broad view of the field is presented from a set of different angles such as pricing policies for long and short life cycle products, or combined inventory and pricing decisions, or pricing in markets with rational customers. However, we preferred to narrow the scope of our work to dynamic pricing models in a Revenue Management context so that we can explicitly present and discuss the main results that have been obtained. In this regard, we believe that our work provides a helpful summary of this field to those readers (researchers or practitioners) interested in getting a general overview of the research that has been done thus far. Nevertheless, we believe that a good survey should not only introduce the field and main results to the nonspecialists but also provide new insights and guidance for future research to the experts. For this purpose, we have complemented our review with some new results, and we have included a list of potential new directions of research in section §4.

The remainder of this paper is organized as follows. We develop in section §2 a generic formulation of the Revenue Management problem that provides a global view of the different elements and their interrelationship. In particular, we present a general pricing problem and describe how the different components such as demand attributes, product characteristics, information, and constraints affect the formulation and its applicability. Next, in section §3 we review the literature and the main results. We approach this review from different angles such as deterministic versus stochastic models and single versus multi products models. Finally, in section §4, we summarize our results and identify open problems and new potential directions of research.

2 A Generic Model

In this section, we describe the Revenue Management model under consideration. The model that we present is sufficiently general to cover the research that we review in section §3 as a special case. Furthermore, some of the elements of our generic formulation in §2.6 have not yet been fully addressed in the literature. In this respect, our motivation for this apparent excess of generality is twofold. First of all, we believe that our generic framework is more appealing to those nonspecialist readers interested in getting the “big picture” behind the Revenue Management problem. Secondly, the contrast between this general model and the specific research presented in §3 can be used to identify potential research opportunity. In section §4, we suggest some new directions.

2.1 Supply

Consider a seller or market player (*e.g.*, an airline, hotel, car rental company, retail store, or an Internet service provider) which has a fixed amount of initial capacity that is used to satisfy a price-sensitive demand[†] during a certain selling period $H = [0, T]$. We model this initial capacity as an m -dimensional vector $C_0 = (c_1(0), \dots, c_m(0))$ of resources where $c_k(0)$ is the initial amount of resource k available. Capacity, in our context, is a rather broad concept that might include number of rooms in a hotel, available seats for a specific origin-destination flight in a given day, or simply the number of white shirts in stock at a garment store.

Under the “standard” Revenue Management problem that we consider, capacity is fixed and any strategic considerations regarding how to acquire the initial level C_0 have been excluded. Capacity is essentially given and the seller is committed exclusively to finding the best way to sell it. This assumption is by no means critical if we consider that in many industries capacity is flexible only in the long run. Moreover, capacity decisions and price decisions take place on different time scales. Issues regarding the size of a hotel or an airplane or the number of shirts to purchase from an overseas supplier are decided long before demand is realized and price policies are implemented.

Critical to the revenue management problem are the characteristics of this available capacity and how it is used to create a set (or *menu*) of final products. As we will see shortly, in some cases much of the complexity of the revenue management problem comes from selecting the correct *menu* of products. From a pricing perspective, two important attributes of the available capacity are its degree of flexibility and its perishability.

Flexibility measures the ability to produce and offer different products using the initial capacity C_0 . We say that capacity is *dedicated* if there is a one-to-one correspondence between capacity and final product. For example, a retailer that purchases 500 white T-shirts to sell during the next summer season has *dedicated* capacity. On the other hand, we say that capacity is *flexible* if it can be used to produce different products or satisfy different customers’ needs. For example, an Internet provider owning bandwidth capacity uses this specific resource to offer a wide range of products from email services to video conferences. In general, flexibility is a continuous attribute ranging from highly dedicated (retailing) to highly flexible (the bandwidth provider).

It should be intuitively obvious that flexibility is a desired feature. In essence, flexible capacity allows the seller to allocate scarce resources efficiently based on observed demand rather than forecasted demand (production postponement). In practice, however, flexibility is not always possible. A retailer buying from an overseas supplier needs to order months before the beginning of the selling season. In the hotel industry, the allocation of the available space into luxury, suite, and standard rooms is essentially decided when the hotel is built.

From a pricing standpoint, flexibility increases the complexity of the problem. As we will discuss later, the action of selling a product has associated two quantities: (i) an immediate revenue equal to the price and (ii) an opportunity cost which is the monetary penalty of using capacity today that could be used to satisfy future demand. When capacity is *dedicated*, selling product i does not affect the ability to supply product j . Thus, the opportunity cost of selling i involves essentially product i and its demand. However, when capacity is flexible selling product i decreases the resources available to produce product j . This interaction among products makes the computation of the opportunity cost and the optimal pricing strategy much harder.

Perishability relates to the (lack of) ability to preserve capacity over time. For example, an

[†]Demand can also depend on other variables controlled by the seller like capacity itself.

empty seat on a departing flight is a unit of capacity that cannot be stocked for a future flight. In general, a distinctive feature of the Revenue Management problem is the perishability of the available capacity. A simple way to treat this perishability is making capacity a time-dependent quantity. For instance, a hotel’s unit of resource might be “Room 106 on Friday night, May 10, 2002” while an airline’s unit of capacity could be “Seat 22B on flight #1243 departing from Boston to Chicago at 4:00pm on Tuesday, May 14, 2002”. How much detail is used to define the units of capacity depends on customers’ preferences and the seller’s ability to profit from their choice. For example, two economic-class seats 22A (window) and 22B (aisle) on a given flight could be considered two different resources and priced differently if customers have significant differences on their preferences for window and aisle seats. In practice, airlines do not discriminate based on this feature and both seats 22A and 22B are considered two units of the same resource: economy-class seats. Retailers, on the other hand, are much more active in this way, charging different prices for a blue shirt and for a red shirt (same model, brand, and size).

From a modelling perspective, perishability increases the dimension of the problem, making capacity, and therefore final products, time-dependent quantities. In our dynamic setting, perishability is an inherent property of the model, although it might be irrelevant in some cases, *e.g.*, when capacity is fully inventoriable and the selling horizon goes to infinity.

As time progresses and resources are consumed (they are sold or they perish), capacity decreases and we denote by $C_t = (c_1(t), \dots, c_m(t))$ the available capacity at time t .

2.2 The Product

Following our previous description of capacity, a product in this context is a sub-collection of the available resources. Based on Gallego and van Ryzin’s [37] production model, we consider an $m \times n$ matrix $A = [a_{ij}]$ such that a_{ij} represents the amount of resource i used to produce one unit of product j . That is, every column j of A represents a different product –say product A_j – and the collection $\mathcal{M} = \{A_1, \dots, A_n\}$ is the *menu* of products offered by the seller. We will consider for the moment that there is no explicit costs associated to the production of the final products. This is, by the way, a common assumption in the literature. In many situations, this assumption is not very restrictive since production costs are negligible, or they are linear and can be incorporated directly into the final price.

Given the available initial capacity C_0 the first important decision of the seller is to define the menu $\mathcal{M}$ of products that will be offered to the end customers. A naïve approach would be to consider any possible subset of C_0 as a product, *i.e.*, $\mathcal{M} = \{a \in \mathbb{R}^m : 0 \leq a \leq C_0\}$. However, even if a demand exists for every conceivable subset of C_0 , the task of setting a different price strategy for every combination is computationally demanding and hard to implement. On one hand, managing a short list of products simplifies the pricing problem. On the other hand, a larger list is more suitable for demand-skimming purposes. The right mix of products should balance this trade-off. For instance, the simplest approach would be to set $A = I_k$, the $(k \times k)$ identity matrix. In this case, every resource is dedicated and offered as a single product. Customers are left with the task of purchasing the appropriate combination of each resource depending on their specific needs. In this case a minimum set of prices is needed, one for each resource. The seller, however, can try to do better by creating *bundles*, which are specific subsets of resources that match specific customers’ needs. By doing so, the seller is able to target the market and increase demand. In this case, a larger set of prices has to be specified with the corresponding increment in management costs.

2.3 Information

Crucial to any dynamic pricing policy is the knowledge of the system and its evolution over time. Real-time pricing necessarily requires real-time demand data, the available capacity, and any other relevant factors (*e.g.*, competitors' strategies, weather). Thus, an information system capable of collecting the right information and making it available at decision points is critical. There is little doubt that one of the major factors that influenced the rapid growth of Yield Management in the airline industry was the development of electronic information systems capable of gathering information about demand and ticket reservation over the large network of travel agencies (*e.g.*, SABRE system for American Airlines, Smith *et al.* [72]). Similarly, as reported by Raman *et al.* [66], retailers are investing large amounts of money (close to \$30 billion a year) to improve IT systems and reduce the systematic problem of inaccurate inventory records.

In our Revenue Management setting, short product life-cycles and perishability impose extra pressure to improve the quality and management of information such as demand forecast and inventory position. For instance, standard forecast methods rely heavily on demand history that is not necessarily available in this short life-cycle environment, for example, retailers selling fashionable products (*e.g.*, Fisher and Ramana [35], Kurawarwala and Matsuo [55]).

Given an initial capacity C_0 , a product menu $\mathcal{M}$, and a demand and price processes, we define the *observed history* $\mathcal{H}_t$ of the selling process as the set of all relevant information available up to t . This history should include at least the observed demand process and available capacity, and it can also include some additional information such as demand forecasts. Most of the research has focused on the simple but tractable Markovian case where $\mathcal{H}_t = C_t$, in which only remaining capacity is relevant for pricing decisions. However, path-dependent models are specially useful when demand distribution is unknown and a learning process is incorporated to improve demand estimates. In general, we expect some degree of information asymmetry between the seller and the buyers. Issues regarding the quality of the product or the level of inventory, for instance, are usually private information held by seller. On the other hand, customers have private information about their product valuations and budget. This asymmetry of information can be modelled using two sub-histories $\mathcal{H}_t^s, \mathcal{H}_t^b \subseteq \mathcal{H}_t$ representing the information available to the seller and customers, respectively, at time t .

2.4 Demand

On the demand side, we divide the set of potential customers into different segments each one having its own set of attributes including needs, budget, and quality expectations. We define a d -dimensional stochastic process $N(t, \mathcal{H}_t) = (N_1(t, \mathcal{H}_t), \dots, N_d(t, \mathcal{H}_t))$ where $N_j(t, \mathcal{H}_t)$ is the cumulative potential demand up to time t from family j given the available information $\mathcal{H}_t$.

Depending on the price (and probably other attributes such as quality) potential customers will decide whether or not to purchase the products. Using Lazear [49] terminology, potential customers are divided into (i) *shoppers* which are those customers that search for products but do not buy because of price or quality considerations and (ii) *buyers* which are those customers that are effectively willing to buy a product. In general, pricing policies should be computed on the bases of both potential customers and buyers. However, in most applications the seller is only capable of collecting information about the set of *buyers* according to sales data[‡].

[‡]One exception is the catalog industry, here the seller controls the population of potential consumers according to the mailing policy (*e.g.*, Bitran and Mondschein [14]). E-commerce is another example since information on shoppers (as opposed to buyers) can be obtained via the Internet, by storing the path customers follows on the website.

In order to model this purchasing process, we define an $n \times d$ matrix $B(P) = [b_{ij}]$ where b_{ij} represents the units of product $i \in \mathcal{M}$ requested by a customer in family $j = 1, \dots, d$; the price process $P_t = \{p_s : s \in [0, t]\}$ is described in detail in section §2.5 below. It is important to note that bundling considerations are directly linked to the structure of this matrix $B(P)$ through its dependence on the product menu $\mathcal{M}$. Combining the vector of potential demand $N(t, \mathcal{H}_t)$ and the matrix $B(P)$, we define an n -dimensional vector $D(t, P, \mathcal{H}) \equiv B(P) N(t, \mathcal{H}_t)$ that represents the effective cumulative demand process in $[0, t]$ at the product level.

Finally, we provide the seller with the ability to partially serve demand if it is profitable to do so. For instance, retailers do not display their entire inventory during promotion days. In the same way, airlines are able to reject low-fare reservations (closing a fare) even if they have available capacity. In light of this, we define an n -dimensional vector $S(t)$ that represents the cumulative sales up to time t . Given the demand, sales, and price processes, the dynamics of the available capacity are governed by the following conditions.

$$C_t = C_0 - AS(t) \quad \text{and} \quad S(t) \leq D(t, P, \mathcal{H}_t) \quad \text{for all } t \in [0, T]. \quad (1)$$

In some contexts, the distinction between sales S and demand D is unnecessary. For instance, if the price can be adjusted continuously and unrestrictedly, the seller will prefer to increase the price rather than reject customers. In this case, the price is the only variable that the seller needs to control. For example, in the yield management literature of seat control, the notion of a *null* price[§] has been introduced to model the accept/reject decision in the context of dynamic pricing policies (see section §3.2.2). We note that if the seller is constrained in the way that s/he can adjust the price (see §2.5 below for some examples of constraints) then the distinction between sales and demand becomes relevant and the accept/reject decision is not necessarily replicable using a dynamic pricing strategy.

In terms of our assumptions, the use of a price-sensitive demand $D(t, P, \mathcal{H}_t)$ implies that the seller has monopolistic market power over the set of buyers. Competition might be present in this formulation, but it is hidden and only the residual demand $N(t, \mathcal{H}_t)$ faced by the seller is considered. We do not incorporate any strategic behavior from the customers' side, demand might depend on the whole observed history of the selling process but we do not model the utility maximization process solved by the customers. Demand in this respect is assumed to be given exogenously. Similarly, customers are assumed to be price takers, meaning they observe the price list offered by the seller and react by buying or not buying some of the products. We will postpone the discussion of other allocation mechanisms such as auction models to section §4.

Certainly, good modeling and forecasting of demand are key for pricing purposes. The alternative formulations available in the literature are unlimited especially in the deterministic demand case. The simplest approach is probably to decompose this deterministic demand into a set of different factors, each one addressing a specific aspect of the problem (*e.g.*, Eliashberg and Jeuland [25], Kalish [44], Jain and Rao [42]):

$$D^{\text{det}}(t, p, \mathcal{H}_t) = \mathcal{D}(t) \mathcal{G}(p) \mathcal{F}(\mathcal{H}_t), \quad (2)$$

where $\mathcal{D}(t)$ is an estimate of the market size as a function of time, $\mathcal{G}(p)$ captures price elasticity effects, and $\mathcal{F}(\mathcal{H}_t)$ models the influence of the available information on customers' purchasing behavior.

From micro economics theory (*e.g.* Mas-Colell *et al.* [54]), the notions of consumers' utility, elasticity, and product substitution form the bases of our understanding and modeling of $\mathcal{G}(p)$. For

[§]A high price that makes demand equal to zero almost surely.

example, exponential demand models are commonly used to model demand in the retail sector (*e.g.*, Smith and Achabal [74]). That is, $\mathcal{G}(p) = \exp(-\eta p)$, where η is a measure of demand elasticity per unit of price. Other models using constant elasticity, $\mathcal{G}(p) = p^{-\eta}$, have also been proposed (*e.g.*, Bitran *et al.* [16]). A functional form for $\mathcal{F}(\mathcal{H}_t)$, for the case $\mathcal{H}_t = C_t$, was developed and empirically tested in Smith and Achabal [74].

On the other hand, the modeling of $\mathcal{D}(t)$ depends on the seasonality of demand and the life cycle of the product. Diffusion models (*e.g.*, Bass [3]) are widely used to model this evolution of demand. In this framework, a population of consumers of size N gradually purchases the product. The rate at which consumers buy the product depends linearly on the number of previous purchases (word-of-mouth or diffusion effects) and the fraction of *innovators* existing in the population. Innovators are those consumers that buy the product independently of the other consumers' actions. In Bass's [3] diffusion model, the rate of purchase at time t is given by

$$\frac{d\mathcal{D}(t)}{dt} = pN + (q - p)\mathcal{D}(t) - \frac{q}{N}\mathcal{D}^2(t), \quad (3)$$

where p is the fraction of innovators and q is a measure of the diffusion effect (imitation). The combination of this diffusion model with price has been proposed in Bass [4] and Jeuland and Doland [43].

The stochastic behavior of the demand has been added to these deterministic models for discrete and continuous time formulations. For the discrete time case, the standard approach is to represent demand as the sum of a deterministic part and a zero-mean stochastic component. Using the notation $dD(t, p, \mathcal{H})$ for the marginal demand in period t , the stochastic additive noise model is given by

$$dD^{\text{stoc}}(t, p, \mathcal{H}_t) = dD^{\text{det}}(p, t, \mathcal{H}_t) + \underbrace{\xi(t, p, \mathcal{H}_t)}_{\text{Random Noise}}. \quad (4)$$

The random noise, which usually follows a zero-mean normal random variable, depends on price and time to reflect the changes on demand uncertainty over the life cycle. Another alternative model is the multiplicative noise model

$$dD^{\text{stoc}}(t, p, \mathcal{H}_t) = dD^{\text{det}}(t, p, \mathcal{H}_t) \xi(t, p, \mathcal{H}_t). \quad (5)$$

In this case, the expected value random noise is normally set to one. Combinations of the additive and multiplicative models can also be used.

For the continuous time case, the most common formulation assumes that demand follows a Poisson process with a deterministic intensity that depends on price and time (*e.g.*, Gallego and van Ryzin [36]-[37], Bitran and Mondschein [15], Feng and Gallego [28]), although it is possible to extend the discrete time formulation above replacing the normally distributed random noise by a continuous time Wiener process (*e.g.*, Raman and Chatterjee [67]).

2.5 Pricing Strategies

In our dynamic setting, a pricing policy $P := P_T = \{p_t : t \in [0, T]\}$ is a collection of mappings $p_t(\cdot|\mathcal{H}_t) : \mathcal{M} \rightarrow \mathbb{R}^+$ where $p_t(i|\mathcal{H}_t)$ is the price of product $i \in \mathcal{M}$ at time t given a current history $\mathcal{H}_t$. Depending on the application, some conditions have to be imposed to ensure that the resulting pricing policy P is consistent with standard practices in industry. The following is a list of the most common constraints that we have come across during our literature review and industrial experience.

- **Finite set of prices:** In many applications the seller can only select prices from a finite list of admissible prices, *i.e.*, $\mathcal{P}_t = \{p_1, \dots, p_{K_t}\}$ (*e.g.*, Chatwin [20], Feng and Xiao [30]-[31], Feng and Gallego [28]). The reasons range from marketing considerations such as customers' perception of prices (\$19.99 versus \$20.00) to managerial aspects since a discrete list of prices is easy to implement and control.
- **Maximum number of price changes:** Most companies restrict the number of price changes during the selling horizon (*e.g.* Feng and Gallego [27]). In some cases, this restriction is not critical since two-price policies have been shown to be asymptotically optimal (*e.g.*, Gallego and van Ryzin [36]). In practice, companies restrict the number of price changes because changing prices too often is difficult and costly from an operational standpoint. We should mention, however, that for the growing Internet-based sale systems, the costs of relabelling the prices of products and those associated with informing customers about these changes are dropping considerably (*e.g.* Brynjolfsson and Smith [18]).
- **Markdowns, Markups, and Promotions:** It is common practice in some industries to enforce a predefined path of the price over time. For instance, retailers usually adopt a markdown policy, or clearance policy, that makes the prices of the products to decrease monotonically over time (*e.g.*, Bitran and Mondschein [15]). In general, these markdown policies are appropriate for those industries which face customers whose willingness to pay for the product diminishes over the selling season such as the retailing. On the contrary, airline companies prefer to mark up their prices in order to discriminate among travellers and business passengers. In this case, customers' willingness to pay increases over time since the more profitable business segment tends to make last minute travel arrangements. Markdowns or markups are rarely advertised, and customers become aware of these variations only through past experiences and word-of-mouth. Promotions, on the other hand, are discounts that companies offer at specific moments in time (such as Mother's Day). These discounts are advertised and reversible.
- **Joint Price Constraints:** In some situations, different products cannot be priced independently. This happens naturally with bundles since the price of the bundle should depend on the price of the different components. For instance, there are practical issues, such as marketing considerations or competitors' strategies, that can force the price of the bundle to be at least $x\%$ (say 10%) cheaper than the sum of the price of the components. In this case, if product $i \in \mathcal{M}$ is a bundle resulting from packing together all the products $j \in \mathcal{B}_i \subseteq \mathcal{M}$, then the bundling constraint on the price is as follows.

$$p_t(i) \leq x \sum_{j \in \mathcal{B}_i} p_t(j) \quad \text{for all } t \in [0, T]. \quad (6)$$

Another case where joint price constraints arise naturally is when the same product is offered at different locations that have independent demand. In this case, it can be argued that the product in location k is different than the product in location l because they face different demand processes and therefore a different price can be set at each location. In practice, however, companies try to avoid this type of geographical discrimination because of image and reputation issues (Bitran *et al.* [16]). In this case, the functional constraint that is added to the model is

$$p_t(k) = p_t(l) \quad \text{for all } t \in [0, T]. \quad (7)$$

Joint price constraints can also occur over time. For example in some industries price is forced to follow a monotonic path. The path might be decreasing, such as permanent markdowns

in the retail sector (Mantrala and Surya [53]), or increasing as it happens in the airline industry. In general, most companies try to avoid pricing policies that may be viewed as “unfair” by the end customers. Situations where two first-class passengers seated together after having paid significantly different prices for their seats can have a negative impact on customers’ perception (especially for the passenger having the expensive ticket). To minimize this problem, the airline company could consider pricing policies that satisfy the following condition.

$$p_t - p_s \leq \xi \quad \text{for all } t, s \in [0, T], \quad (8)$$

where ξ is an upper bound on the variability of the pricing policy.

- **Cost-Based Pricing:** Although capacity is a sunk cost in our setting and in most of the applications that we consider, there is still some tendency in practice to set prices based on costs. The reason is probably a mixture of managers’ incentives based on margins and the classical advice from economic theory where marginal cost plays a central role in pricing decisions. It should be intuitively obvious, however, that once capacity is determined, then pricing policies should only try to maximize the revenue without any cost consideration (except for production costs that we have assumed are negligible). In some cases, however, there are legal restrictions to price below cost, and so cost-based pricing is exogenously imposed. A simple cost-based pricing constraint (that we have come across working in the apparel industry) is given by $p_t \geq (1 + x)r$, where r is the unit cost of the product and x is a minimum margin contribution imposed on the product. Note that if x and r are fixed, then we can redefine the price as $p_t \leftarrow p_t - (1 + x)r$. Under this net margin price formulation, the cost-based constraint above reduces to the nonnegativity of the pricing policy which is always satisfied in our revenue maximization context.

In general, we will denote by $\mathcal{P}$ the set of all admissible pricing policies, those that satisfy all the relevant constraints.

2.6 Revenue Management Formulation

Given the available capacity C_0 , the cumulative demand process $N(t)$, the menu of available products $\mathcal{M}$, and a set of admissible policies $\mathcal{P}$, the seller’s objective is to find a pricing strategy P_t that maximizes the total revenue collected from selling the products to the customers. In addition, the seller has the ability to partially serve demand, and so the selling process S_t is also part of the decision variables. The problem faced by the seller is to find the solution to the following optimal control problem.

$$\sup_{P, S} \mathbb{E}_N \left[\int_0^T p_t dS(t) \right] \quad (9)$$

$$\text{subject to: } C_t = C_0 - A S(t) \geq 0 \quad \text{for all } t \in [0, T], \quad (10)$$

$$0 \leq S(t) \leq D(t, P, \mathcal{H}_t) \quad \text{for all } t \in [0, T], \quad (11)$$

$$P \in \mathcal{P}, \quad \text{and} \quad S(t) \in \mathcal{H}_t. \quad (12)$$

We first note that the model corresponds to a *revenue maximization* problem. The objective (9) is simply the expected revenue collected from selling the products over the available selling period $[0, T]$. As we mentioned in §2.1 all considerations associated with acquiring the initial level of capacity C_0 have been excluded.

Another key element of this formulation is the implicit risk-neutral behavior of the seller. The seller's objective in (9) is to maximize expected revenue without any consideration on the variability of the resulting output. We made this assumption to stay in line with the literature where risk neutrality is by far the most commonly used formulation. In those situations where the seller is constantly solving this revenue management problem (*e.g.*, airlines managing thousands of flights a year or retailers selling thousands of SKU's every season) the risk-neutral formulation is certainly appropriate. Mathematical tractability is another reason for this simple modelling of the seller's preferences. We will return to this assumption later in section §4 where we discuss the extension of this standard revenue management formulation to the more general case of *utility management*. Finally, we point out that the single source of uncertainty in this formulation is on the demand side. We conclude this section, with a pictorial representation on Figure 1 of the general Revenue Management network that we consider.

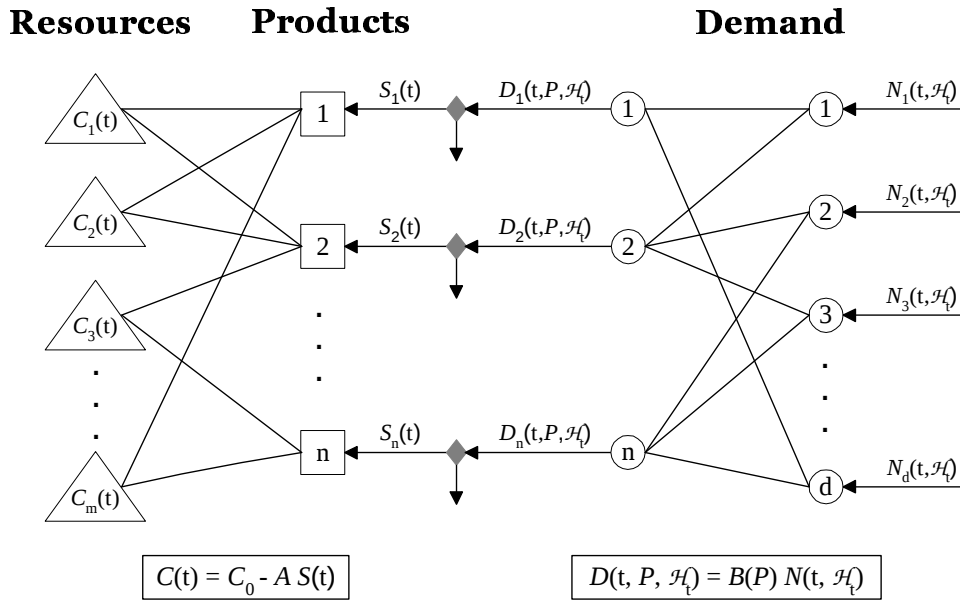


Figure 1: The Revenue Management network.

3 Main Results and Related Literature

We now proceed to a systematic review of the research, publications, and main results on the pricing problem in (9)-(12). The goal of this section is to understand the structure and properties of an optimal solution to this generic problem by examining the different models that have been studied. Certainly, there is no single way to approach the task of reviewing the literature and main results. From an exposition perspective, we find it convenient to start with a basic partition between deterministic and stochastic models.

3.1 Deterministic Models

The deterministic models that we consider in this section assume that the seller has perfect information about the demand process. This is, of course, a major simplification especially for those applications where demand is hardly predictable at the beginning of the season, *e.g.*, new products

or fashion goods. Furthermore, we have argued in the Introduction that Revenue Management techniques are particularly useful for industries facing stochastic demand. There are two important reasons that explain why we have decided to review deterministic models. First of all, deterministic models are easy to analyze and they provide a good approximation for the more realistic yet complicated stochastic models. Moreover, as we will show shortly, deterministic solutions are in some cases asymptotically optimal for the stochastic demand problem (*e.g.*, Gallego and van Ryzin [36]-[37], Cooper [22]). The second reason is that deterministic models are commonly used in practice.

In terms of the literature, deterministic models form the basis of the classic economic model on monopolistic pricing, which is essentially the departing point of the research that is currently done in marketing and operations. It is not in our interest, however, to review the vast economic literature on pricing which mainly focuses on static equilibrium (or steady-state) pricing where marginal cost equals marginal revenues. The reader is referred to Nagle [61] for a comprehensive discussion of the economic literature on pricing theory.

As we argued above, deterministic models are good “first order” approximations (asymptotically optimal in some cases) for more sophisticated stochastic models. In particular, they provide valuable insight on how optimal pricing policies depend on the different parameters of the model.

3.1.1 Single Product Case

The simplest model in this deterministic setting considers the case of a monopolist selling a single product to a price sensitive demand during a fixed period $[0, T]$ (*i.e.*, $|\mathcal{M}| = 1$). The initial inventory is C , demand is deterministic with time dependent and price sensitive intensity $\lambda(p, t)$. In addition, the instantaneous revenue function $r(p, t) = p \lambda(p, t)$ is assumed to be concave as in most real situations. The Revenue Management problem (9)-(12) can be written in this case as follows.

$$\max_{P \in \mathcal{P}} \int_0^T p_t \lambda(p_t, t) dt \quad (13)$$

$$\text{subject to } \int_0^T \lambda(p_t, t) dt \leq C. \quad (14)$$

This is a standard problem in calculus of variations. Let $H(p_t, t) = (p_t - \eta) \lambda(p_t, t)$ be the corresponding Hamiltonian function where $\eta \geq 0$ is the Lagrangian multiplier for (14). The optimality condition (*e.g.*, Gelfand [38]) is given by

$$p_t^* = \eta - \frac{\lambda(p_t^*, t)}{\lambda_p(p_t^*, t)}, \quad (15)$$

where λ_p is the partial derivative of λ with respect to the price. Let $\varepsilon(p, t) = p \left(\frac{\lambda_p(p, t)}{\lambda(p, t)} \right)$ be the *elasticity* of demand with respect to price at time t . Then, condition (15) (together with the fact that $\eta \geq 0$) asserts that at optimality $\varepsilon(p_t^*, t) \leq -1$. That is, demand is *elastic*[†] at the monopolist’s optimal price. We note that the myopic solution p_t^m to (13)-(14) that maximizes the instantaneous rate of return solves

$$p_t^m = - \frac{\lambda(p_t^m, t)}{\lambda_p(p_t^m, t)}. \quad (16)$$

Therefore, if $\eta = 0$, *i.e.*, the capacity constraint (14) is not active, then the optimal strategy p^* is equal to the myopic strategy p^m . On the other hand, if (14) is active then $\eta \geq 0$ and the myopic

[†]We say that $\lambda(p)$ is elastic at price $\bar{p}$ if $\varepsilon(\bar{p}, t) = \bar{p} \left(\frac{\lambda_p(\bar{p}, t)}{\lambda(\bar{p}, t)} \right) \leq -1$.

solution is a lower bound on the optimal strategy. From standard duality theory, η is the shadow price associated with a unit of capacity. Thus, we can think of η as the opportunity cost of selling a unit of product and so necessarily the optimal strategy must satisfy $p_t^* \geq \eta$.

For the case of a time homogeneous demand intensity ($\lambda(p, t) = \lambda(p)$) a fixed price solution can be shown to be optimal over the entire selling period $[0, T]$. In order to characterize this solution let $p^m = \operatorname{argmax}\{p \lambda(p) : p \geq 0\}$ be the “myopic” price policy that maximizes the revenue rate and $\lambda^m = \lambda(p^m)$ be the corresponding demand intensity. Similarly, let $\tilde{p}$ be the solution to $\lambda(\tilde{p})T = C$ and $\tilde{\lambda} = \lambda(\tilde{p})$ be the corresponding demand intensity[‡]. Then, the following is a straightforward application of the Karush-Kuhn-Tucker (KKT) optimality conditions (e.g., Bazaraa *et al.* [5]).

Proposition 1 *Consider the single product Revenue Management problem (13)-(14) with homogeneous demand intensity $\lambda(p)$, and concave revenue rate $r(p) = p \lambda(p)$.*

Case 1. Abundant Capacity: *If $\lambda^m T \leq C$ then the optimal price is p^m and the optimal revenue is equal to $p^m \lambda^m T$.*

Case 2. Scarce Capacity: *If $\lambda^m T > C$ then the optimal price is $\tilde{p}$ and the optimal revenue is equal to $\tilde{p} C$.*

This result is also shown in Gallego and van Ryzin [36] (Proposition 2) and it is used as a building block for constructing heuristics and bounds for the stochastic counterpart. As a direct corollary of proposition 1 we have two important properties of the optimal price strategy: namely, the optimal price is (i) nonincreasing in the initial capacity C and (ii) nondecreasing in the selling period T . Graphically, these results are illustrated in Figure 2. From Figure 2 we can see that the optimal

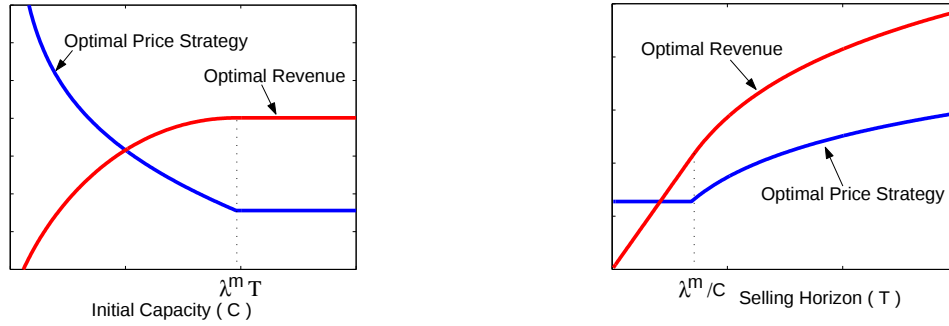


Figure 2: Optimal price strategy for the single product deterministic case. Both figures are drawn using an multinomial demand rate $\lambda(p) = (\exp(p) + 1)^{-1}$.

revenue is evidently a nondecreasing function of both the initial inventory (C) and the length of the selling season (T). In terms of the initial inventory, there is an optimal level $C^m = \lambda^m T$ that maximizes the revenue. Above this threshold additional units of capacity will not increase revenue. Managers should then try to target this optimal value C^m when determining the initial level of capacity. On the other hand, the optimal revenue is monotonically increasing in T reflecting the fact that as the selling horizon increases the seller faces a larger population of potential buyers and therefore he can target the available capacity to those customers having higher valuation for the product. In the limit, if $\bar{p} = \sup\{p : \lambda(p) > 0\}$ then the seller can obtain a maximum revenue of $\bar{p} C$ as T goes to infinity.

[‡]Notice that the existence of $\tilde{p}$ is not guaranteed for large C .

Most extensions of this single product deterministic demand problem generalize some aspect of the functional form of the demand process. For example, Smith and Achabal [74] studied the case where demand intensity depends on price as well as on the level of inventory, *i.e.* $\lambda(p_t, C_t, t)$. The idea (which naturally arises in the retail sector, for instance) is that demand decreases as the inventory is depleted. Customers are less likely to find the product they want (*e.g.*, in terms of size, color, quality, etc.) when available inventory is low. In this setting, the authors derive optimality conditions for the price similar to (15), and closed form solutions are reported for the special case of a *multiplicative separable* demand rate with exponential price sensitivity, *i.e.*, $\lambda(p, C, t) = k(t) y(C) \exp(-\gamma p)$.

Another extensive stream of research coming especially from marketing (*e.g.*, Doland and Jeuland [24]-[43], Kalish [44], Mesak and Berg [57], Mesak and Clark [58], Parker [64], among many others) considers the case of a price sensitive diffusion model (*à la* Bass [3]) to describe the dynamics of the demand. On the Revenue Management context, Feng and Gallego [28] use a diffusion model to characterize the intensity of the demand process. The Bass diffusion model is generally used for durable goods, for which demand at time t depends on the number of units sold previous to t and the size of the population of potential customers. More specifically, the demand rate $\lambda(t)$ at time t is a function of the current price $p(t)$, the amount sold by that time $D(t)$, and the population size N , that is,

$$\lambda(t) := \frac{\partial D(t)}{\partial t} = \lambda(p(t), D(t), N). \quad (17)$$

In general the diffusion effect, *i.e.*, the dependence of the demand rate λ on the cumulative sale $D(t)$, is not uniform over time. Upon introduction, we expect a positive effect (meaning $\partial\lambda/\partial D \geq 0$) due to factors such as word of mouth, improved reputation, or exclusivity. On the other hand, as time passes and the number of sold units increases, we expect market saturation and obsolescence effects to generate a negative impact on demand (*i.e.*, $\partial\lambda/\partial D \leq 0$). According to Kalish [44] results, the evolution of price over time can follow three generic paths: (i) monotonically increasing if word-of-mouth effects have a positive impact on demand, (ii) unimodal: increasing at the beginning, reaching a maximum at some intermediate time, and then decreasing for the rest of the selling period. This situation occurs when there is a positive effect of word-of-mouth at the beginning followed by demand saturation. Finally, (iii) the price is monotonically decreasing over the entire horizon if there is a negative effect of penetration on demand. For a complete review of these single product Bass diffusion models we refer the reader to section §2 in Elmaghraby and Keskinocak [26]. In a different context, Rajan *et al.* [65] and Abad [1] derive optimal pricing policies for the case where inventory deteriorates continuously and deterministically over time at a rate proportional to the inventory position. The special cases of linear demand and exponential decay are studied in more detail.

3.1.2 Multiple Product Case

The case of multiple products ($|\mathcal{M}| \geq 2$) has received considerably less attention. The reason is probably because of the higher degree of complexity attached to these multi-product formulations especially to characterize demand correlation and product substitution effects. In the economics literature, Wilson [85] studies deterministic, multi-product models in which the seller objective it to design an optimal menu of prices and products.

The selection of an appropriate consumers' choice model such as the multinomial logit or multinomial probit (*e.g.* Ben-Akiva and Lerman [9]) to characterize customers' preferences becomes a critical component of the problem's formulation (*e.g.*, Talluri and van Ryzin [79]). We notice that

in the case when capacity is *dedicated* and the price of product i does not affect the demand for product $j \neq i$ (independent demands) then the multi-product case reduces trivially to a set of disconnected single-product problems. The interesting cases arise when capacity is *flexible* and/or demand process depends on the whole vector of prices (substitute or complementary products).

In general, a similar result to proposition (1) can be derived in this multi dimensional case. For exposition purposes, we consider here the simple case of time homogenous demand processes. In this setting, it is not hard to show that a fixed price solution can be used without any sacrifice on performance. Let $D_i(P) = \lambda_i(P) T$ be the cumulative demand for product $i \in \mathcal{M}$ given a vector of prices $P = (p_1, \dots, p_n)$ (λ_i is the time homogenous demand rate). Let $\Lambda(P) = (\lambda_1(P), \dots, \lambda_n(P))$ be the vector of demand intensities and $T \Lambda(P)' P$ be the revenue function (primes (') denote vector transpose). In this case, it is convenient to introduce for each product $i \in \mathcal{M}$ the inverse demand function $P_i(\Lambda)$ that represents the price of product $i \in \mathcal{M}$ given a vector of cumulative intensities Λ . We assume then that $P(\Lambda)$ is a real valued function that is continuous and differentiable and such that the revenue function $P(\Lambda)' \Lambda$ is strictly concave. The Revenue Management problem (9)-(12) can be written in this case as:

$$\begin{aligned} & \max_{\Lambda \geq 0} T P(\Lambda)' \Lambda \\ & \text{subject to } T A \Lambda \leq C. \end{aligned} \tag{18}$$

This is a multidimensional nonlinear programming problem which has a unique solution given the concavity assumption on the revenue function. Similar to proposition (1), two cases characterize the optimal solution. Let Λ^m be the vector of cumulative demands that maximize $P(\Lambda)' \Lambda$. Then Λ^m is optimal if and only if $T A \Lambda^m \leq C$. If this condition is not satisfied, then the optimal solution is a boundary point $\tilde{\Lambda}$ that satisfies the corresponding KKT optimality conditions. The following proposition characterizes the multi-product case.

Proposition 2 *Consider the multi product Revenue Management problem (18)-(19) with homogenous inverse demand function $P(\Lambda)$, and concave revenue function $T P(\Lambda)' \Lambda$.*

Case 1. Abundant Capacity: *If $T A \Lambda^m \leq C$ then the optimal price is $P^m = P(\Lambda^m)$.*

Case 2. Scarce Capacity: *If $T A \Lambda^m \not\leq C$ then let $\tilde{\Lambda}$ be the unique solution to the following Karush-Kuhn-Tucker optimality conditions*

$$\begin{aligned} & \nabla_{\Lambda}[P(\Lambda)' \Lambda] - A' \beta = 0 \\ & \beta'(T A \Lambda - C) = 0 \\ & \Lambda \geq 0 \quad \beta \geq 0, \end{aligned} \tag{20}$$

where ∇_{Λ} is the gradient operator with respect to Λ and β is an m -dimensional vector of Lagrangian multipliers. The optimal price in this case is $\tilde{P} = P(\tilde{\Lambda})$.

Let $\partial_{\Lambda} P(\Lambda)$ be the Jacobian matrix associated to the price vector $P(\Lambda)$. That is, the ij element of this matrix is given by $[\partial_{\Lambda} P(\Lambda)]_{ij} = \frac{\partial P_j(\Lambda)}{\partial \lambda_i}$. Thus, the first KKT condition above implies that the optimal price vector satisfies:

$$P(\Lambda) = A' \beta - \partial_{\Lambda} P(\Lambda) \Lambda. \tag{21}$$

Similar to the single product case, β is the vector of shadow prices associated with the available capacity C , and $A' \beta$ represents the vector of opportunity cost. Therefore, additional capacity is valuable only if it is scarce, *i.e.* $\beta \geq 0$.

It is also important to notice that for the multi-product case it is possible that the optimal price increases with the level of capacity. For instance, consider a simple example with two products where (19) is given by two constraints: $\lambda_1 + \lambda_2 \leq C_1$ and $\lambda_1 \leq C_2$. Suppose that the current level of capacity is $(C_1 = 1, C_2 = 0)$ and that the optimal solution is $\lambda_1^* = 0, \lambda_2^* = 1$. If we increase C_2 to a new value $\tilde{C}_2 = 1$, then under regular conditions on the revenue function the new solution will satisfy $\tilde{\lambda}_1 > 0$ and $\tilde{\lambda}_2 < 1$. That is, an increase in C_2 might induce a decrease in λ_2 . Thus, the optimal price of product 2 will increase after the increase of C_2 . This example raises the question of what the conditions are that will ensure that the optimal price is in fact nonincreasing in capacity. We partially answer this question in the following proposition based on the work by Topkis [80] and Milgrom and Roberts [59] on supermodularity and complementarity.

Proposition 3 *Suppose that the inverse demand function is monotone, that is, if $\Lambda_1 \geq \Lambda_2$ then $P(\Lambda_1) \leq P(\Lambda_2)$. Suppose, moreover, that the objective $P(\Lambda)' \Lambda$ is supermodular and the sets $\mathcal{L}(C) = \{\Lambda \geq 0 : T A \Lambda \leq C\}$ are sublattices[§]. Then, the optimal solution Λ^* to (18)-(19) is nondecreasing in C and the optimal price is nonincreasing in C .*

The proof follows directly from Theorem 5 in Milgrom and Roberts [59]. The monotonicity of $P(\Lambda)$ is the extension of the classical “downward sloping demand function” condition to this multi-dimensional case. We expect the supermodularity assumption to hold when there are product substitution effects (such as for airline seats or hotel rooms) since in these cases the marginal return on product i should be increasing on the price of product j . The requirement of $\mathcal{L}(C)$ being a sublattice is more restrictive. This result holds trivially when capacity is dedicated, *i.e.*, $A = I$ the identity matrix.

Similarly, in the time homogenous case an increase in the selling period (*i.e.*, $T \uparrow$) can be interpreted as a decrease in the level of capacity[†]. Thus, under the same set of assumptions of proposition 3, we expect the optimal price to be an increasing function of T .

3.2 Stochastic Models

Pricing policies with stochastic demand are more complex and harder to compute than their deterministic counterparts. For instance, in this setting a single price solution is rarely optimal unless we restrict ourselves to this type of static policies. On the other hand, stochastic models are clearly used more appropriately to describe real life situations where the paths of demand and inventory are unpredictable over time and managers are forced to react dynamically by adjusting prices as uncertainty reveals itself.

The natural way to tackle a problem of this type is by using stochastic dynamic programming (SDP) techniques. At every decision point during the selling season, the manager collects all relevant information about the current inventory positions and sales and establishes the prices at which the products should be sold. With a few exceptions, most of the research has been done for the single product case under Markovian assumptions on the demand process. In this setting, the inventory levels are the only relevant information that managers need to make pricing decisions.

[§]A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is *supermodular* if for all $x, x' \in \mathbb{R}^n$, $f(x) + f(x') \leq f(\min\{x, x'\}) + f(\max\{x, x'\})$. A set $\mathcal{L} \subseteq \mathbb{R}^n$ is a *sublattice* if for all $x, x' \in \mathcal{L}$, $\min\{x, x'\} \in \mathcal{L}$ and $\max\{x, x'\} \in \mathcal{L}$.

[†]Notice that in the time homogenous case the feasible region $T A \Lambda \leq C$ is equivalent to $A \Lambda \leq \frac{C}{T}$.

3.2.1 Single Product Case

In the single product case ($|\mathcal{M}| = 1$), we can assume without any loss of generality that the initial capacity $C = C_0$ is a scalar representing the number of units of the product that are available at time $t = 0$. Using an SDP formulation, we define $V_t(C_t)$ to be the value function at time t if the inventory is C_t , that is, $V_t(C_t)$ is the optimal expected revenue from time t to the end of the season given that the current inventory position at time t is C_t . Time t has been modelled in the literature as either a continuous or discrete variable. From a practical perspective, we expect that managers will revise their price decisions only at discrete points in times. However, the explosive growth of the Internet and E-commerce make the continuous time model much more suitable for practical uses.

Single Price Models:

The simplest approach to the problem is the single price solution. In this case, we restrict the pricing policy to be a fixed price during the entire season, *i.e.*, $p_t = p$ for all $t \in [0, T]$. This type of static policy is appropriate for products having one or more of the following characteristics: (i) short selling period, (ii) high costs of changing prices, (iii) legal regulations that force the price to be fixed. The fixed price model is simple and easy to implement and control. Hence, even if price changes are possible, managers often choose to use the static fixed price approach. We will see shortly that the fixed price model is asymptotically optimal in some situations. In this single product fixed price model, formulation (9)-(12) is given by

$$V(C, T) = \max_{p \geq 0} V(C, p, T) = \max_{p \geq 0} E[p \min\{D(p, T); C\}], \quad (22)$$

where $D(p, T)$ is the random variable representing the cumulative demand in $[0, T]$ at a price p . Closed-form solutions for this problem are not available for the general case of an arbitrary distribution of $D(p, T)$ but we can characterize the optimal price in terms of the demand elasticity as follows. Let $f(D; p, T)$ be the probability mass function of $D(p, T)$ (or the density function if demand is modelled as a continuous variable). Let also $F(D; p, T)$ be the probability distribution function of $D(p, T)$. We define the demand elasticity with respect to price as $\varepsilon(D, p, T) = \frac{p f_p(D, p, T)}{f(D, p, T)}$ where $f_p(D, p, T)$ is the partial derivative of $f(D, p, T)$ with respect to p .

Proposition 4 *The first order optimality condition for the solution of (22) is*

$$\frac{E[\min\{D; C\} \varepsilon(D, p, T)]}{E[\min\{D; C\}]} = -1. \quad (23)$$

Proof: See the appendix. We note that proposition 4 extends the well-known condition in economics that says that the demand elasticity has to be equal to -1 at the optimal monopolistic price. In the stochastic case, we have that the weighted expected value of the elasticity has to be equal to -1, where the weight is given by the level of sales $\min\{D, C\}$. Similar to the deterministic case, we can show that the optimal price for this single price model is nonincreasing on the initial level of capacity C . We summarize this observation for the case of a continuous demand distribution[†] in the next proposition. We introduce the following definition: A function $g(x, y)$ satisfies *increasing differences* in (x, y) if $g(x^H, y) - g(x^L, y)$ is nondecreasing in y for all $x^H \geq x^L$.

Proposition 5 *Suppose that $F(D; p, T)$ satisfies increasing differences in (D, p) and $-F(D; p, T)$ satisfies increasing differences in (p, T) . If there is a unique optimal solution $p^*(C)$ for (22) then the solution is nonincreasing in C and nondecreasing in T .*

[†]The result and proof for the discrete case follows exactly the same line of arguments.

Proof: See the appendix. As a direct consequence of the proposition we get an upper and a lower bound for the optimal price $p^*(C)$, namely, $p^{\min} := p^*(\infty) \leq p^*(C) \leq p^*(1) := p^{\max}$. Figure 3 shows a numerical example of the behavior of the upper and lower bound for the case where the demand process is Poisson with rate $100 \exp(-0.0044p^2)$. We can see from the figure on the

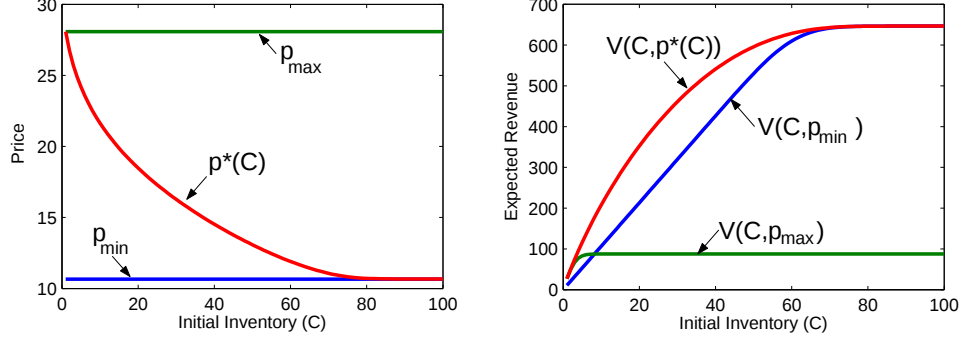


Figure 3: Bounds for the single price model. Following Bitran and Mondschein [15], the demand process used in this numerical computation is Poisson with rate $\lambda = 100 \exp(-0.0044p^2)$, i.e., customers's reservation price follows a Weibull distribution with parameters $(2, 0.0044)$.

left the asymptotical optimality of $p^{\min}$ and $p^{\max}$. The figure on the right plots the behavior of the value function when the optimal price and the two bounds $p^{\min}$ and $p^{\max}$ are used. For the numerical example in the figure, the lower bound $p^{\min}$ performs better when it is used heuristically. This observation is consistent with a set of numerical experiments that we have performed using different demand distributions.

It is also possible to approximate the single price solution using the deterministic version of the model (or *certainty equivalent policy*). Specifically, we can replace the value function in (22) by

$$V^{\det}(C, T) = \max_{p \geq 0} V^{\det}(C, p, T) = \max_{p \geq 0} p \min\{E[D(p, T)], C\} \geq \max_{p \geq 0} E[p \min\{D(p, T), C\}]. \quad (24)$$

The inequality follows from the concavity of the function $f(x) = \min\{x, C\}$ and Jensen's inequality. We conclude that $V(C, T) \leq V^{\det}(C, T)$. Following closely the work by Gallego and van Ryzin [36] (Section §3.3), let $p^{\det}$ be the optimal deterministic price which can be easily computed using proposition 1. Let $\nu^{\det}(C, T)$ be the coefficient of variation of $D(p^{\det}, T)$ then it follows that

$$\begin{aligned} V(C, p^{\det}, T) &= p^{\det} E[D(p^{\det}, T) - (D(p^{\det}, T) - C)^+] \geq p^{\det} E[D(p^{\det}, T)] \left(1 - \frac{\nu^{\det}(C, T)}{2}\right) \\ &\geq V^{\det}(C, T) \left(1 - \frac{\nu^{\det}(C, T)}{2}\right) \geq V(C, T) \left(1 - \frac{\nu^{\det}(C, T)}{2}\right). \end{aligned} \quad (25)$$

The critical step in this derivation is to show that $E[(D(p^{\det}, T) - C)^+] \leq \nu^{\det}(C, T) E[D(p^{\det}, T)]/2$ which follows after some straightforward manipulations from proposition 1 above and equation (18) in Gallego and van Ryzin [36]. Thus, the relative error of using the deterministic price instead of the optimal solution is never greater than $\frac{1}{2} \nu^{\det}(C, T)$, i.e.,

Proposition 6

$$\frac{V(C, p^{\det}, T)}{V(C, T)} \geq 1 - \frac{\nu^{\det}(C, T)}{2}. \quad (26)$$

It is interesting to notice that the quality of the deterministic approximation depends on the coefficient of variation rather than the variance itself. For instance, if the selling horizon increases we should expect that the variance of the cumulative demand will also increase but the coefficient of variation will probably decrease. Similarly, products having high-volume demand are more likely to have a small coefficient of variation. Gallego and van Ryzin [37] derive a similar inequality for the case where the demand intensity is time-varying. We will return to this point later when we discuss the multi-period problem.

The first extension of the single period model is to allow the manager to revise the price only once during the selling horizon. Lazear [49] considers a model of a retailer selling a single unit ($C = 1$) to a population (N) of potential customers whose valuation (reservation price R) for the product is unknown to the seller. The selling horizon is divided in two periods. The retailer's problem is to set the price for the good during the first and second periods, p_1 and p_2 respectively. Lazear's model is of incomplete information. If the product does not sell in the first period at price p_1 , then the retailer can update his/her initial estimate of R to compute p_2 . In this stylized setting, Lazear shows that the price is monotonically decreasing with time, $p_1 > p_2$, and that the magnitude of the markdown $p_1 - p_2$ increases with N . This suggests that prices of high-demand goods (*i.e.*, N is large) adjust more rapidly to time on the market during which the good remains unsold.

In a different setting, Feng and Gallego [27] study a single product two price model where the prices in both periods are fixed and the only decision is when to switch from one to the other. Three cases are studied (*i*) the markdown case when $p_1 > p_2$ (*e.g.*, the retail model), (*ii*) the markup case $p_1 < p_2$ (*e.g.*, the airline model), and (*iii*) the general case $p_1 \leq$ or $\geq p_2$. Under the assumption that demand at price p_i is a Poisson process of intensity λ_i the authors derive structural properties of the optimal stopping time problem. In particular, they show that the optimal policy is of a threshold type. For example, for case (*i*) they show that there is an increasing sequence $\{x_n : n = 1, 2, \dots\}$ of time thresholds such that if the inventory process is C_t then it is optimal to markdown the items to p_2 as soon as the time-to-go ($T - t$) is less than the time threshold x_{C_t} . Similar threshold policies are derived for case (*ii*) and (*iii*). Feng and Xiao [29] extend the two price formulation in Feng and Gallego [27] to the case of a risk-sensitive seller that penalizes the variance of revenue linearly. Other related work on the timing of sales and promotions can be found in Courty and Li [23], Krider and Weinberg [47], Warner and Barsky [82], Kinberg and Rao [46].

Dynamic Price Models:

One of the first papers that addresses the general issue of how to price a perishable product dynamically is the work by Kincaid and Darling [45]. Their setting is a continuous time model where demand follows a Poisson process with fixed intensity λ . An arriving customer at time t has a reservation price r_t for the product, *i.e.*, the maximum price the customer is willing to pay. From the seller perspective, the reservation price r_t is a random variable with distribution $F(r, t)$. Kincaid and Darling consider two cases. In the first case, the seller does not post prices but receives offers from potential incoming buyers, which he/she either accepts or rejects. It is assumed that arriving customers offer their reservation price r_t , *i.e.*, it is assumed that customers do not act as strategic players. In the second case, the seller posts the price p_t and arriving customers purchase the product only if $r_t \geq p_t$. The demand process in this situation is Poisson with intensity $\lambda(1 - F(p_t, t))$. Optimality conditions for the value function $V_t(C_t)$ and the optimal price $p_t(C_t)$ are derived for both cases, and closed-form solutions are reported for the special case $F(r, t) = 1 - \exp(-r)$. When prices are posted, the optimality condition (Hamilton-Jacobi Bellman

equation) is given by

$$\frac{\partial V_t(C_t)}{\partial t} = \max_{p \geq 0} \left\{ \lambda(1 - F(p, t)) \left[p - (V_t(C_t) - V_t(C_t - 1)) \right] \right\} \quad (27)$$

According to this condition, it is not hard to see that the optimal price satisfies $p_t(C_t) \geq V_t(C_t) - V_t(C_t - 1)$. The difference $V_t(C_t) - V_t(C_t - 1)$ represents the opportunity cost of selling a unit of capacity at time t when the available inventory is C_t . In the Yield Management literature $V_t(C_t) - V_t(C_t - 1)$ is referred to as the *bid price* for the inventory level C_t at time t . Note that the maximization in (27) guarantees that the optimal price $p_t(C_t)$ is larger than or equal to this bid price and therefore the value function $V_t(C_t)$ is nondecreasing in t . Under some mild restrictions on $F(p, t)$ and its density $f(p, t)$ [‡], the first order condition characterizes the optimal price $p_t(C_t)$ as follows:

$$p_t(C_t) = \frac{1 - F(p_t(C_t), t)}{f(p_t(C_t), t)} + V_t(C_t) - V_t(C_t - 1). \quad (28)$$

Thus, the problem of computing an optimal price strategy reduces to the computation of the opportunity cost $V_t(C_t) - V_t(C_t - 1)$. In general, there are no exact closed-form solution for the optimal price strategy in (27). One exception reported in Kincaid and Darling [45] is the case of exponential reservation price distribution, that is, $\lambda(p) = \lambda \exp(-\alpha p)$. Condition (27) is also useful to compute a lower bound on $p_t(C_t)$.

Proposition 7 *Suppose that demand is a Poisson process with price sensitive intensity $\lambda(1 - F(p, t))$. Then, the optimal price $p_t(C_t)$ is bounded below by $p^{\min}$, the solution to*

$$p^{\min} = \frac{1 - F(p^{\min}, t)}{f(p^{\min}, t)}. \quad (29)$$

Proof: See the appendix. Similar to Kincaid and Darling's paper are the formulations by Gallego and van Ryzin [36], Bitran and Mondschein [15], Bitran *et al.* [16], Feng and Xiao [30]-[31], Zhao and Zheng [87] where a Poisson process is also used to model demand in a single product dynamic price setting.

In a continuous time formulation, Gallego and van Ryzin [36] re-derive the optimality condition (27) and prove that the value function $V_t(C_t)$ is increasing and concave in both t and C_t and that the optimal price $p_t(C_t)$ is increasing in t and decreasing in C_t . Schematically, Figure 4 shows the path of an optimal price strategy and its corresponding inventory level for the continuous time model. We can see from Figure 4 that the optimal price path is decreasing almost everywhere having discrete positive jumps at every sale epoch. In addition, the slope at which the price decreases over time tends to increase as the selling horizon gets shorter. From a practical perspective, this erratic behavior of the continuous time solution is difficult to implement and control. Gallego and van Ryzin [36] address this issue and show that the fixed-price heuristic that uses the deterministic version of the problem to compute the price is asymptotically optimal as T or C_0 go to infinity. More specifically, suppose that the demand process is replaced by its deterministic counterpart, *i.e.*, the demand rate is simply the expected value of the Poisson process. Let $p^{\det}(C_t)$ be the optimal deterministic price from time t on if the current inventory is C_t and let $V_t^{\det}(C_t)$ be the corresponding value function if $p^{\det}$ is used instead of the optimal price. Clearly, by the optimality

[‡]Bitran and Mondschein [15] show that if $\frac{(1 - F(p, t))^2}{f(p, t)}$ is decreasing function of t then the first order optimality condition is also sufficient.

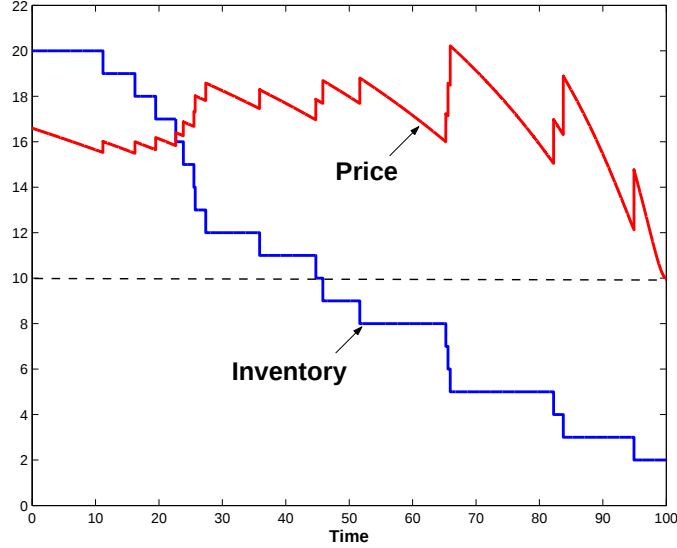


Figure 4: Path of an optimal price policy and its inventory level. Demand is a time homogeneous Poisson process with intensity $\lambda(p) = \exp(-0.1p)$, the initial inventory is $C_0 = 20$, and the selling horizon is $H = 100$. The dashed line corresponds to the minimum price $p^{\min} = 10$.

of $V_t(C_t)$ we must have that $V_t^{\det}(C_t) \leq V_t(C_t)$. However, Gallego and van Ryzin [36] show that

$$\frac{V_t^{\det}(C_t)}{V_t(C_t)} \geq 1 - \frac{1}{2\sqrt{\lambda(p^{\det})(T-t)}}, \quad (30)$$

where $\lambda(p^{\det})$ is the demand intensity at price $p^{\det}$. According to this result, the deterministic price heuristic is asymptotically optimal as T goes to infinity. Note, moreover, that a fixed price policy minimizes management cost since it does not need to keep track of the evolution of the inventory or the demand over time.

Bitran and Monschein [15] consider a periodic pricing review policy where prices are revised only at a finite set of decision times. A distinctive element in their formulation is the inclusion of a markdown constraint that forces prices to be nonincreasing over time, which is a commonly encountered constraint in the retail sector. Similar to Kincaid and Darling, the demand model is the combination of a Poisson arrival process of customers and a purchasing process based on a reservation price which is unknown to the seller. Using a set of numerical experiments, the authors argue that (i) the value function with and without the markdown constraint does not differ significantly (less than 0.7%), (ii) the initial price is increasing on the variability of the reservation price, and (iii) the expected revenue varies significantly with the number of periods at which the prices are allowed to change.

Bitran *et al.* [16] extend the single product periodic review formulation in Bitran and Mondschein [15] to the cases of a retail chain. In this situation, the same product is sold at different locations with each one having its own Poisson demand process. Under the constraint that at every moment in time the price must be the same at all the locations (coordinated price policy), the authors derive optimality conditions and a set of heuristics for the cases when inventory transfers among stores are and are not allowed. The heuristics are constructed using a *rolling horizon* approach, whereby at every decision point the price is computed assuming that this is the last time that the price will be revised. Computational experiments show that this type of heuristics performs quite

well with an average error of 2%-3%. The paper also includes a set of numerical experiments that were conducted using real data collected from a retail chain store.

Zhao and Zheng [87] study the single product pricing problem for the case where the arrival process of customers is a time dependent Poisson process. They also use a reservation price formulation similar to Kincaid and Darling [45] or Bitran and Mondschein [15] to model the purchasing decisions of the customers. For this problem, Zhao and Zheng derive optimality condition equivalent to (27) and show that the value function is concave on both the level of inventory and the duration of the selling season. They also prove that the optimal price is nonincreasing in the level of inventory and find a sufficient condition on the reservation price distribution that guarantees that the optimal price is nondecreasing for the duration of the selling horizon.

Another variation to the basic single-product Poisson demand problem that has received some attention is the case where there is a finite set of predetermined prices $\{p_1, \dots, p_k\}$ from which the seller can choose. Gallego and van Ryzin [36] discuss this issue and show that the deterministic solution, which involved using at most two different prices from the list, is again asymptotically optimal as the initial capacity and selling horizon increase. Independently, Chatwin [20] and Feng and Xiao [31] provide a systematic analysis of the pricing policy and value function for the problem with a finite set of prices. In these papers it is shown that the value function is concave on both the initial inventory and duration of the selling horizon and that the optimal price is nonincreasing in the inventory and decreasing in the time remaining. An upper bound on the maximum numbers of price changes is also reported. In addition, Feng and Xiao [31] show that there is a maximal subset $\mathcal{P}_0 \subseteq \{p_1, \dots, p_k\}$ such that the revenue rate is increasing and concave within $\mathcal{P}_0$ and the optimal price at any time belongs necessarily to $\mathcal{P}_0$. This observation is particularly useful since it narrows down the set of potential optimal prices making the computation of the optimal pricing strategy much easier. Feng and Xiao [30] impose the additional constraint that prices have to change monotonically and both the markdown and markup cases are considered.

The stochastic version of Kalish [44] is studied by Raman and Chatterjee [67]. Specifically, the authors consider a discounted infinite horizon problem where cumulative demand $D(t)$ follows a stochastic differential equation

$$dD(t) = f(D(t), p(t)) dt + \sigma(D(t)) dw(t). \quad (31)$$

$f(D, p)$ is a deterministic function of cumulative sales and price and $w(t)$ is a Wiener process. For this formulation, Raman and Chatterjee derive the Hamilton-Jacobi-Bellman optimality equation and show that for the linear demand case, $f(D, p) = a - bp$, the optimal price strategy is linearly decreasing in D and monotonically increasing in the demand uncertainty (σ). Similar results are derived for two alternative demand formulations: the multiplicatively separable demand function and the simple price-timing model.

Inspired by the results in Gallego and van Ryzin [36] related to the deterministic fixed-price heuristic, we conclude this single product section extending their results to the discrete time formulation. Specifically, we consider the case of a periodic review model with N periods. In each period $n = 1, \dots, N$ a fixed price p_n is charged and $D_n(p_n)$ is the corresponding (random) demand. Let $\{p_n^{\text{det}} : n = 1, \dots, N\}$ be the optimal deterministic solution, *i.e.*, the solution to the following problem.

$$V_1^{\text{det}}(C_0) = \max_{p_1, \dots, p_N} \sum_{n=1}^N p_n E[D_n(p_n)] \quad (32)$$

$$\text{subject to} \quad \sum_{n=1}^N E[D_n(p_n)] \leq C_0. \quad (33)$$

In order to ensure feasibility, we assume that there is a price p^∞ such that $\sum_{n=1}^N E[D_n(p^\infty)] < C_0$. The following result provides an estimate of the quality of the deterministic solution. We define $\mathcal{D}_n^{\text{det}} := \sum_{i=1}^n D_i(p^{\text{det}})$ to be the cumulative demand up to period n and σ_n^2 to be the variance of $\mathcal{D}_n^{\text{det}}$. We also define $\eta_n^{\text{det}}(C_0)$ as follows.

$$\eta_n^{\text{det}}(C_0) := \frac{\sqrt{\sigma_n^2 + (C_0 - E[\mathcal{D}_n^{\text{det}}])^2} - (C_0 - E[\mathcal{D}_n^{\text{det}}])}{2}. \quad (34)$$

Proposition 8 *Suppose that the deterministic problem (32)-(33) is feasible with concave objective and convex feasible region. Then, the optimal expected revenue $V_1(C_0)$ is bounded above by the deterministic solution, that is, $V_1(C_0) \leq V_1^{\text{det}}(C_0)$. In addition, let $V_1(p^{\text{det}}, C_0)$ be the expected revenue that is obtained using the deterministic prices $\{p_n^{\text{det}} : n = 1, \dots, N\}$. Then, we have that*

$$1 \geq \frac{V_1(p^{\text{det}}, C_0)}{V_1(C_0)} \geq \frac{1}{V_1^{\text{det}}(C_0)} \sum_{n=1}^N p_n^{\text{det}} E[D_n(p_n^{\text{det}})] \left(1 - \frac{\eta_n^{\text{det}}(C_0)}{E[D_n(p_n^{\text{det}})]} \right) \geq 1 - \max_n \frac{\eta_n^{\text{det}}(C_0)}{E[D_n(p_n^{\text{det}})]}. \quad (35)$$

Finally, in the time homogeneous case $E[D_n(p)] = T_n \lambda(p)$, where $\lambda(p)$ is time invariant demand intensity and T_n is the duration of period n , a single price p^{det} solves (32)-(33) and

$$1 \geq \frac{V_1(p^{\text{det}}, C_0)}{V_1(C_0)} \geq 1 - \frac{\nu(C_0)}{2}, \quad (36)$$

where $\nu(C_0)$ is coefficient of variation of $\mathcal{D}_N^{\text{det}}$ the cumulative demand over the entire selling horizon.

Proof: See the appendix. We note that if there is no uncertainty on the demand, then the bounds are tied. Proposition 8 shows that it is the coefficient of variation of the demand and not the variance that regulates the quality of the deterministic price heuristic. For instance, consider the time homogeneous case where $E[D_n(p^{\text{det}})] = T_n \lambda(p^{\text{det}})$ and $\text{Var}(D_n(p^{\text{det}})) = T_n \sigma^2(p^{\text{det}})$ [§]. In this case, the results in proposition 8 imply that

$$1 \geq \frac{V_1(p^{\text{det}}, C_0)}{V_1(C_0)} \geq 1 - \frac{\nu^{\text{det}}}{2\sqrt{T}} \quad \text{where} \quad \nu^{\text{det}} = \frac{\sigma(p^{\text{det}})}{\lambda(p^{\text{det}})} \quad \text{and} \quad T = \sum_{n=1}^N T_n. \quad (37)$$

This result says that the relative error of the deterministic solution is proportional to the inverse of the square root of the selling horizon T . A similar result is reported by Gallego and van Ryzin [36] for the case of a Poisson process.

3.2.2 Multiple Product Case

Similar to the deterministic case, the research on optimal pricing policy in a multiple product setting is narrower in scope than the single product counterpart. There is an important case, however, that has received considerable attention especially in the Yield Management literature on airline seat control. The setting is that of a single resource that is used to satisfy a set of different demand

[§]Stochastic processes like the Poisson process or the Brownian motion with stationary and independent increments satisfy these assumptions.

classes. The example of airline seats is of course one where companies use the same resource to accommodate leisure and business travellers. Other applications are in the management of hotel rooms or bandwidth capacity of an Internet provider. In order to price the same units of capacity differently, companies need to identify characteristics that differentiate their potential customers and classify them accordingly. For example, airlines differentiate leisure passengers from business passengers according to their willingness to stay a Saturday night at their destinations.

Static Price Models:

Much of the research on this problem takes a static view of the price. Specifically, prices or fares for the different classes are predetermined and the seller's problem is to accept or reject the incoming requests of customers dynamically as functions of time, unsold capacity, and the type (fare). We can still, however, view this problem as a dynamic pricing problem. In fact, for each class j the seller has two possible prices to set at each moment of time. One is the fixed and predefined price p_j and the other is a high price $p_j^{\text{null}} \gg p_j$, the *null price*, for which class- j demand at that price is almost surely zero. The null price can be thought of as an artificial way to model customers' rejections in the context of pricing policies.

For the sake of space, we do not provide here an exhaustive review of the literature that has addressed this problem. Readers are referred to the survey papers by Belobaba [7] and McGill and van Ryzin [56] for a more systematic review of the inventory (seat) control literature.

The general multiple product version of this problem can be stated as follows.

1. **Data:** An initial m -dimensional vector of resource C_0 , a menu of product $\mathcal{M}$ having fixed and predetermined prices p_j ($j = 1, \dots, \mathcal{M}$), and a consumption matrix $A = [a_{ij}]$ such that a_{ij} is the amount of resource i consumed by one unit of product j .
2. **Problem:** Find an admission/rejection policy $u(t, C_t, j) \in \{0, 1\}$ such that if the current inventory at time t is C_t and there is a customer demanding product j then $u = 1$ if the order is accepted and $u = 0$ if the order is rejected.

In our pricing setting, given an inventory C_t at time t we can redefine step 2 as follows: charge price p_j for product j if $u(t, C_t, j) = 1$ otherwise charge price p_j^{null} . The equivalence between rejections and null prices implicitly assumes that the effects on the demand process of rejecting customers or charging the null prices are the same.

As we already mentioned, much of the results on this problem have been obtained for the single resource (or single leg) case using (i) a continuous time formulation with Poisson demand processes (e.g. Liang [51], Zhao and Zheng [88], Feng and Xiao [32]-[33]), or (ii) a discrete time model with arbitrary demand but with the restriction that at most one customer can arrive per period (e.g., Lee and Hersh [50], Subramanian *et al.* [76]). Let $V_t(C)$ be the optimal value function at time t if the current inventory is C and let $\beta_t(C) = V_t(C) - V_t(C - 1)$ be the corresponding *bid price* function. Then, it is optimal to sell a unit of product j at time t if and only if $p_j \geq \beta_t(C)$ in the continuous time case or if $p_j \geq \beta_{t-1}(C)$ in the discrete time version. Otherwise, the null price p_j^{null} (i.e., rejection) is recommended. If we assume that for all j $p_j^{\text{null}} > \beta_t(C)$ for all t and C , then the optimal pricing policy for class j at time t , given an inventory C , is to set the minimum price $p \in \{p_j, p_j^{\text{null}}\}$ such that $p \geq \beta_t(C)$. This property of the optimal solution, which follows directly from the Bellman equation, emphasizes the importance of the opportunity cost of capacity described by the bid prices. Not surprisingly, then, much of the research has focused on characterizing the main properties of $\beta_t(C)$ such as its monotonicity, i.e., nonincreasing in both t and C . In this single resource case, these properties imply a *nested* structure of the optimal policy.

That is, if it is optimal at time t to accept class j with price p_j then it is optimal to accept any class k with $p_k \geq p_j$.

The multi resource problem (or network Revenue Management) does not differ much from the single resource case in terms of optimality condition. The main difference is that bid prices in this case are dependent on the class of the demand. More specifically, if at time t the current vector of capacity is C and there is a request for product j and if we accept the offer we collect a price p_j and inventory decreases to a new level $C - A_{.j}$, where $A_{.j}$ is the j^{th} column of the consumption matrix A , *i.e.*, the amount of resources used by product j . Thus, if $V_t(C)$ is the value function at time t when the inventory is C then it would be optimal to accept a product j request if $p_j \geq V_t(C) - V_t(C - A_{.j}) := \beta_t(C, j)$. Network Revenue Management models have been studied by Glover *et al.* [40], Talluri and van Ryzin [77]-[78], You [86], de Boer *et al.* [17], Günther *et al.* [41], Bertsimas and Popescu [10], Feng and Lin [34], Cooper [22]. In general, in these papers[†] the seller does not control prices but he/she is only able to accept or reject incoming orders based on their type and the bid prices $\beta_t(C, j)$.

The problem of computing, or estimating, bid prices has been addressed from different angles. The first, and probably most straightforward, approach is to use a deterministic formulation to estimate the value function. In particular, if the prices for the different products $\{p_j\}$ are fixed, then the static deterministic version of our Revenue Management problem (9)-(12) is given by

$$V_t^{\text{det}}(C) = \max_x \sum_j p_j x_j \quad (38)$$

$$\text{Subject to} \quad Ax \leq C \quad (39)$$

$$0 \leq x \leq E[D(t, T)], \quad (40)$$

where $E[D(t, T)]$ is the expected cumulative demand vector from t to the end of the horizon in T . Williamson [84] was one of the first to compute bid prices using the deterministic network flow formulation above. In particular, the dual variables associated with constraint (39) represent the marginal change in $V_t^{\text{det}}(C)$ as a function of the RHS vector C . Thus, if we call δ the vector of dual variables, it is possible to approximate $\beta_t(C, j) = V_t(C) - V_t(C - A_{.j})$ by $\delta' A_{.j}$. Regarding the quality of this approximation, we can first note that δ captures only the marginal change of $V_t^{\text{det}}(C)$ with respect to C , that is, $\delta = \nabla V_t^{\text{det}}(C)$. Thus, if the solution to (38)-(40) is sensitive to the values of C then we should expect $\delta' A_{.j}$ to be a bad approximation for $\beta_t(C, j)$. This problem, however, can be easily corrected by computing directly the difference $V_t^{\text{det}}(C) - V_t^{\text{det}}(C - A_{.j})$ solving two times the deterministic problem (38)-(40) for the levels C and $C - A_{.j}$ of the capacity (*e.g.*, Bertsimas and Popescu [10]). In addition to this, the deterministic solution relies only on the expected value of the demand. This is clearly another source of error especially when demand variability is high. Talluri and van Ryzin [78] use a randomized version of (38)-(40) to solve this problem. Specifically, they propose to estimate the value of δ simulating N independent values of $D(t, T)$, for each value they compute the corresponding dual vector and finally estimate δ as the average of the different dual vectors obtained from the simulations (a Monte Carlo simulation approach is also proposed in Bertsimas and Popescu [10]).

Dynamic Price Models:

Dynamic pricing models for network Revenue Management have been studied by Gallego and van Ryzin [37]. Using a Poisson demand formulation, the authors derive the HJB equation similar to

[†]One exception is the paper by Gallego and van Ryzin [37].

(27). For this multiple product case, the optimality condition is given by

$$\frac{\partial V_t(C_t)}{\partial t} = \sup_p \left\{ \sum_{j=1}^n \lambda_j(p) [p_j - (V_t(C_t) - V_t(C_t - A_{\cdot j}))] \right\}, \quad (41)$$

where $\lambda_j(p)$ is the demand intensity for product j for a vector of prices p . Closed-form solutions for this differential equation are rarely available. For this reason, Gallego and van Ryzin study two heuristics based on the deterministic solution. Suppose that $p^{\det}(t)$ and $\lambda^{\det}(t)$ are the optimal price path and demand intensity, respectively, for the deterministic version of the problem, *i.e.*, the problem that results from replacing the stochastic demand by its expected value. The value of $p^{\det}(t)$ can be obtained using the results in section §3.1.2. The first heuristic, Make-to-Stock policy, computes the expected number of orders for product j , say z_j , under the deterministic solution. That is,

$$z_j = \left\lfloor \int_0^T \lambda_j^{\det}(t) dt \right\rfloor \quad (42)$$

and it preassembles in advance exactly z_j units of product j at the beginning of the selling period. The price of the products is set to be the deterministic solution $p^{\det}(t)$. In this Make-to-Stock heuristic, once the z_j units of product are built there is no future rearrangement of capacity. The second heuristic, Make-to-Order policy, also follows the deterministic price path, $p^{\det}(t)$, but it does not assemble any product in advance. Rather, this policy waits for orders to arrive and builds products as they are requested. Gallego and van Ryzin show that these two heuristics are asymptotically optimal as the expected sales volume goes to infinity. They also extend their model to include overbooking and no-shows.

4 Summary of Results and New Directions

In terms of results, much of the research on dynamic pricing has focused on the problem faced by a monopolist selling a single product and having perfect information about the demand distribution. The deterministic solution of this problem, where demand is replaced by its expected value, can be obtained using standard optimization techniques (section §3.1) and represents a good approximation for the more complicate stochastic version (proposition 8), especially for high-volume demand products having long selling horizons. On the other hand, the exact analysis of the stochastic case and its optimal pricing strategy requires the solution of the Hamilton-Jacobi-Bellman equation (27). This is in general a complex differential equation for which closed form solutions are rarely available; one exception is the case of exponential demand intensity (see Kincaid and Darling [45]). From the HJB equation, the difference $V_t(C_t) - V_t(C_t - 1)$, which represents the opportunity cost of selling one unit of product at time t when the available inventory is C_t , is the key component that has to be computed in order to determine the optimal pricing policy. Not surprisingly, the approximation of this opportunity cost has been the departing point for many approximations and heuristics like the well-known EMSR method proposed by Belobaba [6] for airline seat control. Static single price formulations, like those presented in section §3.2.1, are also sources of good approximations specially when they are used on a rolling-horizon basis (*e.g.*, Bitran *et al.* [16]).

The multiple product dynamic pricing problem has received considerably less attention. Much of the research on this subject has assumed a static view of prices, where the main decision is whether to accept or reject a given customer's request as a function of time, available inventory, and the corresponding fixed price attached to the incoming order. Again, the deterministic version of the

problem (*certainty equivalent control*) has been the starting point for most approximations. Gallego and van Ryzin [37] show that two heuristics based on the deterministic solution are asymptotically optimal as the volume of sales increases. In this multiple product case, the opportunity cost of selling a unit of product j at time t is given by $V_t(C_t) - V_t(C_t - A_{.j})$. Approximations for this quantity form the basis of the so called *bid price* control methods. These bid prices are generally computed at the resource level and they represent the opportunity cost of selling one unit of a resource. Using a deterministic formulation like (38)-(40), bid prices can be interpreted as the shadow prices for the capacity constraint. The connection between bid prices and opportunity costs is studied in Talluri and van Ryzin [77]. In this paper, using a simple example, the authors show that, while bid price policies are generally suboptimal, they are asymptotically optimal as capacity increases (see also Cooper [22]).

As we compare our generic formulation in (9)-(12) with the results presented in section §3 a number of research opportunities become evident. First of all, from the demand side there are at least two important considerations: (i) demand learning and (ii) demand substitution effects. In general, the research that we have presented in this paper assumes that the demand probability distribution is perfectly known to the seller at the beginning of the selling horizon. In practice, this is hardly the case especially for new products for which there are no historical sales data. It would be interesting to extend the dynamic pricing models to capture demand learning as the selling process evolves, for instance using Bayesian updating. In this case, the pricing strategy can be used to control not only revenues but also the speed at which the seller learns about customers' preferences.

On the demand substitution side, we believe there is an opportunity to combine the ongoing efforts in marketing research to understand consumer choice behavior (*e.g.*, Roberts and Lillien [68]) with the modeling and solution techniques that we have presented in this review. In general, the Revenue Management literature has tangentially discussed this connection. Talluri and van Ryzin [79] analyze the yield management problem of capacity control under a generic customer choice model. However, the issue of setting a dynamic pricing strategy for a given choice model remains open. Some results have been obtained using the traditional multinomial logit model.

Closely related to the issue of demand substitution is the problem of product design and bundling. As we briefly mentioned in section §2, the seller owns an initial vector of capacity C_0 that he can use to produce a set of final products. All the research that we have discussed assumes that the set of products is exogenously given. However, in practice the seller has the ability to decide which specific configurations will be available to customers. The optimal decision of which products to offer is strongly connected to the pricing strategy.

The problem of computing an optimal pricing strategy for the multiple product case with periodic price reviews is technically challenging and certainly of practical importance. In this setting, and because of product substitution effects, the whole path of the demand is needed to compute revenues. This technicality makes the solution of the HJB equation particularly hard. In this area, Awad *et al.* [2] have obtained some preliminary results using approximate dynamic programming.

Incorporating rationality on the behavior of customers is another interesting field of research. For instance, how should the seller set the prices if customers act strategically. The natural way to tackle this problem is using a game theoretical approach. For example, Besanko and Winston [11] solve the problem of a seller having unlimited capacity that sets the price of the product during T discrete time periods. In every period each consumer, who has not yet bought the product, decides whether or not to make a purchase at the price posted by the seller. Besanko and Winston solve for the subgame perfect Nash equilibria. A different setting occurs when the seller prefers to conduct an auction to sell the units. Depending on the nature of the initial capacity and the menu of products

offered by the seller the auction might be simple or combinatorial. Vulcano *et al.* [81] investigate the case of a seller having k units to auction during T discrete time periods. Every period a random number of consumers show-up at the auction and are assumed to be independent, in terms of their valuation, of all other customers in the game. Thus, dynamic strategies from the customers' side are not captured by their model. Based on the *revenue equivalent theorem* the authors formulate a dynamic programming model that solves the problem. We believe that incorporating the strategic behavior of consumers –extending the models by Besanko and Winston or Vulcano *et al.*– is an important topic especially for Internet based applications. The Internet offers customers the ability to keep track of the evolution of the selling process and therefore to modify their buying behavior accordingly. In addition, auction mechanisms are particularly suitable for the Internet.

Including market competition is another important extension to the model. For instance, a senior executive of a major retailer chain in Latin America recognizes that price competition among retailers is today the main driver in their selection of a particular pricing policy.

Finally, we conclude this list of potential new directions pointing at the seller's risk neutrality. Essentially all the models that we have discussed assume that the seller is risk neutral. Feng and Xiao [29] is one of the exceptions. However, most product managers in charge of these dynamic pricing policies present some degree of risk aversion. It would be interesting to measure the impact of adding risk aversion to the Revenue Management formulation. Presumably, the quality of the deterministic solution should be even better.

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Appendix

Proof of Proposition 4: We will prove the result for the case of a continuous demand function. The proof of the discrete case is identical. From equation (22) we have

$$V(C_0) = p \int_0^\infty \min\{D, C_0\} f(D, p) dD.$$

Thus, the first order condition is given by

$$\begin{aligned} 0 &= \int_0^\infty \min\{D, C_0\} (f(D, p) + p f_p(D, p)) dD \\ &= \int_0^\infty \min\{D, C_0\} (1 + \varepsilon(D, p)) f(D, p) dD \\ &= E [\min\{D, C_0\} (1 + \varepsilon(D, p))]. \end{aligned}$$

From this last equality the result of the proposition follows directly. ■

Proof of Proposition 5: The value function $V(C_0)$ in (22) can be written as

$$V(C) = p \int_0^C (1 - F(D; p, T)) dD.$$

The first order optimality condition (FOC) is given by $H(C, p, T) := \int_0^C (1 - F(D; p, T) - p F_p(D; p, T)) dD = 0$ and let $p^*(C)$ be the unique solution. Note that $H(D, 0, T) \geq 0$ for all $D > 0$ and so we must have $1 - F(C; p^*(C), T) - p^*(C) F_p(C; p^*(C), T) \leq 0$. In addition, we have that

$$\begin{aligned} H(C+1, p^*(C), T) &= H(C, p^*(C), T) + \int_C^{C+1} (1 - F(D; p^*(C), T) - p^*(C) F_p(D; p^*(C), T)) dD \\ &= \int_C^{C+1} (1 - F(D; p^*(C), T) - p^*(C) F_p(D; p^*(C), T)) dD. \end{aligned}$$

From the increasing difference property of $F(D; p, T)$ on (D, p) it follows that for all $D \geq C$ we have $1 - F(D; p^*(C), T) - p^*(C) F_p(D; p^*(C), T) \leq 1 - F(C; p^*(C), T) - p^*(C) F_p(C; p^*(C), T) \leq 0$. Thus, $H(C+1, 0, T) \geq 0$ and $H(C+1, p^*(C), T) < 0$ and so under the uniqueness assumption of the optimal price we must have $0 \leq p^*(C+1) \leq p^*(C)$.

By similarly arguments, under the increasing difference property of $-F(D, p, T)$ on (p, T) , it is straightforward to show that $H(C, p, T)$ is nondecreasing in T which implies that $p^*(C)$ is increasing in T . ■

Proof of Proposition 7: From condition (27) the optimal price $p_t(C_t)$ satisfies:

$$p_t(C_t) = \operatorname{argmax}_p \{ \lambda (1 - F(p, t))(p - V_t(C_t) + V_t(C_t - 1)) \}.$$

Let us define $R(p, a) = \lambda (1 - F(p, t))(p - a)$. We first note that $R(p, a)$ is supermodular, *i.e.*, $\frac{\partial^2 R(p, a)}{\partial p \partial a} \geq 0$. Therefore, the set of optimizers $p(a) = \operatorname{argmax}_p R(p, a)$ is nondecreasing in a (see for example Theorem 5 in Milgrom and Roberts [59]). Finally, the result follows from the fact that $V_t(C_t) - V_t(C_t - 1) \geq 0$. ■

Proof of Proposition 8: First we note that $V_1^{\det}(C_0)$ define by (32)-(33) is concave in C_0 . This follows directly from the concavity of (32) and the convexity of (33). Let us now show that $V_1(C_0) \leq V_1^{\det}(C_0)$ using induction over N , the number of periods. For $N = 1$, we have that

$$V_1(C_0) = \max_p E[p \min\{D_1(p), C_0\}] \leq \max_p p \min\{E[D_1(p)], C_0\} = V_1^{\det}(C_0),$$

where the inequality follows from Jensen's inequality and the concavity of the function $f(x) = \min\{x, C_0\}$.

Suppose the result holds for the case of $N - 1$ periods. Let us prove the result for N . In fact, let us recall that $V_2(C)$ and $V_2^{\det}(C)$ are the optimal and deterministic value functions for the last $N - 1$ periods. Therefore, we have that

$$\begin{aligned} V_1(C_0) &= \max_{p_1} E[p_1 \min\{D_1(p_1), C_0\} + V_2(C_0 - \min\{D_1(p_1), C_0\})] \\ &\leq \max_{p_1} E[p_1 \min\{D_1(p_1), C_0\} + V_2^{\det}(C_0 - \min\{D_1(p_1), C_0\})] \\ &\leq \max_{p_1} E[p_1 \min\{D_1(p_1), C_0\}] + V_2^{\det}(C_0 - E[\min\{D_1(p_1), C_0\}]), \end{aligned}$$

where the first inequality follows from the step of induction and the second inequality from Jensen's inequality and the concavity of $V_2^{\det}(C)$. Using the definition of $V_2^{\det}(C)$ we have that

$$\begin{aligned} V_1(C_0) &\leq \max_{p_1, \dots, p_N} \left\{ p_1 E[\min\{D_1(p_1), C_0\}] + \sum_{n=2}^N p_n E[D_n(p_n)] \right\} \\ \text{subject to} \quad &E[\min\{D_1(p_1), C_0\}] + \sum_{n=2}^N E[D_n(p_n)] \leq C_0. \end{aligned}$$

Let β be the Lagrangian multiplier for the constraint, then by weak duality we have that

$$V_1(C_0) \leq \min_{\beta \geq 0} \left\{ \beta C_0 + \max_{p_1, \dots, p_N} \left\{ (p_1 - \beta) E[\min\{D_1(p_1), C_0\}] + \sum_{n=2}^N (p_n - \beta) E[D_n(p_n)] \right\} \right\}.$$

We note that for a given β the solution of the maximization ensures that $p_1 \geq \beta$. Thus, by Jensen's inequality and the concavity of $f(x) = \min\{x, C_0\}$ we get

$$V_1(C_0) \leq \min_{\beta \geq 0} \left\{ \beta C_0 + \max_{p_1, \dots, p_N} \left\{ (p_1 - \beta) \min\{E[D_1(p_1)], C_0\} + \sum_{n=2}^N (p_n - \beta) E[D_n(p_n)] \right\} \right\}.$$

In addition, for a given β it is never optimal to choose p_1 such that $E[D_1(p_1)] > C_0$ since increasing p_1 will necessarily increase the objective, therefore we have

$$V_1(C_0) \leq \min_{\beta \geq 0} \left\{ \beta C_0 + \max_{p_1, \dots, p_N} \left\{ (p_1 - \beta) E[D_1(p_1)] + \sum_{n=2}^N (p_n - \beta) E[D_n(p_n)] \right\} \right\} \equiv \mathcal{L}.$$

Finally, we note that right hand side correspond to the dual of the following problem

$$\begin{aligned} V_1^{\det}(C_0) &= \max_{p_1, \dots, p_N} \left\{ \sum_{n=1}^N p_n E[D_n(p_n)] \right\} \\ \text{subject to} & \quad \sum_{n=1}^N E[D_n(p_n)] \leq C_0. \end{aligned}$$

Given the concavity and convexity assumptions and the existence of an interior solution p^∞ , we can apply the strong duality theorem (e.g., Bazaraa *et al.* [5]) to conclude that $\mathcal{L} = V_1^{\det}(C_0)$ which implies $V_1(C_0) \leq V_1^{\det}(C_0)$.

To prove the second part of the proposition, we note that

$$\begin{aligned} V_1(p^{\det}, C_0) &= \sum_{n=1}^N p_n^{\det} E \left[D_n(p_n^{\det}) - \left(D_n(p_n^{\det}) - (C_0 - \mathcal{D}_{n-1}^{\det})^+ \right)^+ \right] \\ &= \sum_{n=1}^N p_n^{\det} E[D_n(p_n^{\det})] \left(1 - \frac{E \left[\left(D_n(p_n^{\det}) - (C_0 - \mathcal{D}_{n-1}^{\det})^+ \right)^+ \right]}{E[D_n(p_n^{\det})]} \right) \\ &\geq \sum_{n=1}^N p_n^{\det} E[D_n(p_n^{\det})] \left(1 - \frac{E \left[(\mathcal{D}_n^{\det} - C_0)^+ \right]}{E[D_n(p_n^{\det})]} \right), \end{aligned}$$

where the inequality follows from $D_n(p_n^{\det}) - (C_0 - \mathcal{D}_{n-1}^{\det})^+ \leq D_n(p_n^{\det}) + -\mathcal{D}_{n-1}^{\det} - C_0 = \mathcal{D}_n^{\det} - C_0$. The result follows now from two observations. First, we have by the previous part that $V_1(C_0) \leq V_1^{\det}(C_0) = \sum_{n=1}^N p_n^{\det} E[D_n(p_n^{\det})]$. Second, using the inequality (*) below we have that $(\mathcal{D}_n^{\det} - C_0)^+ \leq \eta_n^{\det}(C_0)$. Finally, for the time homogenous case, it clear that a single fixed price $p^{\det}$ is optimal for all periods. Then, we have that

$$\begin{aligned} V_1(p^{\det}, C_0) &= p^{\det} E[\min\{\mathcal{D}_N^{\det}, C_0\}] = p^{\det} (\mathcal{D}_N^{\det} - (\mathcal{D}_N^{\det} - C_0)^+) \\ &= p^{\det} E[\mathcal{D}_N^{\det}] \left(1 - \frac{E[(\mathcal{D}_N^{\det} - C_0)^+]}{E[\mathcal{D}_N^{\det}]} \right) \geq V_1^{\det}(C_0) \left(1 - \frac{\nu(C_0)}{2} \right) \\ &\geq V_1(C_0) \left(1 - \frac{\nu(C_0)}{2} \right) \end{aligned}$$

The first inequality is a consequence of the following result by Gallego (1992) (σ_N^2 is the variance of $\mathcal{D}_N(p^{\det})$)

$$\frac{E[(\mathcal{D}_N^{\det} - C_0)^+]}{E[\mathcal{D}_N^{\det}]} \leq \frac{\sqrt{\sigma^2 + (C_0 - E[\mathcal{D}_N^{\det}])^2} - (C_0 - E[\mathcal{D}_N^{\det}])}{2E[\mathcal{D}_N^{\det}]} \leq \frac{\nu(C_0)}{2}, \quad (*)$$

the last inequality follows from the fact that for the deterministic price solution $E[\mathcal{D}_N^{\det}] \leq C_0$. ■