The Term Structure of Real Rates and Expected Inflation*

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Abstract

Changes in nominal interest rates must be due to either movements in real interest rates or expected inflation, or both. We develop a term structure model with regime switches, time-varying prices of risk and inflation to identify these essentially unobserved components of the nominal yield curve. Our full specification has separate regimes in a real factor and inflation and we cannot reject that the real and inflation regimes are independent. We find that the unconditional real rate curve is fairly flat at 1.44%, but slightly humped. In one regime, the real term structure is steeply downward sloping. Conditional means of inflation, but not conditional volatilities, differ significantly across regimes, and inflation is negatively correlated with real rates. We find that expected inflation drives about 80% of the variation of nominal yields at both short and long maturities, but during normal times almost all of the variation of the long term spread is due to movements in expected inflation.

1 Introduction

The real interest rate and expected inflation are two key economic variables; yet, their dynamic behavior is essentially unobserved. A large empirical literature has yielded surprisingly few generally accepted stylized facts. For example, whereas theoretical research often assumes the real interest rate to be constant, empirical estimates for the real interest rate process vary between constancy (Fama, 1975), mean-reverting behavior (Hamilton, 1985), or a unit root process (Rose, 1988). There seems to be more consensus on the fact that real rate variation, if it exists at all, should only affect the short end of the term structure but that the variation in long-term interest rates is primarily affected by shocks to expected inflation (see Mishkin, 1990; and Fama, 1990, among others). However, Pennacchi (1991) finds the exact opposite result.

Our approach differs substantially from the previous literature. First, we use a no-arbitrage term structure model with both (nominal) bond yields and inflation data to identify the real and nominal term structure. This not only increases statistical efficiency but also mitigates the well-known peso problems inherent in rational expectations estimations (see Evans, 1999; and Bekaert, Hodrick and Marshall, 2001, for more discussion). Second, an attractive feature about the model is that it accommodates regime-switching (RS) behavior but still produces closed-form solutions for the term structure of interest rates. This allows inference for both real and nominal yields at all maturities in a manner consistent with no-arbitrage. The empirical case for RS behavior in interest rates is strong (see below), but we go a step beyond the extant literature by attempting to identify the real and nominal sources of the regime switches. Third, the model accommodates flexible time-varying risk premiums crucial for matching time-varying bond premia. (see, for example, Dai and Singleton, 2002).

We briefly describe the related literature in Section 2. Section 3 develops the model, including the derivation of the bond prices implied by the RS term structure model. In Section 4, we briefly describe how to estimate the model with maximum likelihood and detail the specification tests we use to select the best model.

In Section 5, we conduct various specification tests and analyze the parameter estimates of the best model. We find that the best-performing model has separate and independent regimes in real rates and inflation. Surprisingly, the inflation process only has a regime-dependent drift; we cannot reject that inflation volatility remains the same across regimes. However, the real rate has a low mean and low volatility in one regime, while in the other regime the real rate mean and volatility is significantly higher.

Section 6 contains the main economic results. The real rate is negatively correlated with both expected and unexpected inflation. We find that, unconditionally, the term structure of real rates assumes a fairly flat shape around 1.44%, with a slight hump, peaking at a 1-year maturity. However, there is one regime in which the real rate curve is downward sloping. For models with separate inflation and real rate regimes, the regime-dependent expected nominal term structure is never downward sloping. The model matches an unconditional upward sloping nominal term structure by fitting an unconditional upward-sloping term structure of expected inflation. Empirically, nominal interest rates (spreads) do not behave pro-cyclically (counter-cyclically) across the business cycle but our model-implied real rates do. The decompositions of nominal yields into real yields and expected inflation at various horizons indicate that variation in expected inflation explains about 80% of the variation in nominal rates at both short and long maturities. Finally, Section 7 concludes.

2 Related Literature

To better appreciate the relative contribution of our article, we link it to three distinct literatures: (i) the extraction of real rates and expected inflation from nominal yields and realized inflation or inflation forecasts, (ii) the theoretical term structure literature and equilibrium affine models in finance and (iii) the empirical regime-switching literature on interest rates and inflation.

Our approach to identify real rates and expected inflation differs substantially from the previous literature. First, we develop a no-arbitrage term structure model and use (nominal) term structure data, with inflation data, to identify the real and expected inflation components of nominal interest rates. This is in contrast with an early literature that uses neither term structure data, nor a pricing model to obtain estimates of real rates and expected inflation. Mishkin (1981) and Huizinga and Mishkin (1986) simply project ex-post real rates on instrumental variables. Hamilton (1985), Fama and Gibbons (1982) and Burmeister et al. (1986) use a low-order ARIMA model and identify expected inflation and real rates under the assumption of rational expectations using a Kalman filter. Whereas implicit assumptions on the time-series process of the forcing variables in the model and rational expectations are still essential to obtain identification in our model, the use of term structure information with time-varying risk premiums to identify the unobserved components is likely to significantly increase efficiency and mitigate peso problems.

Second, our model accommodates regime switches but still produces closed-form solutions for the term structure of interest rates. The empirical evidence for RS behavior in interest rates is very strong and confirmed in many articles (see, among many others, Hamilton, 1988; Gray, 1996; Sola and Driffill, 1994; Bekaert, Hodrick and Marshall, 2001; and Ang and

Bekaert, 2002a). However, articles that have used term structure information and a pricing model to obtain estimates of real rates and expected inflation have so far ignored RS behavior. This includes papers by Pennacchi (1991), Sun (1992) and Boudoukh (1993) for US data and Buraschi and Jiltsov (2001), Risa (2001), Barr and Campbell (1997), Remolona et al. (1998) and Evans (1998) for UK data. This is curious, because the early literature implicitly demonstrated the importance of accounting for regime or structural changes. For example, the Huizinga-Mishkin (1986) projections are unstable over the 1979-1982 period, and the slope coefficients of regressions of future inflation onto term spreads in Mishkin (1990) are substantially different pre- and post-1979.

Finally, there are a number of articles that have formulated term structure models accommodating regime switches (see Naik and Lee, 1994; Hamilton, 1988; Veronesi and Yared, 1999; Bekaert, Hodrick and Marshall, 2001; Bansal and Zhou, 2002), but all of these models (with the exception of Veronesi and Yared, 1999) are only concerned with nominal interest rate data. Moreover, most of these models are more restrictive than the one we propose below or they do not accommodate closed-form solutions and must use linearizations. The tractability of our proposed model also simplifies estimation in that the likelihood function can be derived and simulation-based estimation methods are unnecessary.

Evans (1999) is probably most closely related to our article. He formulates a dynamic pricing model with regime switches for UK real and nominal yields and inflation. However, he does not obtain closed-term solutions for bond prices and only accommodates two regimes driving all dynamics. The most general model we estimate has two separate regime variables each having two possible realizations, the first variable primarily drives real rates, the second regime variable only affects inflation. Hence, there are four regimes in total. Earlier work has stressed either inflation regimes (Evans and Wachtel, 1993; and Evans and Lewis, 1995) or real interest rate regimes (Garcia and Perron, 1996). In this article, we separately identify the contributions of real and nominal factors to regime changes.

3 The Model

We derive a parsimonious model that accommodates regime-switches and is consistent with the dynamics of both term structure and inflation data. From the term structure literature (see, for example, Dai and Singleton, 2000), we know that affine term structure models need three factors to match term structure dynamics. Therefore, we start with a 3-factor representation of yields. While most term structure studies use only unobservable factors, our first factor is an observed factor, namely inflation, which switches regimes. Ang and Piazzesi (2002) show that incorporating macro-economic factors significantly improves the ability of standard term structure models to fit the dynamics of yields. A second factor represents time-variation in the price of risk. Fisher (1998), Dai and Singleton (2002) and Cochrane and Piazzesi (2002) demonstrate that, in the context of affine models, time-varying prices of risk succesfully capture the dynamics of term premia. Finally, a third factor represents a latent RS term structure factor.

In our first model (Sections 3.1 and 3.2), we accommodate two possible regimes, as is customary in the literature on regime switches in nominal short rates (see, for example, Hamilton, 1988; Gray, 1996; and Ang and Bekaert, 2002a). However, we also consider models with separate regimes for inflation and a real rate factor (Section 3.3). An alternative RS model (Section 3.4) introduces an additional unobserved factor that represents expected inflation. This model generalizes classic ARMA-models of real rates and expected inflation (see Fama and Gibbons, 1982; and Hamilton, 1985).

3.1 The Benchmark Regime-Switching Model

Denote the state variables as $X_t = (q_t \; f_t \; \pi_t)'$, where q_t and f_t are unobserved state variables and π_t is observable inflation, follow the RS process:

$$X_{t+1} = \mu(s_{t+1}) + \Phi X_t + \Sigma(s_{t+1})\varepsilon_{t+1}, \tag{1}$$

where s_t indicates the regime and:

$$\mu(s_t) = \begin{bmatrix} \mu_q \\ \mu_f(s_t) \\ \mu_\pi(s_t) \end{bmatrix}, \quad \Phi = \begin{bmatrix} \Phi_{qq} & 0 & 0 \\ \Phi_{fq} & \Phi_{ff} & 0 \\ \Phi_{\pi q} & \Phi_{\pi f} & \Phi_{\pi\pi} \end{bmatrix}, \quad \Sigma(s_t) = \begin{bmatrix} \sigma_q & 0 & 0 \\ 0 & \sigma_f(s_t) & 0 \\ 0 & 0 & \sigma_\pi(s_t) \end{bmatrix}. \tag{2}$$

While the conditional mean and volatility of f_t and π_t switch regimes, the conditional mean and volatility of q_t does not. The reduced-form process for inflation is quite complex and involves moving average terms. This is important because the autocorrelogram of inflation at the quarterly frequency is relatively well approximated by the autocorrelogram of an ARMA(1,1) process. Note that the mean-reversion of all variables in Φ is not regime-dependent.

The real short rate follows the affine literature by assuming that the short rate is affine in the state variables:

$$r_t = \delta_0 + \delta_1' X_t. \tag{3}$$

The q_t parameter in X_t also determines the price of risk (see below). This means that the time-varying price of risk can directly influence the real rate as it would in any equilibrium model with growth. For example, when risk aversion increases, the real rate of interest rate may either increase or decrease. The real rate may increase because of the increased desire to smooth consumption and borrow from the future, which causes higher interest rates. The real rate may decrease because of the precautionary savings motive. The model also allows for arbitrary correlation between the real rate and inflation. Note that since f_t and π_t in X_t change across regimes, the real rate process (3) also inherits the RS structure of the state variables.

The regime variable $s_t = 1, 2$ follows a Markov chain with transition probability matrix:

$$\Pi = \left[\begin{array}{cc} p_{11} = Pr(s_t = 1 | s_{t-1} = 1) & p_{12} = 1 - p_{11} \\ p_{21} = 1 - p_{22} & p_{22} = Pr(s_t = 2 | s_{t-1} = 2) \end{array} \right]. \quad (4)$$

We denote the stable probabilities of the Markov chain implied by Π as $\pi_i = Pr(s_t = i)$. Finally, we specify the real pricing kernel to take the form:

$$\widehat{m}_{t+1} = \log \widehat{M}_{t+1} = -r_t - \frac{1}{2}\lambda_t(s_{t+1})'\lambda_t(s_{t+1}) - \lambda_t(s_{t+1})'\varepsilon_{t+1} \quad (5)$$

where the prices of risk $\lambda_t(s_t)$ are given by:

$$\begin{aligned} \lambda_t(s_t) &= (\gamma_t \ \lambda(s_t)')' \\ \gamma_t &= \gamma_0 + \gamma_1 q_t \\ &= \gamma_0 + \gamma_1 e_1' X_t, \end{aligned} \quad (6)$$

where e_i represents a vector of zero's with a 1 in the ith position and $\lambda(s_t) = (\lambda_f(s_t) \ \lambda_\pi(s_t))'$. In this formulation, the prices of risk of f_t and π_t change across regimes. The variable q_t controls the time-variation of the price of risk in γ_t in (6) and does not switch regimes.[1]

Our model significantly extends several existing specifications. Naik and Lee (1994) and Landén (2000) present models with constant prices of risk and regime switches where bond prices have affine solutions. Veronesi and Yared (1999) use Liptser and Shiryaev (1979) filtering techniques to obtain closed-form solutions for bond prices, but they can only handle switching in the mean $\mu(s_t)$. Kim and Nelson (1999) comment that since conditional means are hard to pin down, it is switching volatility which largely identifies the regimes. Evans (1999) and Bansal and Zhou (2002) allow switching in the mean reversion parameters, covariances and means, but both articles use linearizations to obtain approximate analytical bond prices. In contrast, in Section 3.5, we show that our specification produces closed-form solutions for bond prices, enabling both efficient estimation and the ability to fully characterize real and nominal yields at all maturities without discretization error.

[1] Allowing γ_t to switch across regimes results in the loss of closed-form solutions for bond prices, which we detail in Section 3.5.

The model has two main caveats. First, Gray (1996), Bekaert, Hodrick and Marshall (2001) and Ang and Bekaert (2002a) show that mean-reversion of the short rate is significantly different across regimes. If we relax this constraint, closed-form bond prices are no longer available.[2]

Second, the model cannot accommodate time-varying transition probabilities. Ang and Bekaert (2002b) document that constant transition probabilities fail to fully account for the non-linear short rate drift and volatility functions estimated by Aït-Sahalia (1996), Stanton (1997) and others. While constant transition probabilities induce some non-linearities, only time-varying transition probabilities with logistic forms (used by Diebold et al., 1994) can reproduce the non-linearities estimated by the non-parametric studies.

Whereas these are important concerns, the numerical difficulties encountered in computing bond prices for these more complex specifications are formidable and the use of term structure information is critical in identifying both the inflation and real rate components in interest rates and the RS parameters. Moreover, our model with a latent term structure factor and a time-varying price of risk, combined with the RS means and variances, is very rich and cannot be identified from inflation and short rate series alone.

3.2 Identification

In a single-regime setting, Dai and Singleton (2000) show that many term structure models with unobserved state variables result in observationally equivalent systems. Hence, restrictions must be imposed on the parameters for identification. In a single-regime Gaussian model, Dai and Singleton show that identification can be accomplished by setting the conditional covariance to be a diagonal matrix and letting the correlations enter through the feedback matrix (Φ), which is parameterized to be lower triangular, which we do here. Note that the process for inflation is influenced by both past inflation, time-varying prices of risk (through q_t) and the term-structure (through f_t).

Since q_t and f_t are latent variables, they can be arbitrarily scaled. Hence, we set $\delta_1 = (\delta_q\, \delta_f\, \delta_\pi)' = (1\, 1\, \delta_\pi)'$ in (3). Setting δ_q and δ_f to be constants allows σ_q and σ_f to be estimated. Because q_t is an unobserved variable, estimating μ_q in (2) is equivalent to allowing γ_0 in (6) or δ_0 in (3) to be non-zero. Hence, q_t must have zero mean for identification. Therefore, we set $\mu_q = 0$, since q_t does not switch regimes. Similarly, because we estimate $\lambda_f(s_t)$, f_t must have zero mean. For identification, we constrain $\mu_f(s_t)$ so that the unconditional mean of f_t is zero.

[2] Dai and Singleton (2003) develop a model where long-term means, mean reversion and variances switch across regimes, with state-dependent transition probabilities, but no-closed form solutions are available for bond prices. Bond prices can still be computed numerically by solving a series of coupled partial differential equations.

Because we only have nominal bonds, it is impossible to identify the inflation risk premium in our model. Articles focusing on US data that attempt to identify the inflation risk premium, such as Buraschi and Jiltsov (2002) and Veronesi and Yared (1999), obtain identification through stylized economic models.[3] Therefore, we follow the literature and set $\lambda_\pi(s_t) = 0$, so that the inflation risk premium is zero.

The resulting model is theoretically identified from the data but we impose two additional restrictions on the benchmark model.[4] First, we set $\Phi_{12} = 0$ in (2). With this restriction, there are, in addition to inflation factors, two separate and easily identifiable sources of variation in interest rates: a regime-switching factor and a time-varying price of risk. Identifying their relative contribution to interest rate dynamics becomes easy with this restriction and it is not immediately clear how a non-zero Φ_{12} would help enrich the model. Second, we set $\gamma_0 = 0$ in (6). Theoretically, affine models allow the identification of $N-1$ prices of risk with the use of N zero-coupon bonds, but empirically, it is very difficult to accurately pin down more than one constant price of risk (see Dai and Singleton, 2000).

To obtain some intuition on identification, consider a two period real bond. The 2-period term spread in an affine model is given by:

$$\widehat{y}_t^2 - r_t = \frac{1}{2}\left(\mathrm{E}_t(r_{t+1}) - r_t\right) - \frac{1}{4}\mathrm{var}_t\left(r_{t+1}\right) + \frac{1}{2}\mathrm{cov}_t\left(\widehat{m}_{t+1}, r_{t+1}\right). \qquad (7)$$

The first term $(\mathrm{E}_t(r_{t+1}) - r_t)$ is an Expectations Hypothesis (EH) term, the second term $\mathrm{var}_t\left(r_{t+1}\right)$ is a Jensen's inequality term and the last term, $\mathrm{cov}_t(\widehat{m}_{t+1}, r_{t+1})$, is the risk premium. In the single-regime affine setting equivalent to our model, this term is given by:

$$\mathrm{cov}_t(-\widehat{m}_{t+1}, r_{t+1}) = \gamma_0\sigma_q + \lambda_f\sigma_f + \gamma_1\sigma_q q_t, \qquad (8)$$

which shows that the effects of γ_0 and λ_f are indistinguishable as they both act as constant terms.

The RS model has a considerably more complex expression for the 2-period real term

[3] The literature using UK indexed gilts often attempts to estimate the inflation risk premium. For the US, similar data is only available from post-1997 in thinly traded markets. Even for the UK, data is available only post-1982.

[4] The more general model actually performs substantially worse than the selected model on the specification tests, but its substantive implications are quite similar. In particular, the variance decompositions of nominal yields are very similar to those implied by our model in Section 6.4.

spread:

$$\begin{aligned}\widehat{y}_t^2(i) - r_t =& \frac{1}{2}(\mathrm{E}_t(r_{t+1}|s_t = i) - r_t) - \frac{1}{2}\left(\gamma_0\sigma_q + \gamma_1\sigma_q q_t\right) \\ &- \frac{1}{2}\log\sum_{j=1}^{K} p_{ij}\exp\Big[-\delta_1'\Big(\mu(j) - \mathrm{E}\left[\mu(s_{t+1})|s_t = i\right]\Big) \\ &+ \frac{1}{2}\delta_1'\Sigma\left(j\right)\Sigma\left(j\right)'\delta_1 + \lambda_f\left(j\right)\sigma_f\left(j\right)\Big],\end{aligned} \tag{9}$$

for K regimes. First, the term spread now switches across regimes, explicitly shown by the dependence of the yield $\widehat{y}_t^2(i)$ on regime $s_t = i$. The EH term $(\mathrm{E}_t(r_{t+1}|s_t = i) - r_t)$ also switches regimes. The time-varying price of risk term, $-\frac{1}{2}(\gamma_0\sigma_q + \gamma_1\sigma_q q_t)$, is the same as in (8) because the process for q_t does not switch regimes. The remaining terms in (9) are highly non-linear, as they involve the log of the sum of an exponential function of regime-dependent terms, weighted by transition probabilities. Within the non-linear expression, the term $\frac{1}{2}\delta_1'\Sigma\left(j\right)\Sigma\left(j\right)'\delta_1$ represents a Jensen's inequality term, which is regime-dependent, and $\lambda_f\left(j\right)\sigma_f\left(j\right)$ represents a RS price of risk term. A new term in (9) that does not have a counterpart in (8) is $-\delta_1'(\mu(j) - \mathrm{E}\left[\mu\left(s_{t+1}\right)|s_t = i\right])$. This is a jump term involving the difference of drifts across regimes. Hence, it is unlikely that adding another constant, γ_0, adds much flexibility to the model.

Finally, we impose one restriction not necessary for identification, but for efficiency gains. The mean level of the real short rate in (3) is determined by the mean level of inflation multiplied by δ_π and the constant term δ_0. We set δ_0 to match the mean of the nominal short rate in the data, improving the fit of the model.

3.3 Incorporating Different Real Rate and Inflation Regimes

The two regime specification in (4) restricts the real rate and inflation to share the same regimes. To incorporate the possibility of different real rate and inflation regimes, we introduce two different regime variables $s_t^r \in \{1, 2\}$ for the f_t process and $s_t^\pi \in \{1, 2\}$ for the inflation process. Because the former is a real factor, we refer to s_t^r as the real rate regime variable. Nevertheless, the reduced form model for the real rate incorporates both the inflation and real rate regimes, since inflation is one of the factors entering the real short rate (3). The δ_π parameter controls how a switch in the inflation regime impacts the real rate. For example, a monetary policy reaction function described by a Taylor-rule implies that high inflation brings about high real rates. Moreover, because f_t enters the conditional mean of inflation, the inflation process is affected by the real rate regime as well. For example, monetary-policy induced increases in real rates could ward off higher inflation next period through a negative $\Phi_{\pi f}$ coefficient.

To incorporate the effects of s_t^π and s_t^r, we define an aggregate regime variable $s_t \in \{1, 2, 3, 4\}$ to account for all possible combinations of $\{s_t^r, s_t^\pi\} = \{(1,1), (1,2), (2,1), (2,2)\}$, following Hamilton (1994). The full transition probability matrix is of dimension 4×4 and has 10 additional parameters relative to the benchmark model. To reduce the number of parameters, we investigate three restricted cases as follows.

Independent Regimes

In the first case, we impose the restriction that the inflation and the real rate regimes evolve independently. In other words,

$$Pr\left[s_{t+1}^r = j, s_{t+1}^\pi = k | s_t^r = m, s_t^\pi = n\right] = Pr\left[s_{t+1}^r = j | s_t^r = m\right] \times Pr\left[s_{t+1}^\pi = k | s_t^\pi = n\right]. \quad (10)$$

This model is very parsimonious, adding only two parameters to the benchmark model. However, the contemporaneous correlation between real rates and inflation regimes is completely determined by the δ_π parameter in (3).

Correlated Real Rate and Inflation Regimes

It is conceivable that real rate regimes are correlated directly with expected inflation regimes. For example, monetary authorities may attempt to sharply increase real rates in response to a hike in expected inflation while focusing on other goals expressed in nominal terms. We consider two cases (Cases A and B) to model correlation between real rate and inflation regimes. We briefly outline these cases, while full details are given in Appendix A.

In Case A, we specify the current real rate regime to depend on the contemporaneous realization of the inflation regime and on the past real rate regime. In this case, we decompose the joint transition probability of real rates and inflation regimes as:

$$\begin{aligned} &Pr\left[s_{t+1}^r = j, s_{t+1}^\pi = k | s_t^r = m, s_t^\pi = n\right] \\ &= Pr\left[s_{t+1}^r = j | s_{t+1}^\pi = k, s_t^r = m, s_t^\pi = n\right] \times Pr\left[s_{t+1}^\pi = k | s_t^r = m, s_t^\pi = n\right] \\ &= Pr\left[s_{t+1}^r = j | s_{t+1}^\pi = k, s_t^r = m\right] \times Pr\left[s_{t+1}^\pi = k | s_t^\pi = n\right] \end{aligned} \quad (11)$$

In the last line, we assume that the past inflation regime does not determine the contemporaneous correlation of the real rate and inflation regime. Mathematically, we assume that $Pr\left[s_{t+1}^r = j | s_{t+1}^\pi = k, s_t^r = m, s_t^\pi = n\right] = Pr\left[s_{t+1}^r = j | s_{t+1}^\pi = k, s_t^r = m\right]$. We also assume that $Pr\left[s_{t+1}^\pi = k | s_t^\pi = n, s_t^r = m\right] = Pr\left[s_{t+1}^\pi = k | s_t^\pi = n\right]$, or that past real rates do not influence future inflation regime realizations. Case A is consistent with monetary policy changing real rates in response to inflation shocks, as is the case in a Taylor-rule.

One short-coming of the Case A specification is that it cannot capture periods when monetary policy has successfully used real rate increases to stave off a regime of high inflation. In Case B, the inflation regime may depend on the previous real rate regime:

$$\begin{aligned} & Pr\left[s^r_{t+1}=j, s^\pi_{t+1}=k | s^r_t=m, s^\pi_t=n\right] \\ & \quad = Pr\left[s^r_{t+1}=j | s^\pi_{t+1}=k, s^r_t=m, s^\pi_t=n\right] \times Pr\left[s^\pi_{t+1}=k | s^r_t=m, s^\pi_t=n\right] \\ & \quad = Pr\left[s^r_{t+1}=j | s^r_t=m\right] \times Pr\left[s^\pi_{t+1}=k | s^r_t=m, s^\pi_t=n\right]. \end{aligned} \tag{12}$$

Here, we assume that $Pr\left[s^r_{t+1}=j | s^\pi_{t+1}=k, s^r_t=m, s^\pi_t=n\right] = Pr\left[s^r_{t+1}=j | s^r_t=m\right]$. Economically, this means that future real rate regimes only depend on current real rate regimes. In contrast, future inflation regimes depend on the stance of the monetary authority's real rate regime as well as the current inflation environment.

3.4 A Regime-Switching Model with Stochastic Expected Inflation

In a final extension, motivated by the ARMA-model literature (see Hamilton, 1985; and references therein), we allow inflation to be composed of a stochastic expected inflation term plus a random shock:

$$\pi_{t+1} = w_t + \sigma_\pi \varepsilon^\pi_{t+1},$$

where $w_t = \mathrm{E}_t\left[\pi_{t+1}\right]$ is the one-period-ahead expectation of future inflation. This can be accomplished in our framework by expanding the state variables to $X_t = (q_t\, f_t\, w_t\, \pi_t)'$ which follow the dynamics of equation (1), except now:

$$\mu(s_t) = \begin{bmatrix} \mu_q \\ \mu_f(s_t) \\ \mu_w(s_t) \\ 0 \end{bmatrix}, \quad \Phi = \begin{bmatrix} \Phi_{qq} & 0 & 0 & 0 \\ \Phi_{fq} & \Phi_{ff} & 0 & 0 \\ \Phi_{wq} & \Phi_{wf} & \Phi_{ww} & \Phi_{w\pi} \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{13}$$

and $\Sigma(s_t)$ is a diagonal matrix with $(\sigma_q\, \sigma_f(s_t)\, \sigma_w(s_t)\, \sigma_\pi(s_t))'$ on the diagonal. Note that both the variance of inflation and the process of expected inflation are regime-dependent. Moreover, past inflation affects current expected inflation through $\Phi_{w\pi}$.

The real short rate and the regime transition probabilities are the same as in the benchmark model (4). The real pricing kernel also takes the same form as (5) with one difference. The regime-dependent part of the prices of risk in equation (6) is now given by:

$$\lambda(i) = (\lambda_f(i)\ \lambda_w(i)\ \lambda_\pi(i))',$$

but we set $\lambda_w(i) = 0$ so that there is no correlation between the real pricing kernel and any inflation shocks.

To gauge the actual contribution of regime switches, we also estimate single-regime counterparts to the benchmark and stochastic expected inflation models. The single-regime equivalent of the benchmark model is a three-factor Gaussian term structure model with one latent variable entering the time-varying price of risk. For the stochastic expected inflation model, the single-regime equivalent is a four-factor model.

3.5 Bond Prices

Despite the non-linearities present in the model structure of equations (1) to (13), it is possible to derive closed-form solutions for both real and nominal yields on zero coupon bonds.

Real Bond Prices

In our model, the real zero coupon bond price of maturity n conditional on regime $s_t = i$, $\widehat{P}_t^n(s_t = i)$, is given by the closed-form expression:

$$\widehat{P}_t^n(i) = \exp(\widehat{A}_n(i) + \widehat{B}_n X_t), \tag{14}$$

where $\widehat{A}_n(i)$ is dependent on regime $s_t = i$, $\widehat{B}_n$ is a $1 \times N$ vector and N is the total number of factors in the model, including inflation. The expressions for $\widehat{A}_n(i)$ and $\widehat{B}_n$ are given in Appendix B. Since the real bond prices are given by (14), it follows that the real yields $\widehat{y}_t^n(i)$ are affine functions of X_t, conditional on regime i:

$$\hat{y}_t^n(i) = -\frac{\log(\widehat{P}_t^n)}{n} = -\frac{1}{n}(\widehat{A}_n(i) + \widehat{B}_n X_t). \tag{15}$$

The technical innovation in deriving (14) is to recognize that the $\widehat{B}_n$ parameter does not switch for two reasons. First, Φ remains constant across regimes. Second, the time-varying price of risk parameter γ_1 also does not switch across regimes. If these parameters become regime-dependent, closed-form bond solutions are no longer possible.

While the expressions for $\widehat{A}_n(i)$ and $\widehat{B}_n$ are complex, some intuition can be gained on how the prices of risk affect each term. The constant price of risk γ_0 enters only the constant term in the yields $\widehat{A}_n(s_t)$, but affects the term in all regimes. A more negative γ_0 causes long maturity yields to be, on average, higher than short maturity yields, as is true in the simplest affine model specification. Hence, γ_0 affects the unconditional average shape of the yield curve. The regime-dependent prices of risk $\lambda(s_t)$ also only affect the $\widehat{A}_n(s_t)$ terms. Unlike the γ_0 term, the $\lambda(s_t)$

term enters non-linearly. However, a non-linear average effect of $\lambda(s_t)$, across the two regimes also moves the real yield curve. In addition, since the $\lambda(s_t)$ terms differ across regimes, $\lambda(s_t)$ also controls the regime-dependent level of the yield curve away from the unconditional shape of the yield curve. Thus, the model can accommodate a potentially upward sloping yield curve in one regime but a downward sloping yield curve in another regime. The prices of risk affect the time-variation in the yields through the parameter γ_1. This term only enters the $\widehat{B}_n(s_t)$ terms. A more negative γ_1 means that long-term yields are higher, and respond more to shocks in the price of risk factor q_t.

Nominal Bond Prices

To compute nominal bond prices and yields, we formulate the nominal pricing kernel in the standard way as $M_{t+1} = \widehat{M}_{t+1} P_{t+1}/P_t$:

$$m_{t+1} = \log M_{t+1} = -r_t - \frac{1}{2}\lambda_t(s_{t+1})'\lambda_t(s_{t+1}) - \lambda_t(s_{t+1})'\varepsilon_{t+1} - e_N' X_{t+1}, \tag{16}$$

where e_N is a vector of zeros with a 1 in the last position, pulling π_t from the $N \times 1$ vector X_t. This allows nominal bond prices to be written as:

$$P_t^n(i) = \exp(A_n(i) + B_n X_t), \tag{17}$$

where the scalar $A_n(i)$ is dependent on regime $s_t = i$ and B_n is an $1 \times N$ vector. Since nominal bond prices are given by (17), it follows that the nominal yields $y_t^n(i)$ are affine functions of X_t, conditional on regime i:

$$y_t^n(i) = -\frac{\log(P_t^n)}{n} = -\frac{1}{n}(A_n(i) + B_n X_t). \tag{18}$$

Appendix C shows that the only difference between the $\widehat{A}_n(i)$ and $\widehat{B}_n$ terms for real bond prices and the $A_n(i)$ and B_n terms for nominal bond prices are due to terms that select inflation from X_t. Positive inflation shocks decrease nominal bond prices. The general solution also allows for an inflation premium, but this is zero under our parameterization.

Examining the one-period nominal yield in detail, we can break the nominal yield into the real rate plus expected inflation implied from bond yields by taking the difference between (18) and (15):

$$y_t^1 = r_t + \pi_{t,1}^e,$$

where expected inflation implied from the nominal bond price, $\pi_{t,1}^e$, over the next period can be shown to be:[5]

$$\pi^e_{t,1}(i) = -\log\left[\sum_{j=1}^{K} p_{ij}\exp\left(-\mu_\pi(j) + \frac{1}{2}\sigma^2_\pi(j) + \sigma_\pi(j)\lambda_\pi(j)\right)\right] + e_N'\Phi X_t. \qquad (19)$$

The last term in the exponential represents the inflation risk premium, which is zero by assumption in our model. The $\frac{1}{2}\sigma^2_\pi(j)$ term is the standard Jensen's inequality term, while $-\mu_\pi(s_t)$ represents the non-linear, regime-dependent part of expected inflation. The last term $e_N'\Phi X_t$ represents the time-varying part of expected inflation, which does not switch across regimes. We label the bond-price implied expected inflation for maturity n as $\pi^e_{t,n}$.

The expected inflation from bond prices, $\pi^e_{t,n}$, is different from $\mathrm{E}_t(\pi_{t+n,n})$, the expected inflation over n periods implied by the dynamics of the factors in (1), where:

$$\mathrm{E}_t(\pi_{t+n,n}) = \frac{1}{n}\mathrm{E}_t(\pi_{t+1} + \cdots + \pi_{t+n}),$$

and $\mathrm{E}_t(\pi_{t+1,1}) \equiv \mathrm{E}_t(\pi_{t+1})$. Long-horizon forecasts of inflation from (1) are given by implied long-horizon forecasts from a RS-VAR. Hence, we refer to $\mathrm{E}_t(\pi_{t,n})$ as the RS-VAR implied inflation forecast. For a horizon of one-period, we can compare $\pi^e_{t,1}$ in (19) with the corresponding expression for $\mathrm{E}_t(\pi_{t+1})$:

$$\begin{aligned}\mathrm{E}_t(\pi_{t+1}|s_t = i) &= e_N'\mathrm{E}[\mu(s_{t+1})|s_t = i] + e_N'\Phi X_t \\ &= \left(\sum_{j=1}^{K} p_{ij}\mu_\pi(j)\right) + e_N'\Phi X_t. \qquad (20)\end{aligned}$$

The constant terms in (19) for $\pi^e_{t,1}$ and (20) for $\mathrm{E}_t(\pi_{t,1})$ are different. The bond-price implied inflation term ($\pi^e_{t,1}$) reflects both a Jensen's inequality term $\frac{1}{2}\sigma^2_\pi(s_t)$ and a non-linear term, driven by taking the log of a sum, weighted by transition probabilities. Because $\exp(.)$ is a convex function, Veronesi and Yared (1999) call this non-linearity effect on the regime-dependent means of inflation $\mu_\pi(s_t)$ a "convexity bias." This also makes $\pi^e_{t,1} < \mathrm{E}_t(\pi_{t+1})$. Note that $\pi^e_{t,1}$ in (19) and $\mathrm{E}_t(\pi_{t+1})$ in (20) have the same time-varying inflation forecast component, $e_N'\Phi X_t$. Hence, for investigating how expected inflation covaries with different variables, for example, real rates, we can look at either $\pi^e_{t,1}$ or $\mathrm{E}_t(\pi_{t+1})$ for a 1-quarter horizon. However, for horizons greater than $n = 1$, $\pi^e_{t,n}$ and $\mathrm{E}_t(\pi_{t+n,n})$ have different time-varying components.

To assess the relative importance of real rates and priced expected inflation, we compute the sample variances for real rates and priced expected inflation and look at the relative contribution

[5] For a single-regime affine model, $\pi^e_{t,1}$ is given by $\left(\mu_\pi - \frac{1}{2}\sigma^2_\pi - \sigma_\pi\lambda_\pi\right) + e_N'\Phi X_t$.

of each component to the variance of the nominal yield (see also Risa, 2002):

$$\tau_{\hat{y}_t^n} = \frac{\text{cov}\left(y_t^n, \hat{y}_t^n\right)}{\text{var}\left(y_t^n\right)} = \frac{\text{var}\left(\hat{y}_t^n\right) + \text{cov}\left(\hat{y}_t^n, \pi_{t,n}^e\right)}{\text{var}\left(y_t^n\right)}$$
$$\tau_{\pi_{t,n}^e} = \frac{\text{cov}\left(y_t^n, \pi_{t,n}^e\right)}{\text{var}\left(y_t^n\right)} = \frac{\text{var}\left(\pi_{t,n}^e\right) + \text{cov}\left(\hat{y}_t^n, \pi_{t,n}^e\right)}{\text{var}\left(y_t^n\right)}. \quad (21)$$

4 Econometrics

4.1 Likelihood Function

To estimate the RS term structure model, we follow standard practice and specify a set of yields that are measured without error to extract the unobserved factors (Chen and Scott, 1993). The other yields are specified to be measured with error and provide over-identifying restrictions for the term structure model. For the benchmark model, we specify the 1-quarter and 20-quarter yields to be measured without error. For the RS model where inflation has a stochastic mean, we additionally specify the 4-quarter yield to be measured without error. Computing the likelihood function is detailed in Appendix D. The likelihood is not simply the likelihood of the yields measured without error multiplied by the likelihood of the measurement errors, which would be the case in a standard affine model estimation. Since we have regime variables, these must be integrated out of the likelihood function. Our model implies a RS-VAR for inflation and yields with complex cross-equation restrictions resulting from the no-arbitrage assumptions.

4.2 Specification Tests

Residual Tests

We report two tests on in-sample scaled residuals ϵ_t of yields and inflation.[6] Following Bekaert and Harvey (1997), we use a GMM test for serial correlation in scaled residuals ϵ_t:

$$E[\epsilon_t\, \epsilon_{t-1}] = 0. \quad (22)$$

We also test for serial correlation in the second moments of the scaled residuals:

$$E[((\epsilon_t)^2 - 1)\,((\epsilon_{t-1})^2 - 1)] = 0. \quad (23)$$

[6] The scaled residuals ϵ_t are not the same as the shocks ε_t in (1). For a variable x_t, the scaled residual is given by $\epsilon_t = (x_t - \text{E}_{t-1}(x_t))/\sqrt{\text{var}_{t-1}(x_t)}$, where x_t are yields or inflation. The conditional moments are computed using our RS model and involve ex-ante probabilities $p(s_t = i|I_{t-1})$.

Moment Tests

A well-specified model should imply unconditional moments close to the sample moments.[7] Because we want to decompose the variation of nominal yields of various horizons into real and expected inflation components, the relative variances and autocorrelograms of term spreads, yields and inflation are of particular importance. To enable comparison across several non-nested models, we introduce the point statistic:

$$H = (h - \bar{h})' \Sigma_h^{-1} (h - \bar{h}), \tag{24}$$

where $\bar{h}$ are sample estimates of unconditional moments, h are the unconditional moments from the estimated model, and Σ_h is the covariance matrix of the sample estimates of the unconditional moments, estimated by GMM (Newey-West, 1987). In this comparison, the moments implied by various models are compared to the data, with the data sampling error Σ_h held constant across the models. The moments we consider are the first and second moments of term spreads and long yields; the first and second moments of inflation; the autocorrelogram of term spreads; and the autocorrelogram of inflation.[8]

5 Estimation Results

We use quarterly frequency data over 1952:Q2 to 2000:Q4. The disadvantages in using monthly inflation data motivates the use of quarterly data. Monthly CPI figures are very seasonal and create a timing problem because prices are collected over the course of the month. The use of a quarterly frequency mitigates both problems. The data on the Consumer Price Index – All Urban Consumers (CPI-U, seasonally adjusted, 1982-84=100) is from the Bureau of Labor Statistics. Our constructed measure of inflation is the quarter on quarter log difference of the CPI. Our bond data are 4-, 12- and 20-quarter maturity zero-coupon yield data from CRSP. The 1-quarter rate is obtained from the CRSP Fama risk-free rate file.

[7] Appendix E shows how to compute conditional and unconditional moments of our model. Timmermann (2000) provides explicit formulae for a similar formulation of (1), except that the conditional mean of X_{t+1} depends on $\mu(s_{t+1}) + \Phi(X_t - \mu(s_t))$ rather than on $\mu(s_{t+1}) + \Phi X_t$. In Timmermann's set-up, $\mathrm{E}(X_t|s_t)$ is trivially $\mu(s_t)$, whereas in our model the computation is more complex. See Appendix E for details.

[8] An alternative way to set up a moment test would be to take into account both the sampling error of the moments in the data and the sampling error of the moments of the model, implied by the uncertainty in the estimates of the parameters. Equation (24) ignores the latter. This allows the same weighting matrix, computed from the data, to be used across different models, similar to Hansen and Jagannathan (1997). If parameter uncertainty is also taken into account, we might fail to reject, not because the model accurately pins down the moments, but because of the large uncertainty in estimating the model parameters.

5.1 Specification Tests

Panel A of Table 1 reports the specification tests for the residuals for six models. Model I is the single regime counterpart of the benchmark RS model, described in Section 3.1, while Model II is the single regime counterpart to the RS model with stochastic inflation described in Section 3.4. Model III represents the benchmark RS model with two regimes, whereas Model IV is the model with two independent real and inflation regime variables. We do not report the tests for the models with correlated regime variables, as the test statistics look almost identical to those for Model IV. Model V (VI) is the stochastic inflation RS model with one regime variable (two independent regime variables). We use this nomenclature throughout the remainder of the paper.

The residuals of the yields and spreads are well behaved for all models. There is slight serial correlation in the residuals of the 5-year spread for Model III and some remaining serial correlation in the second moments for the 1-quarter yield residuals for Model I, but there is not a single 5% rejection of the null of no serial correlation. In stark contrast, only two of the six models pass the specification tests for the inflation residuals: Models IV and VI. All our models feature complex reduced-form inflation models with MA components which can theoretically match this pattern. It is only by introducing two separate regime variables (in models IV and VI) that we can remove the serial correlation in the inflation residuals.

Panels B and C of Table 1 summarize the fit of the models with respect to some salient moments of the data. We turn first to Panel B, which reports the goodness-of-fit tests in (24), with respect to four sets of moments: the mean and variance of the spread and the long rate (recall that all models fit the mean of the short rate by construction in the estimation procedure), the mean and variance of inflation, and the autocorrelogram of the spread and the autocorrelogram of inflation. Model IV is the only model that passes all of the specification tests. This is important, because we need to accurately match the sample moments of yields and inflation in the data in order to make reliable inferences about unobserved real rates and expected inflation.

It is informative to consider how good the fit really is by examining the reported moments in Panel C of Table 1. They reveal a phenomenal fit. Of the 13 moments reported, Model IV comes within 1 standard error of 10 data moments, and is within two standard errors of the remaining 3. The close fit with the moments of the spread is particularly remarkable. All the other models produce some moments that lie outside 2 standard error bounds of the data estimate. The autocorrelogram of inflation is perhaps the most difficult to match. Inflation features a relatively low first order autocorrelation coefficient with very slowly decaying higher-order autocorrelations, and only Model IV produces a near-perfect fit.

5.2 Parameter Estimates

To conserve space, we report the parameter estimates for all the models in the Appendix. Here we focus on Model IV, the benchmark RS inflation model with independent real rate and inflation regimes, which is the best performing model.

Table 2 reports the relative contributions of the different factors driving the short rate, long yield, and term spread and inflation dynamics in the model. The time-varying price of risk factor q_t is relatively highly correlated with both inflation and the short rate, but shows little correlation with the spread. In other words, q_t is a level factor. The RS term structure factor f_t is more highly correlated with the spread, in absolute value, than with either the short rate or inflation. Hence, f_t is a slope factor. The factor f_t is also less variable and less persistent than q_t. Partly because of this, it does not play a large role in the dynamics of the real rate, only accounting for 7% of its variation. The most variable factor, however, is inflation and it accounts for 47% of the variation of the real rate. The model produces a 68% correlation between inflation and the short rate, which matches the data correlation of 70% almost perfectly. The table also reports that the model-implied correlation between the 5-year spread and inflation is -41%, well within two standard errors of the data moment (-37%)

Table 3 reports some parameter estimates for Models III and IV that illustrate the importance of regimes and at the same time characterize what differentiates one regime from another. Panel A reports Model III, where real rates and inflation have the same two regimes. The first real rate regime is characterized by a low f factor mean and low f factor standard deviation. Whereas the means in the two regimes are only significantly different at the 10% significance level, the standard deviations differ substantially from one another both in economic and statistical terms. For the inflation process, both the mean and standard deviations differ across the two regimes, with the difference significant at the 5% level. The first regime has a higher drift of inflation but lower volatility of innovations in inflation. Finally, the prices of risk for the f factor are border-line significantly different, at the 5% level, across the two regimes.

In Panel B of Table 3, we estimate separate independent regimes of real rates and inflation. For the f factor, the inference is the same as in Panel A. However, for the inflation process, there is now no significant difference across regimes in innovation variances. With four regimes, heteroskedasticity in inflation does not seem to pose a major issue. This does not mean that there is no inflation heteroskedasticity in this model. As is well known, the conditional variance of a RS process embeds a jump term due to the difference in conditional means in the two regimes (see Gray, 1996). Moreover, the f factor not only affects the conditional mean of inflation, but also affects the conditional variance, so that there is heteroskedastic inflation across real rate regimes. Regime differentiation is much clearer only from the drifts: the first inflation regime is a high inflation regime while the second inflation regime is a low inflation regime, and the difference in drift is significant at the 1% level. The two-regime estimation assigns inflation to have different volatilities because it cannot differentiate between periods of high and volatile real rates and high and volatile inflation. Finally, the prices of risk for the f factor do not differ significantly across the two regimes, even though the price of risk is negative in the first regime and significantly positive in the second regime. Hence, the addition of the separate regimes for real rates and inflation has important consequences for the behavior of real rates and expected inflation that are not picked up by restricting the estimation to only one regime variable.

We next examine the smoothed regime probabilities over the sample period. We start with the benchmark one regime variable-two regimes model (Model III) in Figure 1, because that model has been the main focus of the RS literature, applied to nominal interest rates. The second regime here is a regime of a high mean-high variability f factor, but low mean inflation factor. The high variability regime occurs from 1973-1975, in 1979-1982 and briefly after the oil shock in the recession of 1975. This is consistent with other studies. For example, Garcia and Perron (1996) find that a high inflation regime prevails during 1973-1982 and that the real rate switches to a more volatile regime after 1973 to the end of their sample in 1986. The second regime also aligns exactly with the monetary targeting period from 1979-1982 found by RS studies on the nominal rate (see, for example, Gray, 1996; Ang and Bekaert, 2002a).

In Figure 2, we plot the smoothed regime probabilities for Model IV, for the two separate and independent regime variables. The top panel graphs the probability of the real rate regime being in the high variability-high mean regime. The 1979-1982 period is still associated with this regime, but it starts a bit later and there is a relapse into this regime around 1985. The period just prior to the 1960 recession, the 4 years before the 1970 recession, and the two years before the 1975 recession and the 1975 recession itself are also classified as being in the high real rate regime. This regime briefly reappears in 1995. Overall the real rate regime variable spends around 21% of the sample period in the high mean-high variability regime.

The inflation regimes look very different, immediately indicating the potential importance of separating the real and inflation regime variables. The inflation regime variable is mostly in regime 1, the high inflation regime, where it spends 78% of the time over the sample. Exceptions include the period right after the 1955 and 1970 recessions, a brief period between the 1980 and 1982 recessions and a more extensive period in the mid-1980's and finally the early 1990's. Some previous attempts at identifying inflation regimes include Evans and Wachtel (1993) and

Evans and Lewis (1995). However, their inflation model is quite different as it features a random walk component in one regime (with no drift) and an AR(1) model in the other. The random walk regime has the most variable innovations and dominates the 1968-1983 period. During that period, their regime variable makes a few dips that coincide rather well with the dips in our regime probability of being in the high inflation regime. However, we classify 1955-1968 as a high inflation regime and they classify this period as a low inflation variability regime. If we study the link between inflation and the regimes more closely, it is perhaps better to characterize the high inflation regime as a "normal regime" and the low inflation regime as a "disinflation regime". The model accommodates rapid decreases in inflation by a transition to the second regime.

In Model IV, the two regime variables are independent. As discussed in Section 3.3, we also estimate RS models with correlated regimes (Cases A and B). The qualitative properties of all three transition probability matrices are largely the same. In particular, all regimes are highly persistent, but the persistence decreases going from regime 1 to regime 4. Given the similarities between the transition probability matrices for Model IV and Cases A and B, it is no surprise that a likelihood ratio test fails to reject independence of the two regime variables from both Cases A and B at the 10% level. Consequently, we focus most of our attention on the independent regime variable model.

6 The Term Structure of Real Rates and Expected Inflation

Several papers estimating RS models on nominal interest rates and spreads find that a high interest rate-high variability regime coincides with periods of low or negative term spreads (see Hamilton, 1988; Sola and Driffill, 1994; and Ang and Bekaert, 2002a). Imposing approximate no-arbitrage restrictions, Bansal and Zhou (2002) also find that the nominal short rate is higher, and their nominal term structure factors are more volatile, in one of their two regimes. In our model, we can decompose this variation of the nominal term structure into its real and inflation components. In this section, we analyze the implications of the benchmark model for the real and nominal term structure, and expected inflation.

We summarize the behavior of real short rates, 1-quarter ahead expected inflation and nominal short rates in Table 4 before examining the term structure of each component in detail. The first regime is a low real rate-high inflation regime, where we spend 77% of the time. In this regime, both real rates and inflation are not very volatile. The second regime has a high mean real short rate (2.20%), combined with low expected inflation (both $\pi^e_{t,1}$ and $\mathrm{E}_t(\pi_{t+1})$ are

2.48%). This is similar to regime 4, except both real rates and expected inflation are much more volatile in regime 4. Regime 3 has slightly higher real rates than regime 1, but is still a low real rate regime. However, both real rate and inflation volatility are high. Regime 3 also has the highest expected inflation mean and volatility.

Hence, we can summarize our regime characteristics as:

		Real Rates	Inflation	% Time
$s_t = 1$	$s_t^r = 1, s_t^\pi = 1$	Low and Stable	High and Stable	77%
$s_t = 2$	$s_t^r = 1, s_t^\pi = 2$	High and Stable	Low and Stable	10%
$s_t = 3$	$s_t^r = 2, s_t^\pi = 1$	Low and Volatile	High and Volatile	11%
$s_t = 4$	$s_t^r = 2, s_t^\pi = 2$	High and Volatile	Low and Volatile	2%

All the levels (low or high) and variability (stable or volatile) are relative statements, so caution must be taken in the interpretation. For example, statistically, the regime-dependent volatility of inflation does not differ across regimes.

Because of the dependence of the real rate on inflation, the terminology "high and low real rate regime" based on the regimes in f turns out to be slightly misleading. In fact, the means of the real rates are driven more by the inflation regime than by anything else. In fact, in regimes 2 and 4, inflation is high but real rates are low. This is because our estimation confirms one of the most robust findings in the literature on real rates and inflation. The coefficient δ_π is estimated to be -0.536 (with a standard error of 0.052), inducing a negative covariance between inflation and real rates. Pennachi (1991), using a two-factor affine model of real rates and inflation, also finds that real rates and inflation are negatively correlated. In contrast, the volatility of both real rates and inflation is driven by the real rate regime.

6.1 The Term Structure of Real Rates

We examine the real term structure in Figure 3. To facilitate comparison with the previous literature, the top panel of Figure 3 graphs the regime-dependent real term structure for the two-regime benchmark model (Model III). Every point on the curve for regime i represents the expected real zero coupon bond yield conditional on regime i, ($\mathrm{E}[\hat{y}_t^n(i)|s_t = i]$).[9] The normal regime has a fairly flat, but slightly humped real term structure with a peak at a maturity around 6 quarters. The second regime has a strongly downward-sloping real yield curve, with a real short rate of 2.7%. Hence, the behavior uncovered by previous RS models seems to be primarily driven by real rate variation. Unconditionally, the term structure is also fairly flat,

[9] It is also possible to compute the more extreme case $\mathrm{E}[\hat{y}_t^n(i)|s_t = i, \forall t]$, that is, assuming that the process never leaves regime i. These curves have similar shapes to the ones shown in the figures but lie at different levels.

but is slightly hump-shaped, starting at a rate of about 1.7%, increasing to just over 1.8% at 4-quarters and then declining to around 1.7% at 20-quarters. Our RS term structure model is very flexible and can easily produce a variety of shapes of the real yield curve, including flat, inverse-humped, upward-sloping or downward-sloping yield curves. Hence, the estimated flat, but slightly humped, shape of the real term structure is not due to any restrictions imposed by the model.

In the bottom panel of Figure 3, we show the real term structure of Model IV. In this model, regime 1 corresponds to the normal low real rate-high inflation regime. Like Model III in the top panel, the real term structure is also fairly flat with a slight hump-shape peaking at 4-quarters. Both regimes 2 and 4, which are the low inflation regimes, have downward sloping real yield curves but they are not as steep as regime 1 for Model III. Finally, regime 3, a low real rate-high inflation and volatile regime, has a very humped, non-linear, real term structure. Unconditionally, the real rate curve does not look very different from that obtained in the two-regime benchmark model but real rates are, on average, 20 to 30 basis points lower.

The top panel of Figure 4 shows that the real rate shows a lot of short-term variation, sometimes decreasing and increasing sharply. Although this could be noise, this variation may have genuine economic causes, for example the action of monetary policy. Note, for example, the sharp decreases of real rates in the 1958 and 1975 recessions and the sharp increases after the two oil shocks. In fact, standard error bands arising from parameter uncertainty (not shown) are very tight. It is also striking that, consistent with the older literature, real rates are generally low from the 1950's until 1980. The sharp increase in the early 1980's up to almost 8% was temporary, but it took until the 1990's before real rates reached the low levels of a little below 2%. This is still slightly higher than the level of the real rate before 1980. In the bottom panel of Figure 4, we graph the 5-year real rate. Not surprisingly, the 5-year real rate shows much less variation, but the same secular effects that drive the variation of the short real rate are visible. Given these patterns, particularly of the real short rate, it is not surprising that the Garcia-Perron (1996) model, which allowed for a finite number of possible ex-ante real rates, provided a reasonable fit to the data (up until the end of their sample in 1986), although it is clear that it misses some important variation and would have a hard time generating the gradual decrease of real rates since the 1980's.

Table 5 reports a number of unconditional characteristics of real yields. The unconditional standard deviation of the real short rate (20-quarter real yield) is 1.60% (0.61%). These moments solidly reject the hypothesis that the real short rate is constant, but at long horizons real yields are much more stable. This is clearly shown by the autocorrelations of the real short rate and 20-quarter real rate, which are 61% and 94%, respectively.

The last three columns of Table 5 show unconditional univariate betas of the real rate with respect to different components of 1-quarter inflation: realized inflation, expected inflation and unexpected inflation. Note that the univariate betas with respect to expected and unexpected inflation are the same for using $\pi^e_{t,1}$ or $\mathrm{E}_t(\pi_{t+1})$ for a 1-quarter horizon because both $\pi^e_{t,1}$ and $\mathrm{E}_t(\pi_{t+1})$ have the same time-varying component of expected inflation only for the 1-quarter horizon. We find that the link between the real rate and realized inflation is negative but the relation is weak, and it becomes weaker for longer maturities. The correlation between real rates and expected inflation is negative, but insignificantly different from zero at short horizons. The negative sign is consistent with previous studies like Fama (1975) and Mishkin (1990, 1991), who examine this relationship without imposing no-arbitrage restrictions. The point estimates of the real rate betas increase with maturity, but remain insignificantly different from zero. However, real rate betas with respect to unexpected inflation are significantly negative at short maturities (-0.488) but insignificantly different from zero at long maturities. When an unexpected shock for inflation hits, it immediately causes the short rate to fall, but it has little effect on the spread. This makes sense if a large part of the variation in inflation shocks is indeed transitory.

6.2 The Term Structure of Expected Inflation

The 1-quarter ahead forecasts of expected inflation produced by the RS-VAR, $E_t(\pi_{t+1})$ rarely exceed 2% in absolute value and are on average -2 basis points, so the model produces unbiased forecast errors. The Root Mean Squared Error (RMSE) of actual versus forecasted expected inflation is 1.9% over the full sample. During NBER recessions, the RMSE is considerably higher at 2.9% but so is actual inflation (at 4.4%). In NBER expansions, the ratio of the RMSE to inflation volatility is actually worse (the RMSE is 1.7% and actual inflation volatility is 2.8%). This is because both the high inflation of the 1975-1980 period and the subsequent disinflation were not entirely expected. We defer a more detailed analysis of the inflation forecasting performance of our model to future work.

Table 6 reports some characteristics of expected inflation ($\pi^e_{t,n}$ and $\mathrm{E}_t(\pi_{t+n,n})$), differentiating the moments across regimes. We focus first on the yield-curve based expected inflation estimates $\pi^e_{t,n}$ to characterize the inflation regimes. The first inflation regime, $s^\pi_t = 1$ is a high inflation regime with a mean of 4.22% as opposed to 2.54% in the second inflation regime. If we look at expected inflation across the real rate regimes, the first real rate regime $s^r_t = 1$ (where the real rate is stable), is associated with lower expected inflation, 3.96%, than the second real rate regime (where the real rate is volatile) at 4.42%. However, we observe an unconditional real rate beta with respect to inflation of close to zero in Table 5. The reason is because the real rate regime $s_t^r = 1$ ($s_t^r = 2$) integrates inflation over $s_t = 1, 2$ ($s_t = 3, 4$), where the regime-dependent means of the real rate are 4.16% and 2.20% (1.50% and 2.28%), producing a very flat unconditional response. Hence, the unconditional real rate beta clouds the regime-dependent relationship.

Perhaps the most striking feature in Table 6 is the upward sloping term structure of $\pi_{t,n}^e$ in all regimes. In particular, the expected inflation spread is 91 basis points in the first inflation regime and 146 basis points in the second inflation regime. Of course, if $\pi_{t,n}^e$ would truly reflect expected inflation, this could not occur. The bottom half of Table 6 reports expected inflation from the RS-VAR, $\mathrm{E}_t(\pi_{t+n,n})$. For this measure of expected inflation, we always approach the unconditional mean of inflation as n increases, in all regimes.

What makes the behavior of $\pi_{t,n}^e$ so different from $\mathrm{E}_t(\pi_{t+n,n})$? Recall that there are three sources of differences between $\pi_{t,n}^e$ and $\mathrm{E}_t(\pi_{t+n,n})$ in our model, illustrated by (19) and (20) for $n = 1$. First, there is the usual Jensen's inequality term due to inflation volatility present in any pricing model with inflation. Second, there is the convexity bias caused by the stochastic nature of $\mu_\pi(s_t)$ across regimes and the non-linearity introduced by the RS model. However, our estimates reveal that the Jensen's bias and the convexity bias are small in magnitude (at most 15 basis points) and so cannot account for the large spread in $\pi_{t,n}^e$. Instead, the upward slope in the term structure of priced expected inflation is due to the correlation between real rates and inflation.

An investor holding a nominal bond is subject to two sources of risk: movements in inflation reducing the value of the nominal payment and movements in real rates. In our model, real rates and inflation are correlated. If real rates increase during periods of low inflation, nominal bond prices are low just when purchasing power is high. This correlation between real rates and inflation arises through two channels in our model: feedback from the real rate factors q_t and f_t to π_t in the companion matrix Φ (equation (1)) and how inflation enters the real rate, through δ_π (equation (3)). It is the feedback in Φ which is mostly responsible for the upward sloping term structure of $\pi_{t,n}^e$. In particular, if Φ is set to be diagonal, the $\pi_{t,n}^e$ curve becomes slightly downward-sloping. An almost flat term structure of $\pi_{t,n}^e$ is produced with a diagonal Φ and a slightly less negative value of δ_π (-0.400) compared to its estimated value of -0.536.

In population, the regime variable is 77% of the time in the first regime (low real rate-high inflation regime), 10% of the time in the second regime (high real rate-low inflation), 11% of the time in the third regime (low real rate-high inflation) and 2% of the time in regime 4 (high real rate-low inflation). In the sample, the actual times spent in each regime are 66%, 16%, 12%, 6%, respectively. Hence, regimes 2 and 4, the low inflation regimes, are visited more often in the sample than in population. This may reflect a mis-specification in the model by assuming that the price of inflation risk is zero. However, it also may reflect a Peso problem, in that people are genuinely expecting higher inflation, which has not occurred often enough during the sample. If we re-estimate our model with positive inflation risk premiums (setting $\lambda_\pi(s_t)$ to be negative), the discrepancy does not disappear, suggesting the Peso problem interpretation.

6.3 Nominal Term Structure

In their two regime model, Bansal and Zhou (2000) show that one regime displays the normal situation of an upward sloping nominal yield curve and the other regime displays a fairly flat nominal yield curve. The top panel of Figure 5 confirms this pattern for Model III, which features only two regimes. However, the much better performing Model IV, shown in the bottom panel of Figure 5, with four regimes, does not produce a downward-sloping or flat nominal yield curve in any regime. Recall that in some regimes the real rate curve is downward sloping, but the upward sloping term structure of expected inflation completely counteracts this effect.

For Model IV, the first regime (low real rate-high inflation regime) displays a nominal yield curve that is nicely upward sloping, with the slope flattening out for longer maturities, matching the unconditional term structure and the data almost exactly. In the second regime, where the inflation drift is lower, the yield curve is steeply upward sloping but rates are lower than in the first regime because of lower expected inflation. In the third regime, the term structure is steeply upward sloping at the short end but then becomes flat and slightly downward sloping for maturities extending beyond 6 quarters. Nominal interest rate are the highest in this regime because of high-expected inflation in this regime, since real rates are about the same level in this regime as they are in regime 1. In the 4th regime, the term structure is J-shaped with rates below the unconditional curve. This is a regime where the real interest rate curve is downward sloping, but at a high level. Inflation, however, is low in this regime, making nominal yields low, and is upward sloping, which starts to counteract the downward real slope at maturities longer than 1 year, causing the slight dip between 1 and 4-quarters.

Interest rates are often associated with the business cycle. According to the conventional wisdom, interest rates are pro-cyclical and spreads counter-cyclical (see, for example, Fama, 1990). Table 7 shows that this is incorrect. In fact, interest rates are overall larger during recessions. However, when we focus on real rates, the conventional story is right. This can only be the case if expected inflation is counter-cyclical. The table shows that this is indeed the case, with expected inflation being strongly counter-cyclically, reaching 5.05% in recessions but only 3.66% in expansions.

One interesting fact that Table 7 illustrates is that recessions are characterized by more uncertainty, in the sense that all interest rates, spreads and inflation are more volatile in recessions than they are in expansions. Whereas these are simply empirical facts, our model shows that this is also the case for real rates, real rate spreads and expected inflation spreads. The bottom part of the table lists the proportions of each regime realized through the whole sample, compared with the proportions of each regime realized in expansions and recessions. The normal regime 1 occurs much more during expansions. In comparison, the volatile real rate and inflation regimes 3 and 4 occur much more often during recessions.

6.4 Variance Decompositions

Table 8 reports the population variance decomposition (21) of the nominal yield into real and expected inflation variation produced by Model IV. We also report the variance decompositions conditional on each regime. The results are striking: expected inflation accounts for 79% of the total variance of nominal rates, with this proportion the same for longer maturities. This is at odds with the folklore wisdom that expected inflation primarily affects long-term bonds (see, among others, Fama, 1975; Mishkin, 1981). However, this result is consistent with the result in Pennacchi (1991), who uses a two-factor affine model (of real rates and inflation), identifying expected inflation from survey data. The decomposition shows little variation across regimes. Expected inflation is slightly less important at short horizons, but the long-horizon decomposition is essentially unchanged.

Looking at the variance decomposition of nominal term spreads, Table 8 shows that, unconditionally, inflation accounts for 52% of the 4-quarter term spread and 83% of the 20-quarter term spread. Hence, for term spread changes, inflation shocks dominate at the long-end of the yield curve. In the normal regime (regime 1), inflation shocks account for almost all (99%) of the movements of the long term spread. In regimes 3 and 4, inflation accounts for relatively little of movements in term spreads. In these regimes, real rates are very volatile, and expected inflation accounts for only 11% of the 4-quarter term spread, increasing to 60% of the 20-quarter term spread. Hence, the conventional wisdom that inflation is more important for the long end of the yield curve holds, not for the level of yields, but for term spreads.

6.5 Extracting Real Rates and Expected Inflation with Other Models

We put a lot of effort in formulating a model that fits the data well along as many dimensions as possible, which led to the choice of Model IV. This effort was grounded in the intuition that the selection of the correct model would have important implications for the identification of real rates and expected inflation. In this section, we demonstrate that this is indeed the case. Table 9 reports an analysis of real rates and expected inflation produced by other models compared to Model IV. On average, all the models produce similar average real rates and expected inflations, with the differences becoming slightly larger for the 20-quarter rates in both cases. The models with 4 regimes produce remarkably close average real rates. Likewise, the average errors are never larger than 12 basis points for the one quarter rates, but the single regime models generate real rates and expected inflation rates over 20 quarter horizons that differ by an average of 50 to 60 basis points from the Model IV rates.

However, in Table 9, it is the range of errors that is most striking, reaching multiple percentage points for many models. In particular, the models with stochastic expected inflation (Models V and VI) have a wide range of quite different predictions for real rates and expected inflation. Not surprisingly, the models that only extended Model IV to a more general transition probability matrix structure (Models IVa and IVb) imply very similar real rate and expected inflation behavior. Perhaps, the best way to summarize the differences is to investigate the RMSE of the real rate and expected inflation predictions of the various models relative to Model IV. For these closely related models, the RMSE never exceeds 10 basis points. This not surprising given that we cannot statistically reject the independence of the real and inflation regimes. The stochastic inflation models V and VI generate RMSE's for both one-quarter real rates and expected inflation in excess of 1.30%. These numbers are slightly smaller for the 20-quarter horizon, where they hover at 56 or 90 basis points. For the other models, the RMSE's are also fairly large.

7 Conclusion

In this article, we develop three and four factor models of the term structure that embed regime switches in both real and nominal factors, and time-varying price of risks. We compare the performance of these models to their affine counterparts. The model that provides the best fit with data is a three-factor model with independent regimes for a real factor and inflation. This four-regime model is substantially different in its implications for the term structure than the standard two-regime model. We cannot reject that the real and inflation regimes are independent, but this is likely due to other non-regime channels between real rates and inflation. For example, inflation enters with a significantly negative coefficient of -0.536 in the real short rate equation.

Because the model fits the data so well, we use it to extract real rates and expected inflation. We find that the real rate curve is fairly flat but slightly humped, with an average real rate of 1.44% and a 20-quarter spread of not even 2 basis points. The real rate has a variability of 1.60% and has an autocorrelation of 61%. In one regime, the real rate curve is even steeply downward sloping. The yield curve of expected inflation implied by bond yields is steeply upward sloping, but this is due to the correlation of real rates and inflation priced into nominal yields. The correlation effect causes long-horizon yield-curve based measures of expected inflation to be larger than expected inflation implied from the factor data-generating process, particularly at long horizons. Hence, expected inflation forecasts from nominal bonds versus real bonds are upward biased.

The standard view that interest rates are pro-cyclical and spreads counter-cyclical is typically based on real economic effects. However, in the data, nominal interest rates are counter-cyclical. Our model generates real rates that are entirely consistent with the standard view. Interestingly, although lower, real rates are substantially more variable in recessions. Finally, our model is not consistent with the folklore wisdom that expected inflation primarily drives long term nominal yields. We find that expected inflation accounts for 80% of the variation in nominal yields at both short and long maturities. However, the standard view re-appears when we investigate spreads rather than levels. Long term spreads are primarily driven by changes in expected inflation, particularly during normal times.

Our work here is only the beginning of a research agenda. First, our model uses term structure information in an efficient way to generate expected inflation. Hence, it is likely that this may be a good candidate as an inflation-forecasting model. Simple approaches that use term structure information without no-arbitrage restrictions to forecast inflation have not proved successful (see Stock and Watson, 2003). Second, our model would allow us to link the often discussed deviations from the Expectations Hypothesis (Campbell and Shiller, 1991, for example) to deviations from the Fisher hypothesis (Mishkin, 1992). Finally, although we have made one step in the direction of identifying the economic sources of regime switches in interest rates, more could be done. In particular, a more explicit examination of the role of business cycle variation and changes in monetary policy as sources of the regime switches is an interesting topic for further research.

Appendix

A Modelling Separate Real Rate and Inflation Regimes

We detail the independent and correlated real rate and inflation regime specifications (Cases A and B) of Section 3.3. The Appendix Table A-4 reproduces the 4×4 transition probability matrices implied by the independent model and Cases A and B.

Independent Regimes

Equation (10) gives rise to a restricted transition probability matrix Π_0:

	$[s_{t+1}=1]$	$[s_{t+1}=2]$	$[s_{t+1}=3]$	$[s_{t+1}=4]$
$[s_t=1]$	$p^r p^\pi$	$p^r(1-p^\pi)$	$(1-p^r)p^\pi$	$(1-p^r)(1-p^\pi)$
$[s_t=2]$	$p^r(1-q^\pi)$	$p^r q^\pi$	$(1-p^r)(1-q^\pi)$	$(1-p^r)q^\pi$
$[s_t=3]$	$(1-q^r)p^\pi$	$(1-q^r)(1-p^\pi)$	$q^r p^\pi$	$q^r(1-p^\pi)$
$[s_t=4]$	$(1-q^r)(1-q^\pi)$	$(1-q^r)q^\pi$	$q^r(1-q^\pi)$	$q^r q^\pi$

(A-1)

Case A

In (11), we parameterize $Pr\left[s^r_{t+1}=j|s^\pi_{t+1}=k, s^r_t=m\right]$ as $p^{\text{“}j\text{”},\text{“}m\text{”}}$, where:

$$j = \begin{cases} A & \text{if } s^r_{t+1}=s^\pi_{t+1}=1 \\ B & \text{if } s^r_{t+1}=s^\pi_{t+1}=2 \end{cases}$$

$$m = \begin{cases} A & \text{if } s^r_t=1 \\ B & \text{if } s^r_t=2. \end{cases}$$

The "j"-component captures (potentially positive) correlation between inflation and real rate regimes. The "m"-component captures persistence in real rate regimes.

With this notation, the transition probability matrix Π_1 assumes the form:

	$[s_{t+1}=1]$	$[s_{t+1}=2]$	$[s_{t+1}=3]$	$[s_{t+1}=4]$
$[s_t=1]$	$p^{AA}p^\pi$	$\left(1-p^{BA}\right)(1-p^\pi)$	$\left(1-p^{AA}\right)p^\pi$	$p^{BA}(1-p^\pi)$
$[s_t=2]$	$p^{AA}(1-q^\pi)$	$\left(1-p^{BA}\right)q^\pi$	$\left(1-p^{AA}\right)(1-q^\pi)$	$p^{BA}q^\pi$
$[s_t=3]$	$p^{AB}p^\pi$	$\left(1-p^{BB}\right)(1-p^\pi)$	$\left(1-p^{AB}\right)p^\pi$	$p^{BB}(1-p^\pi)$
$[s_t=4]$	$p^{AB}(1-q^\pi)$	$\left(1-p^{BB}\right)q^\pi$	$\left(1-p^{AB}\right)(1-q^\pi)$	$p^{BB}q^\pi$

(A-2)

This model has four additional parameters relative to the benchmark model. We can test the null of independent real rate and inflation regimes versus correlated regimes by:

$$H_0 : p^{BA} = 1 - p^{AA} \text{ and } p^{BB} = 1 - p^{AB}.$$

Case B

In (12), we parameterize $Pr\left[s^\pi_{t+1}=k|s^r_t=m, s^\pi_t=n\right]$ as $p^{\text{“}j\text{”},\text{“}m\text{”}}$, where:

$$j = \begin{cases} A & \text{if } s^\pi_{t+1}=s^r_t=1 \\ B & \text{if } s^\pi_{t+1}=s^r_t=2 \end{cases}$$

$$m = \begin{cases} A & \text{if } s^\pi_t=1 \\ B & \text{if } s^\pi_t=2. \end{cases}$$

Then the transition probability matrix Π_2 assumes the form:

$$\begin{array}{c|c|c|c|c|} & [s_{t+1}=1] & [s_{t+1}=2] & [s_{t+1}=3] & [s_{t+1}=4] \\ \hline [s_t=1] & p^r p^{AA} & p^r\left(1-p^{AA}\right) & \left(1-p^r\right)\left(1-p^{BA}\right) & \left(1-p^r\right)p^{BA} \\ \hline [s_t=2] & p^r p^{AB} & p^r\left(1-p^{AB}\right) & \left(1-p^r\right)\left(1-p^{BB}\right) & \left(1-p^r\right)p^{BB} \\ \hline [s_t=3] & \left(1-q^r\right)p^{AA} & \left(1-q^r\right)\left(1-p^{AA}\right) & q^r\left(1-p^{BA}\right) & q^r p^{BA} \\ \hline [s_t=4] & \left(1-q^r\right)p^{AB} & \left(1-q^r\right)\left(1-p^{AB}\right) & q^r\left(1-p^{BB}\right) & q^r p^{BB} \\ \hline \end{array} \qquad \text{(A-3)}$$

We refer to this parameterization as the partially-correlated regimes Case B. We can test Case B against the null of the independent model by:

$$H_0 : p^{BA} = 1 - p^{AA} \text{ and } p^{BB} = 1 - p^{AB}.$$

B Real Bond Prices

Let N_1 be the number of unobserved state variables in the model ($N_1 = 3$ for the stochastic inflation model, $N_1 = 2$ otherwise) and $N = N_1 + 1$ be the total number of factors including inflation. The following proposition describes how our model implies closed-form real bond prices.

Proposition B.1 *Let $X_t = (q_t\, f_t\, \pi_t)'$ or $X_t = (q_t\, f_t\, w_t\, \pi_t)'$ follow (1), with the real short rate (3) and real pricing kernel (5) with prices of risk (6). The regimes s_t follow a Markov chain with transition probability matrix $\Pi = \{p_{ij}\}$. Then the real zero coupon bond price for period n conditional on regime i, $\widehat{P}_t^n(s_t = i)$, is given by:*

$$\widehat{P}_t^n(i) = \exp(\widehat{A}_n(i) + \widehat{B}_n X_t). \qquad \text{(B-1)}$$

The scalar $\widehat{A}_n(i)$ is dependent on regime $s_t = i$ and $\widehat{B}_n$ is a $1 \times N$ vector that is partitioned as $\widehat{B}_n = [\widehat{B}_{nq}\, \widehat{B}_{nx}]$, where $\widehat{B}_{nq}$ corresponds to the q variable and $\widehat{B}_{nx}$ corresponds to the other variables in X_t. The coefficients $\widehat{A}_n(i)$ and $\widehat{B}_n$ are given by:

$$\begin{aligned} \widehat{A}_{n+1}(i) &= -\left(\delta_0 + \widehat{B}'_{nq}\sigma_q\gamma_0\right) + \log\sum_j \pi_{ij}\exp\Bigg(\widehat{A}_n(j) + \widehat{B}_n\mu(j) \\ &\quad - \widehat{B}_{nx}\Sigma_x(j)\lambda(j) + \frac{1}{2}\widehat{B}_n\Sigma(j)\Sigma(j)'\widehat{B}'_n\Bigg) \\ \widehat{B}_{n+1} &= -\delta'_1 + \widehat{B}_n\Phi - \widehat{B}_{nq}\sigma_q\gamma_1 e'_1, \end{aligned} \qquad \text{(B-2)}$$

where e_i denotes a vector of zero's with a 1 in the ith place and $\Sigma_x(i)$ refers to the lower $N_1 \times N_1$ matrix of of $\Sigma(i)$ corresponding to the non-q_t variables in X_t. The starting values for $\widehat{A}_n(i)$ and $\widehat{B}_n$ are:

$$\begin{aligned} \widehat{A}_1(i) &= -\delta_0 \\ \widehat{B}_1 &= -\delta'_1. \end{aligned} \qquad \text{(B-3)}$$

Proof:

We first derive the initial values in (B-3):

$$\begin{aligned} P_t^1(i) &= \sum_j p_{ij}\mathrm{E}_t\left[\widehat{M}_{t+1}|S_{t+1} = j\right] \\ &= \sum_j p_{ij}\exp\left(-r_t - \frac{1}{2}\lambda_t(j)'\lambda_t(j) - \lambda_t(j)'\varepsilon_{t+1}\right) \\ &= \exp\left(-\delta_0 - \delta'_1 X_t\right) \end{aligned} \qquad \text{(B-4)}$$

Hence:

$$\widehat{P}_t^1(i) = \exp(\widehat{A}_1(i) + \widehat{B}_1 X_t),$$

where $A_1(i)$ and B_1 take the form in (B-3).

We prove the recursion (B-2) by induction. We assume that (B-1) holds for maturity n and examine $\widehat{P}_t^{n+1}(i)$:

$$\begin{aligned}
\widehat{P}_t^{n+1}(i) =& \sum_j p_{ij} \mathrm{E}_t \exp\left[-r_t - \frac{1}{2}\lambda_t(j)'\lambda_t(j) - \lambda_t(j)'\varepsilon_{t+1} + \widehat{A}_n(j) + \widehat{B}_n X_{t+1}\right], \\
=& \sum_j p_{ij} \mathrm{E}_t \exp\left[-\delta_0 - \delta_1' X_t - \frac{1}{2}\lambda_t(j)'\lambda_t(j) - \lambda_t(j)'\varepsilon_{t+1} + \widehat{A}_n(j)\right. \\
&\left. + \widehat{B}_n(\mu(j) + \Phi X_t + \Sigma(j)\varepsilon_{t+1})\right] \qquad \text{(B-5)}
\end{aligned}$$

Evaluating the expectation, we have:

$$\begin{aligned}
\widehat{P}_t^{n+1}(i) =& \sum_j p_{ij} \exp\left[-\delta_0 - \delta_1' X_t - \frac{1}{2}\lambda_t(j)'\lambda_t(j) + \widehat{A}_n(j) + \widehat{B}_n\mu(j)\right. \\
&\left. + \widehat{B}_n \Phi X_t + \frac{1}{2}\left(\widehat{B}_n\Sigma(j) - \lambda_t(j)'\right)\left(\widehat{B}_n\Sigma(j) - \lambda_t(j)'\right)'\right] \\
=& \exp\left[-\delta_0 + \left(\widehat{B}_n\Phi - \delta_1'\right)X_t\right] \\
& \times \sum_j p_{ij} \exp\left[\widehat{A}_n(j) + \widehat{B}_n\mu(j) - \widehat{B}_n\Sigma(j)\lambda_t(j) + \frac{1}{2}\widehat{B}_n\Sigma(j)\Sigma(j)\widehat{B}_n'\right] \qquad \text{(B-6)}
\end{aligned}$$

But we can write:

$$\begin{aligned}
\widehat{B}_n\Sigma(j)\lambda_t(j) =& [\widehat{B}_{nq}\ \widehat{B}_{nx}] \begin{bmatrix} \sigma_q(\gamma_0 + \gamma_1 e_1' X_t) \\ \Sigma_x(j)\lambda(j) \end{bmatrix} \\
=& \widehat{B}_{nq}\sigma_q(\gamma_0 + \gamma_1 e_1' X_t) + \widehat{B}_{nx}\Sigma_x(j)\lambda(j). \qquad \text{(B-7)}
\end{aligned}$$

Expanding and collecting terms, we can write:

$$\widehat{P}_t^n(i) = \exp(\widehat{A}_n(i) + \widehat{B}_n X_t),$$

where $\widehat{A}_n(i)$ and $\widehat{B}_n$ take the form of (B-2). ■

C Nominal Bond Prices

Following the notation of Appendix B, let N_1 be the number of unobserved state variables in the model ($N_1 = 3$ for the stochastic inflation model, $N_1 = 2$ otherwise) and $N = N_1 + 1$ be the total number of factors including inflation. The following proposition describes how our model implies closed-form nominal bond prices.

Proposition C.1 *Let $X_t = (q_t\, f_t\, \pi_t)'$ or $X_t = (q_t\, f_t\, w_t\, \pi_t)'$ follow (1), with the real short rate (3) and real pricing kernel (5) with prices of risk (6). The regimes s_t follow a Markov chain with transition probability matrix $\Pi = \{p_{ij}\}$. Then the nominal zero coupon bond price for period n conditional on regime i, $P_t^n(s_t = i)$, is given by:*

$$P_t^n(i) = \exp(A_n(i) + B_n X_t), \qquad \text{(C-1)}$$

where the scalar $A_n(i)$ is dependent on regime $s_t = i$ and B_n is an $N \times 1$ vector:

$$\begin{aligned}
A_{n+1}(i) =& -\left(\delta_0 + B_{nq}'\sigma_q\gamma_0\right) + \log\sum_j \pi_{ij}\exp\left(A_n(j) + (B_n - e_N')\mu(j)\right. \\
&\left. - \left(B_{nx} - e_{N_1}'\right)\Sigma_x(j)\lambda(j) + \frac{1}{2}(B_n - e_N')\Sigma(j)\Sigma(j)(B_n - e_N')'\right) \\
B_{n+1} =& -\delta_1' + (B_n - e_N')\Phi - B_{nq}\sigma_q\gamma_1 e_1', \qquad \text{(C-2)}
\end{aligned}$$

where e_i denotes a vector of zero's with a 1 in the ith place, $A(i)$ is a scalar dependent on regime $s_t = i$, B_n is a row vector, which is partitioned as $B_n = [B_{nq}\, B_{nx}]$, where B_{nq} corresponds to the q variable and $\Sigma_x(i)$ refers to the lower $N_1 \times N_1$ matrix of of $\Sigma(i)$ corresponding to the non-q_t variables in X_t. The starting values for $A_n(i)$ and B_n are:

$$\begin{aligned} A_1(i) &= -\delta_0 + \log \sum_j \pi_{ij} \exp\left(-e_N'\mu(j) + \frac{1}{2} e_N' \Sigma(j) \Sigma(j)' e_N + e_{N_1}' \Sigma_x(j) \lambda(j)\right) \\ B_1 &= -\left(\delta_1' + e_N' \Phi\right). \end{aligned} \tag{C-3}$$

Proof:

We first derive the initial values (C-3) by directly evaluating:

$$\begin{aligned} P_t^1(i) &= \sum_j p_{ij} E_t\left[\widehat{M}_{t+1} | S_{t+1} = j\right] \\ &= \sum_j p_{ij} \exp\left(-r_t - \frac{1}{2}\lambda_t(j)'\lambda_t(j) - \lambda_t(j)'\varepsilon_{t+1} - e_N'\left(\mu(j) + \Phi X_t + \Sigma(j)\varepsilon_{t+1}\right)\right) \\ &= \exp\left(-\delta_0 - \delta_1' X_t - e_N' \Phi X_t\right) \\ &\quad \times \sum_j p_{ij} \exp\left(-e_N'\mu(j) - e_N'\Sigma(j)\varepsilon_{t+1} - \frac{1}{2}\lambda_t(j)'\lambda_t(j) - \lambda_t(j)'\varepsilon_{t+1}\right) \\ &= \exp\left(-\delta_0 - \delta_1' X_t - e_N' \Phi X_t\right) \\ &\quad \times \sum_j p_{ij} \exp\left(-e_N'\mu(j) + \frac{1}{2} e_N'\Sigma(j)\Sigma(j)' e_N + e_N'\Sigma(j)\lambda_t(j)\right). \end{aligned} \tag{C-4}$$

Note that $e_N'\Sigma(j)\lambda_t(j) = e_{N_1}'\Sigma_x(j)\lambda(j)$. Hence:

$$P_t^1(i) = \exp\left(A_1(i) + B_1 X_t\right)$$

where $A_1(i)$ and B_1 are given by (C-3).

To prove the general recursion we use proof by induction:

$$\begin{aligned} P_t^{n+1}(i) &= \sum_j p_{ij} E_t\Big[\exp\left(-r_t - \frac{1}{2}\lambda_t(j)'\lambda_t(j) - \lambda_t(j)'\varepsilon_{t+1} - e_N' X_{t+1}\right) \\ &\qquad \exp\left(A_n(j) + B_n X_{t+1}\right)\Big] \\ &= \sum_j p_{ij} E_t\Big[\exp\Big(-\delta_0 - \delta_1' X_t - \frac{1}{2}\lambda_t(j)'\lambda_t(j) - \lambda_t(j)'\varepsilon_{t+1} + A_n(j) \\ &\qquad + \left(B_n - e_N'\right)\left(\mu(j) + \Phi X_t + \Sigma(j)\varepsilon_{t+1}\right)\Big)\Big] \\ &= \sum_j p_{ij} \exp\Big(-\delta_0 - \delta_1' X_t - \frac{1}{2}\lambda_t(j)'\lambda_t(j) + A_n(j) + \left(B_n - e_N'\right)\mu(j) \\ &\qquad + \left(B_n - e_N'\right)\Phi X_t + \frac{1}{2}\left(\left(B_n - e_N'\right)\Sigma(j) - \lambda_t(j)'\right)\left(\left(B_n - e_N'\right)\Sigma(j) - \lambda_t(j)'\right)'\Big) \\ &= \exp\left(-\delta_0 + \left(\left(B_n - e_N'\right)\Phi - \delta_1'\right) X_t\right) \sum_j p_{ij} \exp\Big(A_n(j) + \left(B_n - e_N'\right)\mu(j) \\ &\qquad - \left(B_n - e_N'\right)\Sigma(j)\lambda_t(j) + \frac{1}{2}\left(B_n - e_N'\right)\Sigma(j)\Sigma(j)\left(B_n - e_N'\right)'\Big) \end{aligned} \tag{C-5}$$

Now note that:

$$
\begin{aligned}
\left(B_n - e_N'\right)\Sigma\left(j\right)\lambda_t\left(j\right) &= \left(B_n - e_N'\right)\left[\begin{array}{c} \sigma_q\left(\gamma_0+\gamma_1 e_1' X_t\right) \\ \Sigma_x\left(j\right)\lambda(j) \end{array}\right] \\
&= \left[\begin{array}{c} B_{nq} \\ B_{nx}-e_{N_1}' \end{array}\right]\left[\begin{array}{c} \sigma_q\left(\gamma_0+\gamma_1 e_1' X_t\right) \\ \Sigma_x\left(j\right)\lambda(j) \end{array}\right] \\
&= B_{nq}\sigma_q\left(\gamma_0+\gamma_1 e_1' X_t\right) + \left(B_{nx}-e_{N_1}'\right)\Sigma_x\left(j\right)\lambda(j) \qquad \text{(C-6)}
\end{aligned}
$$

where $B_n = [B_{nq}\ B_{nx}]$.

Hence, collecting terms and substituting (C-6) into (C-5), we have:

$$P_t^{n+1}\left(i\right) = \exp\left[A_{n+1}\left(i\right) + B_{n+1}X_t\right],$$

where: $A_n(i)$ and B_n are given by (C-2).

Note that the $A_n(i)$ term allows for an inflation premium, captured through $\Sigma_x(s_t)$ and $\lambda(s_t)$, but this is zero under our formulation. ■

D Likelihood Function

We specify the set of nominal yields without measurement error as Y_{1t} ($N_1 \times 1$) and the remaining yields as Y_{2t} ($N_2 \times 1$). There are as many yields measured without error as there are latent factors in X_t. The complete set of yields are denoted as $Y_t = (Y_{1t}'\, Y_{2t}')'$ with dimension $M \times 1$, where $M = N_1 + N_2$. Note that the total number of factors in X_t is $N = N_1 + 1$, since the last factor, inflation, is observable.

Given the expression for nominal yields in (17), the yields observed without error and inflation, $Z_t = (Y_{1t}'\, \pi_t)'$, take the form:

$$Z_t = \mathcal{A}_1(s_t) + \mathcal{B}_1 X_t, \qquad \text{(D-1)}$$

where:

$$\mathcal{A}_1(s_t) = \begin{bmatrix} \mathcal{A}_n(s_t) \\ 0 \end{bmatrix} \qquad \mathcal{B}_1 = \begin{bmatrix} \mathcal{B}_n \\ e_N' \end{bmatrix}, \qquad \text{(D-2)}$$

where $\mathcal{A}_n(s_t)$ is the $N_1 \times 1$ vector stacking the $-A_n(s_t)/n$ terms for the N_1 yields observed without error, and $\mathcal{B}_n$ is a $N_1 \times N$ matrix which stacks the $-B_n/n$ vectors for the two yields observed without error. Then we can invert for the unobservable factors:

$$X_t = \mathcal{B}^{-1}(Z_t - \mathcal{A}_1(s_t)) \qquad \text{(D-3)}$$

Substituting this into (D-1) and using the dynamics of X_t in (1), we can write:

$$Z_t = c(s_t, s_{t-1}) + \Psi Z_{t-1} + \Omega(s_t)\epsilon_t, \qquad \text{(D-4)}$$

where:

$$
\begin{aligned}
c(s_t, s_{t-1}) &= \mathcal{A}_1(s_t) + \mathcal{B}_1\mu(s_t) - \mathcal{B}_1\Phi\mathcal{B}_1^{-1}\mathcal{A}_1(s_{t-1}) \\
\Psi &= \mathcal{B}_1\Phi\mathcal{B}_1^{-1} \\
\Omega(s_t) &= \mathcal{B}_1\Sigma(s_t)
\end{aligned}
$$

Note that our model implies a RS-VAR for the observable variables with complex cross-equation restrictions.

The yields Y_{2t} observed with error have the form:

$$Y_{2t} = \mathcal{A}_2(s_t) + \mathcal{B}_2 X_t + u_t, \qquad \text{(D-5)}$$

where $\mathcal{A}_2$ and $\mathcal{B}_2(s_t)$ follow from Proposition C.1 and u is the measurement error, $u_t \sim N(0, V)$, where V is a diagonal matrix. We can solve for u_t in equation (D-5) using the inverted factor process (D-3). We assume that u_t is uncorrelated with the errors ε_t in (1).

Following Hamilton (1994), we redefine the states s_t^* to count all combinations of s_t and s_{t-1}, with the corresponding re-defined transition probabilities $p_{ij}^* = p(s_{t+1}^* = i | s_t^* = j)$. We re-write (D-4) and (D-5) as:

$$
\begin{aligned}
Z_t &= c(s_t^*) + \Psi Z_{t-1} + \Omega(s_t^*)\epsilon_t, \qquad \text{(D-6)} \\
Y_{2t} &= \mathcal{A}_2(s_t^*) + \mathcal{B}_2 X_t + u_t.
\end{aligned}
$$

Now the standard Hamilton (1989 and 1994) and Gray (1996) algorithms can be used to estimate the likelihood function. Since (D-6) gives us the conditional distribution $f(\pi_t, Y_t^1|s_t^* = i, I_{t-1})$, we can write the likelihood as:

$$
\begin{aligned}
\mathcal{L} &= \prod_t \sum_{s_t^*} f(\pi_t, Y_t^1, Y_t^2|s_t^*, I_{t-1}) Pr(s_t^*|I_{t-1}) \\
&= \prod_t \sum_{s_t^*} f(Z_t|s_t^*, I_{t-1}) f(Y_t^2|\pi_t, Y_{1t}, s_t^*, I_{t-1}) Pr(s_t^*|I_{t-1}) \qquad \text{(D-7)}
\end{aligned}
$$

where:

$$
f(Z_t|s_t^*, I_{t-1}) = (2\pi)^{-(N_1+1)/2}|\Omega(s_t^*)\Omega(s_t^*)'|^{-1/2} \\
\exp\left(-\frac{1}{2}(Z_t - c(s_t^*) - \Psi Z_{t-1})'[\Omega(s_t^*)\Omega(s_t^*)']^{-1}(Y_{2t} - c(s_t^*) - \Psi Z_{t-1})\right)
$$

is the probability density function of Z_t conditional on s_t^* and

$$
f(Y_{2t}|\pi_t, Y_{1t}, s_t^*, I_{t-1}) = \\
(2\pi)^{-N_2/2}|V|^{-1/2} \exp\left(-\frac{1}{2}(Y_{2t} - \mathcal{A}_2(s_t^*) - \mathcal{B}_2 X_t)' V^{-1}(Y_{2t} - \mathcal{A}_2(s_t^*) - \mathcal{B}_2 X_t\right)
$$

is the probability density function of the measurement errors conditional on s_t^*.

The ex-ante probability $Pr(s_t^* = i|I_{t-1})$ is given by:

$$
Pr(s_t^* = i|I_{t-1}) = \sum_j p_{ji}^* Pr(s_{t-1}^* = j|I_{t-1}), \qquad \text{(D-8)}
$$

which is updated using:

$$
\begin{aligned}
Pr(s_t^* = j|I_t) &= \frac{f(Z_t, s_t^* = j|I_{t-1})}{f(Z_t|I_{t-1})} \\
&= \frac{f(Z_t|s_t^* = j, I_{t-1}) Pr(s_t^* = j|I_{t-1})}{\sum_k f(Z_t|s_t^* = k, I_{t-1}) Pr(s_t^* = k|I_{t-1})}
\end{aligned}
$$

E Computing Moments of the Regime-Switching Model

The formulae given here assume that there are K regimes $s_t = 1, \dots K$.

Conditional First Moments $\mathrm{E}(X_t|s_t)$

Starting from (1), and taking expectations conditional on s_{t+1}, we have:

$$
\mathrm{E}(X_{t+1}|s_{t+1}) = \mathrm{E}(\mu(s_{t+1})|s_{t+1}) + \Phi \mathrm{E}(X_t|s_{t+1}) \qquad \text{(E-1)}
$$

To evaluate $\mathrm{E}(X_t|s_{t+1})$ we use Bayes Rule:

$$
\mathrm{E}(X_t|s_{t+1} = i) = \sum_{j=1}^{K} \mathrm{E}(X_t|s_t = j) Pr(s_t = j|s_{t+1} = i). \qquad \text{(E-2)}
$$

The probability $Pr(s_t = j|s_{t+1} = i)$ is the transition probability of the 'time-reversed' Markov chain that moves backward in time. These backward transition probabilities are given by:

$$
Pr(s_t = j|s_{t+1} = i) \triangleq b_{ij} = p_{ji}\left(\frac{\pi_j}{\pi_i}\right),
$$

where $p_{ji} = Pr(s_{t+1} = i|s_t = j)$ are the forward transition probabilities in (4) and $\pi_i = Pr(s_t = i)$ is the stable probability of regime i. Denote the backward transition probability matrix as $B = \{b_{ij}\}$

Using the backward transition probabilities, (E-1) can be rewritten:

$$\mathrm{E}(X_{t+1}|s_{t+1} = i) = \mu(i) + \Phi \sum_{j=1}^{K} \mathrm{E}(X_t|s_t = j) b_{ji}. \tag{E-3}$$

Assuming stationarity, that is $\mathrm{E}(X_{t+1}|s_{t+1} = i) = \mathrm{E}(X_t|s_t = i)$, and defining the $K \times 1$ vectors:

$$\vec{\mathrm{E}}(X_t|s_t) = \begin{bmatrix} \mathrm{E}(X_t|s_t = 1) \\ \vdots \\ \mathrm{E}(X_t|s_t = K) \end{bmatrix} \quad \text{and} \quad \vec{\mu}(s_t) = \begin{bmatrix} \mu(1) \\ \vdots \\ \mu(K) \end{bmatrix},$$

we can write:

$$\vec{\mathrm{E}}(X_t|s_t) = \vec{\mu}(s_t) + \Phi \vec{\mathrm{E}}(X_t|s_t) B'.$$

Hence, we can solve for $\vec{\mathrm{E}}(X_t|s_t)$ as:

$$\vec{\mathrm{E}}(X_t|s_t) = (I - B \otimes \Phi)^{-1} \vec{\mu}(s_t) \tag{E-4}$$

Conditional Second Moments $\mathrm{E}(X_t X_t'|s_t)$

Starting from (1), we can write:

$$X_{t+1}X_{t+1}' = (\mu(s_{t+1}) + \Phi X_t + \Sigma(s_{t+1})\varepsilon_{t+1})(\mu(s_{t+1}) + \Phi X_t + \Sigma(s_{t+1})\varepsilon_{t+1})', \tag{E-5}$$

and taking expectations conditional on s_{t+1}, we have:

$$\begin{aligned} \mathrm{E}(X_{t+1}X_{t+1}'|s_{t+1}) &= \mu(s_{t+1})\mu(s_{t+1})' + \Sigma(s_{t+1})\Sigma(s_{t+1})' \\ &\quad + \mu(s_{t+1})(\Phi \mathrm{E}(X_t|s_{t+1})' + \Phi \mathrm{E}(X_t|s_{t+1})\mu(s_{t+1})' + \Phi \mathrm{E}(X_t X_t|s_{t+1})\Phi'. \end{aligned} \tag{E-6}$$

We can evaluate the term $\mathrm{E}(X_t|s_{t+1})$ from (E-2). Hence, we can define an $N \times N$ matrix $G(i)$:

$$G(i) = \mu(i)\mu(i)' + \Sigma(i)\Sigma(i)' + \mu(i)(\Phi \mathrm{E}(X_t|s_{t+1} = i)' + \Phi \mathrm{E}(X_t|s_{t+1} = i)\mu(i)'. \tag{E-7}$$

Substituting (E-7) into (E-6) and using Bayes' Rule, we have:

$$\begin{aligned} \mathrm{E}(X_{t+1}X_{t+1}'|s_{t+1} = i) &= G(i) + \sum_{j=1}^{K} \Phi[\mathrm{E}(X_t X_t'|s_t = j) Pr(s_t = j|s_{t+1} = i)]\Phi' \\ &= G(i) + \sum_{j=1}^{K} b_{ij} \Phi \mathrm{E}(X_t X_t'|s_t = j)\Phi' \end{aligned}$$

Taking vec's of both sides, we obtain:

$$\mathrm{vec}(\mathrm{E}(X_{t+1}X_{t+1}'|s_{t+1} = i)) = G(i) + (\Phi \otimes \Phi) \sum_{j=1}^{K} \mathrm{vec}(\mathrm{E}(X_{t+1}X_{t+1}'|s_{t+1} = i)) b_{ij} \tag{E-8}$$

If we define the $KN^2 \times 1$ vectors:

$$\vec{\mathrm{E}}(X_t X_t'|s_t) = \begin{bmatrix} \mathrm{vec}(\mathrm{E}(X_{t+1}X_{t+1}'|s_{t+1} = 1)) \\ \vdots \\ \mathrm{vec}(\mathrm{E}(X_{t+1}X_{t+1}'|s_{t+1} = K)) \end{bmatrix} \quad \text{and} \quad \vec{G} = \begin{bmatrix} \mathrm{vec}(G(1)) \\ \vdots \\ \mathrm{vec}(G(K)) \end{bmatrix}$$

we can write (E-8) as:

$$\vec{\mathrm{E}}(X_t X_t'|s_t) = \vec{G} + (\Phi \otimes \Phi)\vec{\mathrm{E}}(X_t X_t'|s_t) B'.$$

Hence, we can solve for $\vec{\mathrm{E}}(X_t X_t'|s_t)$ as:

$$\vec{\mathrm{E}}(X_t X_t'|s_t) = (I_{KN^2} - B \otimes (\Phi \otimes \Phi))^{-1}\vec{G}. \tag{E-9}$$

Unconditional Moments

The first unconditional moment $\mathrm{E}(X_t)$ is solved simply by taking unconditional expectations of (1), giving

$$\mathrm{E}(X_t) = (I - \Phi)^{-1}\sum_{i=1}^{K}\pi_i \mu(i). \tag{E-10}$$

To solve the second unconditional moment $\mathrm{var}(X_t)$, we use:

$$\begin{aligned}\mathrm{var}(X_t) &= \mathrm{E}(X_t X_t') - \mathrm{E}(X_t)\mathrm{E}(X_t)' \\ &= \mathrm{E}(\mathrm{E}(X_t X_t'|s_t)) - \mathrm{E}(X_t)\mathrm{E}(X_t)' \\ &= \sum_{i=1}^{K}\{\mathrm{var}(X_t|s_t=i) + \mathrm{E}(X_t|s_t=i)\mathrm{E}(X_t|s_t=i)'\}\,\pi_i - \mathrm{E}(X_t)E(X_t)'\end{aligned} \tag{E-11}$$

Moments of Yields

Bond yields are affine functions of X_t, from Propositions B.1 and C.1. Hence, they can be written as $Y_t = A + BX_t$ for some choice of A and B. Then, moments of Y_t are given by:

$$\begin{aligned}\mathrm{E}(Y_t|s_t) &= A + B\mathrm{E}(X_t|s_t) \\ \mathrm{var}(Y_t|s_t) &= B\mathrm{var}(X_t|s_t)B'\end{aligned} \tag{E-12}$$

$$\begin{aligned}\mathrm{E}(Y_t) &= A + B\mathrm{E}(X_t) \\ \mathrm{var}(Y_t) &= B\mathrm{var}(X_t)B'\end{aligned} \tag{E-13}$$

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Table 1: Residual and Moment Tests

Panel A: Residual Tests

		Models					
		I	II	III	IV	V	VI
1-qtr yield	Serial Correlation	0.30	0.70	0.35	0.06	0.99	0.24
	Heteroskedasticity	0.26	0.15	0.07	0.09	0.09	0.09
5-year spread	Serial Correlation	0.20	0.30	0.05	0.60	0.83	0.14
	Heteroskedasticity	0.21	0.28	0.14	0.09	0.13	0.06
Inflation	Serial Correlation	0.00**	0.03*	0.02*	0.21	0.01*	0.41
	Heteroskedasticity	0.14	0.11	0.55	0.32	0.90	0.26

Panel B: χ^2 Tests on Moments (p-values)

	Models					
	I	II	III	IV	V	VI
Mean/var spread and long rate	0.48	0.00**	0.00**	0.48	0.00**	0.00**
Mean/var inflation	0.00**	0.19	0.54	0.71	0.76	0.00**
1,5,10 autocorrelations spread	0.25	0.33	0.05	0.42	0.17	0.05
1,5,10 autocorrelations inflation	0.04*	0.00**	0.04*	0.16	0.00**	0.00**

Panel C: Model-Implied and Sample Moments

		Models						Data	
		I	II	III	IV	V	VI	Moment	SE
stdev y_t^1		3.15	3.30†	2.93	3.10	3.32	5.07†	2.86	0.15
mean y_t^{20}		6.46	6.59	6.25	6.46	5.80†	4.71†	6.39	0.20
stdev y_t^{20}		2.60	2.54	2.28†	2.66	2.68	3.62†	2.74	0.14
mean $y_t^{20} - y_t^1$		1.00	1.13†	0.78†	1.00	0.34	-0.75†	0.93	0.07
stdev $y_t^{20} - y_t^1$		1.27†	1.48†	1.52†	1.09	1.55†	2.83†	1.03	0.05
mean π_t		3.68	4.02	3.74	4.03	4.21	5.30	3.88	0.23
stdev π_t		4.87†	3.72†	3.38	3.47	3.37	5.22†	3.19	0.16
Spread Autocorrelations	$\rho(1)$	0.81	0.76	0.65	0.77	0.69	0.66	0.73	0.05
	$\rho(5)$	0.36	0.36	0.36	0.29	0.26	0.33	0.32	0.12
	$\rho(10)$	0.14	0.16	0.17	0.08	0.09	0.16	-0.04	0.12
Inflation Autocorrelations	$\rho(1)$	0.90†	0.49†	0.73	0.80	0.39†	0.61†	0.77	0.06
	$\rho(5)$	0.57	0.27†	0.31†	0.46	0.20†	0.32†	0.59	0.10
	$\rho(10)$	0.34	0.18	0.18	0.34	0.12†	0.20	0.36	0.09

Panel A reports p-values of scaled residual tests in (22) and (23). The first entry reports the p-value of a GMM-based test of $\mathrm{E}(\epsilon_t \epsilon_{t-1}) = 0$ and the second row reports the p-value of a GMM-based test of $\mathrm{E}[(\epsilon_t^2 - 1)(\epsilon_{t-1}^2 - 1)] = 0$. Panel B reports p-values of goodness-of-fit χ^2 tests (equation (24)) for various moments implied by different models. The long rate refers to the 20-quarter nominal rate y_t^{20} and the spread refers to $y_t^{20} - y_t^5$. For each line, the moments specified in the first column are used in the H point-statistic. Panel C reports moments of 5-year spreads and inflation π_t implied by various models, compared with the sample estimates in data and standard errors in the last two columns, computed using GMM with 4 Newey-West (1987) lags. We denote the ith correlation as $\rho(i)$ at a quarterly frequency. Means and standard deviations are in percent. In Panels A and B, p-values less than 0.05 (0.01) are denoted by * (**). In Panel C, moments outside plus or minus two standard errors of the data moment are denoted by †. Models I and II are the single-regime equivalents of the inflation and expected inflation models in Sections 3.1 and 3.4, respectively. Model III denotes the benchmark RS model with two regimes. Model IV denotes the benchmark RS model with separate and independent regimes for real rates and inflation. The results of the inflation model allowing for correlated real rates and inflation regimes (Case I and II) are almost identical to Model IV and so are omitted. Model V denotes the RS expected inflation model with two regimes. Model VI denotes the RS expected inflation model with separate and independent regimes for real rates and inflation.

Table 2: Factor Behavior

	Correlation with Inflation	Short Rate	Spread	Stdev	Auto	Contribution to Real Rate Variance
q	0.59	0.89	-0.12	1.78	0.98	0.47
	(0.12)	(0.06)	(0.05)	(0.62)	(0.01)	(0.35)
f	0.25	0.44	-0.98	0.72	0.75	0.07
	(0.08)	(0.12)	(0.01)	(0.21)	(0.01)	(0.09)
π	1.00	0.68	-0.41	3.47	0.80	0.46
	–	(0.09)	(0.06)	(0.43)	(0.05)	(0.36)

The table reports various unconditional moments of the three factors: the time-varying price of risk factor q_t, the regime-switching factor f_t and inflation π_t, from the benchmark model with independent regimes in real rates and inflation (Model IV). The short rate refers to the 1-quarter nominal yield and the spread refers to the 20-quarter nominal term spread. The numbers between parentheses are standard errors reflecting parameter uncertainty from the estimation, computed using the delta-method. In the last row, the correlation between inflation and the short rate (spread) is 0.67 (-0.37) in the data. The variance decomposition of the real rate is computed as cov$(r_t, z_t)/$var(r_t), with z_t respectively q_t, f_t and $\delta_\pi \pi_t$.

Table 3: Selected Regime-Switching Parameters

	Regime 1	Regime 2	P-value Test of Equality
Panel A: Model III			
$\mu_f(s_t) \times 100$	-0.016	0.096	0.09
	(0.015)	(0.088)	
$\sigma_f(s_t) \times 100$	0.077	0.258	0.00**
	(0.020)	(0.099)	
$\mu_\pi(s_t) \times 100$	0.453	0.199	0.00**
	(0.078)	(0.120)	
$\sigma_\pi(s_t) \times 100$	0.400	0.973	0.03*
	(0.018)	(0.381)	
λ_f	-0.482	-0.005	0.05*
	(0.104)	(0.353)	
Panel B: Model IV			
$\mu_f(s_t^r) \times 100$	-0.006	0.039	0.10
	(0.004)	(0.023)	
$\sigma_f(s_t^r) \times 100$	0.078	0.246	0.00**
	(0.020)	(0.020)	
$\mu_\pi(s_t^\pi) \times 100$	0.435	0.219	0.00**
	(0.079)	(0.095)	
$\sigma_\pi(s_t^\pi) \times 100$	0.479	0.471	0.88
	(0.027)	(0.052)	
λ_f	-0.523	0.335	0.23
	(0.100)	(0.157)	

We report selected parameters from the benchmark model with two regimes (Panel A, Model III) and independent real rate and inflation regimes (Panel B, Model IV). All parameters are unscaled and not annualized. The p-values relate to Wald χ^2 tests of parameter equality across regimes. P-values less than 0.05 (0.01) are denoted by * (**).

Table 4: Real Rates, Expected Inflation, Nominal Rates Across Regimes

		Regime			
		$s_t = 1$	$s_t = 2$	$s_t = 3$	$s_t = 4$
Real Short Rates	Mean	1.31	2.20	1.50	2.39
		(0.43)	(0.53)	(0.40)	(0.50)
	Std Dev	1.54	1.53	1.82	1.81
		(0.23)	(0.27)	(0.28)	(1.03)
1-qtr Priced Expected Inflation $\pi^e_{t,1}$	Mean	4.16	2.48	4.62	2.94
		(0.42)	(0.65)	(0.48)	(0.71)
	Std Dev	2.78	2.78	3.17	3.17
		(0.52)	(0.52)	(0.54)	(0.54)
1-qtr RS-VAR Expected Inflation $\mathrm{E}_t(\pi_{t+1})$	Mean	4.17	2.48	4.62	2.94
		(0.42)	(0.65)	(0.48)	(0.71)
	Std Dev	2.78	2.78	3.17	3.17
		(0.52)	(0.52)	(0.54)	(0.54)
Nominal Short rate	Mean	5.47	4.68	6.12	5.32
		(0.06)	(0.24)	(0.26)	(0.35)
	Std Dev	2.99	2.99	3.72	3.72
		(0.80)	(0.80)	(0.69)	(0.69)

We report means and standard deviations for real short rates, 1-quarter expected inflation and nominal short rates implied by the benchmark model with independent real rate and inflation regimes (Model IV), across each of the four regimes. The regime $s_t = 1$ corresponds to $(s^r_t = 1, s^\pi_t = 1)$, $s_t = 2$ to $(s^r_t = 1, s^\pi_t = 2)$, $s_t = 3$ to $(s^r_t = 2, s^\pi_t = 1)$ and $s_t = 4$ to $(s^r_t = 2, s^\pi_t = 2)$. Standard errors reported in parentheses are computed using the delta-method.

Table 5: Characteristics of Real Rates

	Unconditional Moments			Real Rate Betas		
Maturity Qtrs	Mean	Stdev	Auto-correlation	Realized Inflation	Expected Inflation	Unexpected Inflation
1	1.44	1.60	0.61	-0.18	-0.07	-0.49
	(0.42)	(0.23)	(0.08)	(0.14)	(0.17)	(0.06)
4	1.59	1.00	0.73	-0.07	0.01	-0.26
	(0.43)	(0.26)	(0.13)	(0.11)	(0.16)	(0.05)
12	1.52	0.67	0.89	0.03	0.08	-0.10
	(0.42)	(0.34)	(0.10)	(0.09)	(0.13)	(0.02)
20	1.46	0.61	0.94	0.05	0.10	-0.06
	(0.43)	(0.36)	(0.06)	(0.08)	(0.12)	(0.02)

The table reports various moments of the real rate, implied from the benchmark model IV with independent real rate and inflation regimes We report the unconditional mean, standard deviation and autocorrelation of real yields of various maturity in quarters, and univariate betas of the real rate, with respect to realized inflation, expected inflation and unexpected inflation. Standard errors reported in parenthesis are computed using the delta-method.

Table 6: Regime-Dependent Means of Expected Inflation

Qtrs	Real Rate Regimes $s_t^r = 1$	$s_t^r = 2$	Inflation Regimes $s_t^\pi = 1$	$s_t^\pi = 2$	Unconditional
Yield-Curve Based Expected Inflation $\pi^e_{t,n}$					
1	3.96	4.42	4.22	2.54	4.02
	(0.42)	(0.49)	(0.42)	(0.42)	(0.74)
4	4.17	4.96	4.49	2.67	4.27
	(0.39)	(0.50)	(0.39)	(0.61)	(0.40)
12	4.71	5.43	4.99	3.42	4.80
	(0.39)	(0.47)	(0.39)	(0.49)	(0.39)
20	4.94	5.42	5.13	4.00	5.00
	(0.41)	(0.47)	(0.42)	(0.44)	(0.41)
RS-VAR Expected Inflation $\mathrm{E}_t(\pi_{t+n,n})$					
1	3.97	4.42	4.22	2.54	4.03
	(0.42)	(0.49)	(0.42)	(0.66)	(0.42)
4	3.95	4.53	4.20	2.75	4.03
	(0.42)	(0.52)	(0.42)	(0.64)	(0.42)
12	3.97	4.41	4.13	3.27	4.03
	(0.42)	(0.48)	(0.42)	(0.54)	(0.42)
20	3.99	4.28	4.09	3.54	4.03
	(0.42)	(0.45)	(0.42)	(0.48)	(0.42)

The table reports means and standard deviations of expected inflation, conditional on the inflation regime variable (s_t^π), implied from the benchmark model IV with independent real rate and inflation regimes. Moments computed conditional on the real regime variable s_t^r (inflation regime s_t^π) integrate out the effect of s_t^π (s_t^r). Standard errors reported in parentheses are computed using the delta-method.

Table 7: Conditional Moments Across Business Cycles

	Maturity (Qtrs)	Mean Expansion	Mean Recession	Std Dev Expansion	Std Dev Recession
Real Rates $\hat{y}_t^n$	1	1.64	1.25	1.32	2.33
		(0.20)	(0.20)	(0.05)	(0.08)
	20	1.47	1.59	0.71	0.95
		(0.41)	(0.39)	(0.21)	(0.28)
Real Spread $\hat{y}_t^{20} - \hat{y}_t^1$		-0.17	0.34	0.95	1.65
		(0.28)	(0.29)	(0.07)	(0.15)
Nominal Rates y_t^n	1	5.30	6.29	2.47	4.15
		(0.05)	(0.11)	(0.22)	(0.37)
	20	6.26	7.12	2.42	3.83
		(0.20)	(0.21)	(0.23)	(0.35)
Nominal Spread $y_t^{20} - y_t^1$		0.96	0.83	1.00	1.14
		(0.19)	(0.19)	(0.26)	(0.26)
Expected Inflation $\pi_{t,n}^e$	1	3.66	5.05	2.26	3.70
		(0.18)	(0.16)	(0.15)	(0.25)
	20	4.79	5.53	1.81	3.02
		(0.39)	(0.39)	(0.38)	(0.58)
Expected Inflation Spread $\pi_{t,20}^e - \pi_{t,1}^e$		1.13	0.48	1.36	2.19
		(0.31)	(0.31)	(0.10)	(0.14)
Actual Inflation		3.54	5.43	2.76	4.40
Ex-post Real Rate		1.74	0.86	1.99	3.61
Ex-post Real Spread		2.70	1.69	2.32	4.01

Regime Realizations Across Business Cycles

	$s_t = 1$	$s_t = 2$	$s_t = 3$	$s_t = 4$
Whole Sample	0.66	0.16	0.12	0.06
Expansions	0.70	0.15	0.10	0.05
Recessions	0.51	0.17	0.20	0.11

The table reports various sample moments of real rates, nominal rates and expected inflation implied from yield-curve forecasts ($\pi_{t,n}^e$) from Model IV, conditional on expansions and recessions, as defined by the NBER. Standard errors reported in parentheses are computed using the delta-method on sample moments. The ex-post real rate (spread) is the nominal rate (spread) minus actual inflation over the sample. The second part of the table reports the proportions of each regime (the number of periods assigned to be in regime $s_t = i$ divided by the total number of observations) across the whole sample, and conditional on NBER expansions and recessions. The regime classification uses smoothed $Pr(s_t = i|I_T)$ probabilities.

Table 8: Variance Decomposition of Nominal Yields and Spreads

	Regime 1		Regime 2		Regime 3		Regime 4		Unconditional	
n	Real	Infl	Real	Infl	Real	Infl	Real	Infl	Real	Infl
Variance Decomposition of Nominal Yields										
1	0.20	0.80	0.20	0.80	0.26	0.74	0.26	0.74	0.21	0.79
	(0.11)	(0.11)	(0.11)	(0.11)	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)
20	0.21	0.79	0.21	0.79	0.20	0.80	0.20	0.80	0.21	0.79
	(0.11)	(0.11)	(0.11)	(0.11)	(0.11)	(0.11)	(0.11)	(0.11)	(0.11)	(0.11)
Variance Decomposition of Nominal Yield Spreads										
4	0.21	0.79	0.21	0.79	0.89	0.11	0.89	0.11	0.48	0.52
	(0.19)	(0.19)	(0.19)	(0.20)	(0.12)	(0.11)	(0.12)	(0.11)	(0.16)	(0.16)
20	0.01	0.99	0.01	0.99	0.40	0.60	0.40	0.60	0.17	0.83
	(0.22)	(0.21)	(0.22)	(0.22)	(0.16)	(0.16)	(0.16)	(0.16)	(0.19)	(0.19)

The table reports variance decompositions of nominal yields and nominal yield spreads into real rate ($\tau_{\hat{y}_t^n}$) and expected inflation ($\tau_{\pi^e_{t,n}}$) components, defined in equation (21) using yield-curve based expectation inflation $\pi^e_{t,n}$. The regime $s_t = 1$ corresponds to $(s_t^r = 1, s_t^\pi = 1)$, $s_t = 2$ to $(s_t^r = 1, s_t^\pi = 2)$, $s_t = 3$ to $(s_t^r = 2, s_t^\pi = 1)$ and $s_t = 4$ to $(s_t^r = 2, s_t^\pi = 2)$. Standard errors reported in parentheses are computed using the delta-method on population moments.

Table 9: Real Rates and Expected Inflation: Comparisons with Other Models

Panel A: Real Rates

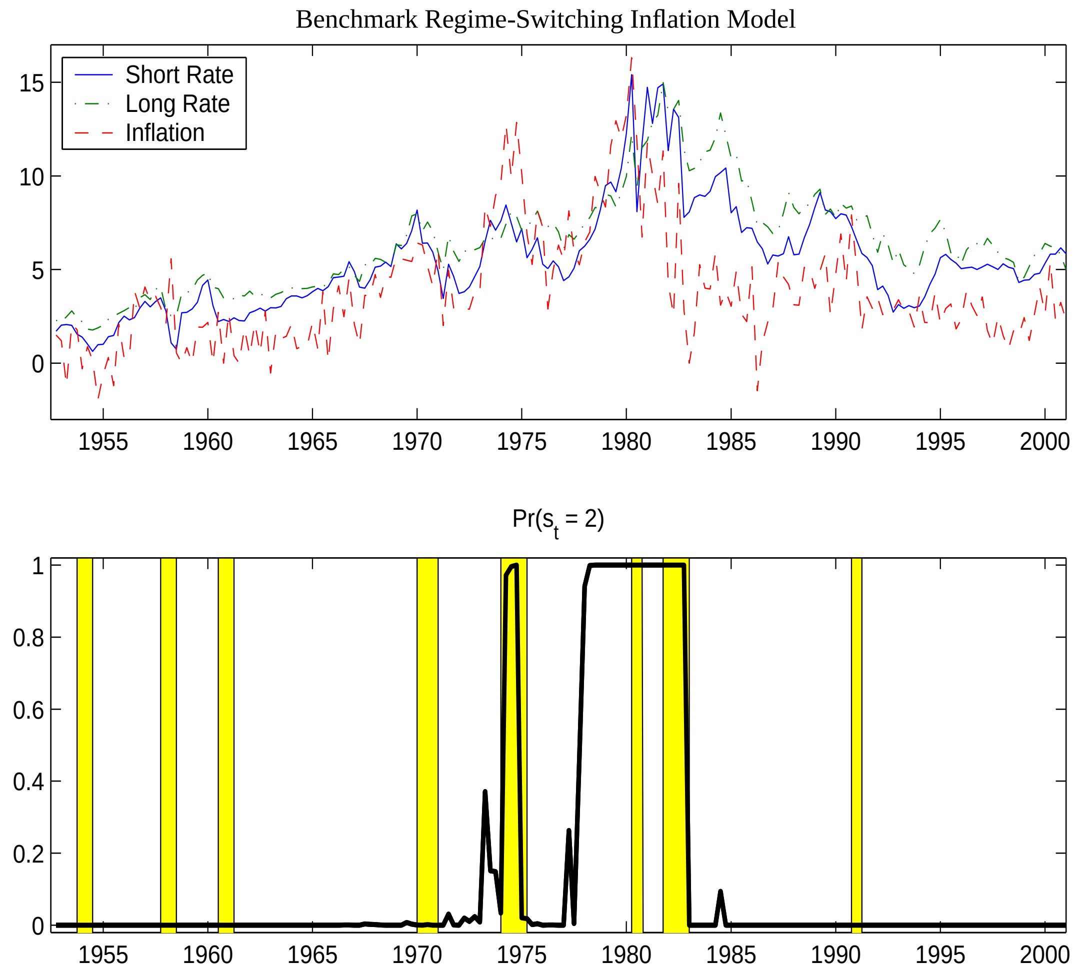
	1-qtr Real Rate				20-qtr Real Rate			
Models	Mean Err	Min Err	Max Err	RMSE	Mean Err	Min Err	Max Err	RMSE
I	0.06	-1.96	1.25	0.48	-0.52	-2.91	0.37	0.75
II	0.27	-3.18	5.99	1.41	-0.10	-1.35	1.59	0.51
III	0.12	-0.49	0.83	0.27	0.29	-0.47	1.14	0.38
IVa	-0.00	-0.05	0.03	0.01	-0.01	-0.14	0.04	0.01
IVb	0.00	-0.35	0.39	0.07	-0.05	-0.44	0.33	0.09
V	0.41	-3.80	3.79	1.28	0.59	-1.04	2.82	0.90
VI	0.07	-4.30	5.28	1.33	-0.19	-1.53	1.35	0.56

Panel B: Price Expected Inflation $\pi^e_{t,n}$

	1-qtr Expected Inflation				20-qtr Expected Inflation			
Models	Mean Err	Min Err	Max Err	RMSE	Mean Err	Min Err	Max Err	RMSE
I	-0.06	-1.25	1.96	0.48	0.52	-0.37	2.91	0.75
II	-0.28	-5.99	3.18	1.41	0.10	-1.59	1.35	0.51
III	-0.12	-0.83	0.49	0.27	-0.29	-1.14	0.47	0.38
IVa	0.00	-0.03	0.05	0.01	0.01	-0.04	0.14	0.01
IVb	-0.00	-0.39	0.35	0.07	0.05	-0.33	0.44	0.09
V	-0.41	-3.79	3.80	1.27	-0.59	-2.82	1.04	0.90
VI	-0.07	-5.29	4.30	1.33	0.19	-1.35	1.53	0.56

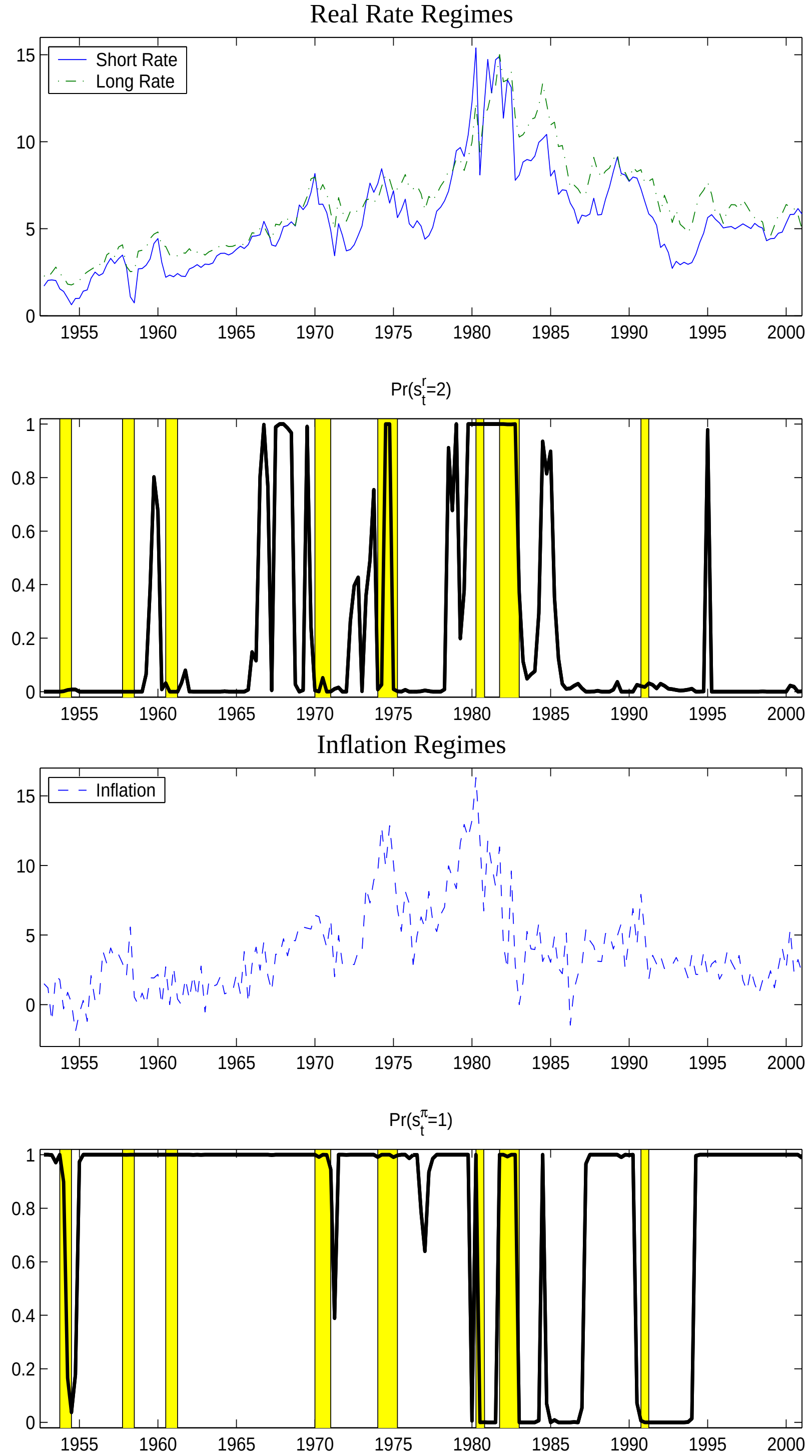
We report real rates (Panel A) and price expected inflation rates (Panel B) produced by other models as compared to Model IV, the benchmark model with independent real rate and inflation regimes. The columns denoted ‘Mean Err’, ‘Min Err’ and ‘Max Err’ compare the extracted real rate or expected inflation series with the real rates or expected inflation from Model IV over the whole sample. We list the RMSE of the errors in the column denoted ‘RMSE’. Models I and II are the single-regime equivalents of the inflation and expected inflation models in Sections 3.1 and 3.4, respectively. Model III denotes the benchmark model with two regimes. The results of the benchmark model allowing for correlated real rates and inflation regimes (Cases A and B in Section 3.3) are denoted as IVa and IVb, respectively. Model V denotes the RS expected inflation model with two regimes. Model VI denotes the RS expected inflation model with separate and independent regimes for real rates and inflation.

Figure 1: Smoothed Regime Probabilities: Benchmark Model III



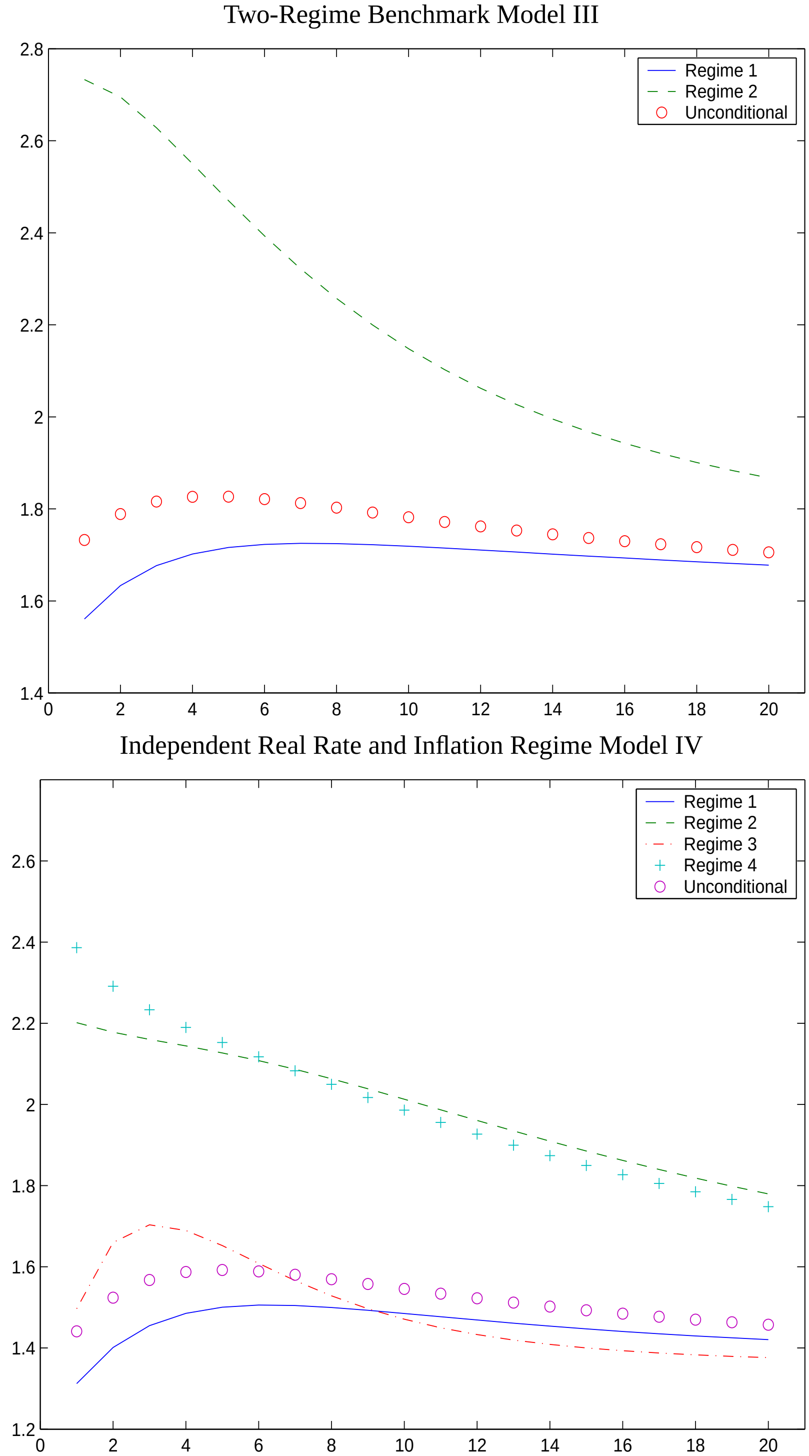
The bottom graph shows the smoothed probabilities of the second regime, $Pr(s_t = 2|I_T)$, of the benchmark model using information over the whole sample, along with short (1-quarter), long (20-quarter) yields and inflation shown in the top panel. NBER recessions are indicated by shaded bars.

Figure 2: Smoothed Regime Probabilities: Independent Real Rate and Inflation Regimes



The figure displays smoothed probabilities, using information over the whole sample, from the benchmark model with separate and independent real rate and inflation regimes (Model IV). The top panel shows the smoothed probabilities of the second real rate regime, $Pr(s_t^r = 2|I_T)$, along with short (1-quarter) and long (20-quarter) yields. In the bottom panel, the smoothed probabilities of the first inflation regime $Pr(s_t^\pi = 1|I_T)$ are shown, together with realized quarterly inflation. NBER recessions are indicated by shaded bars.

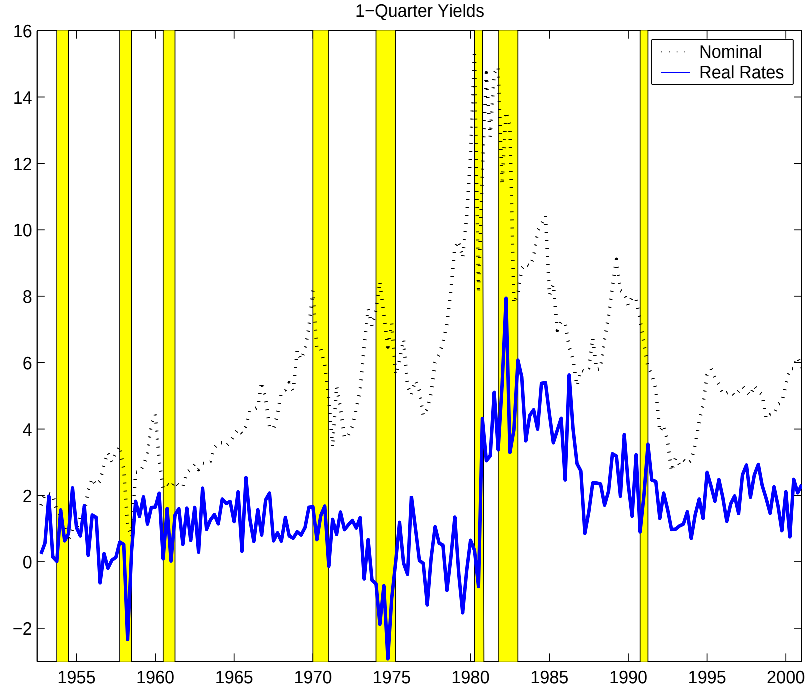
Figure 3: Real Term Structure



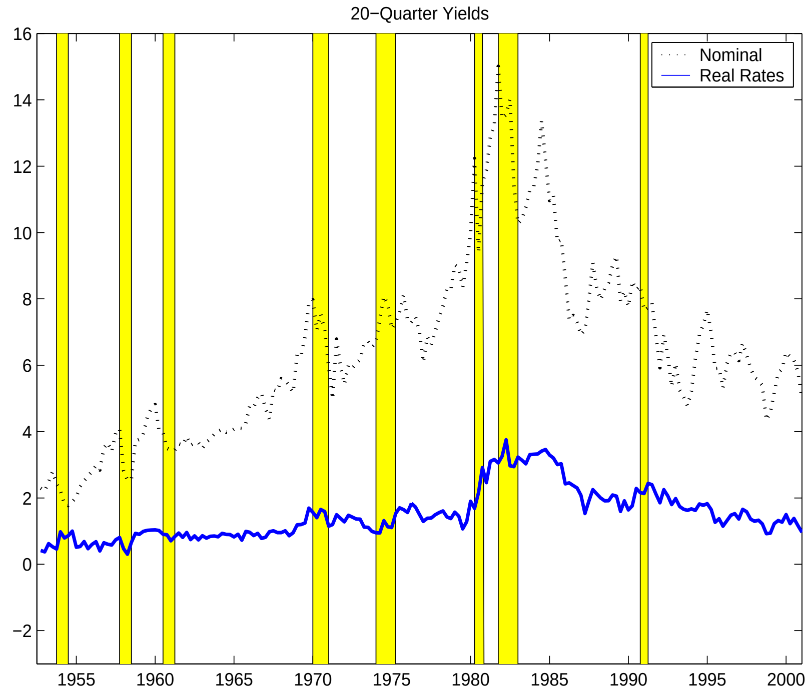
The figure graphs the real yield curve, conditional on each regime and the unconditional real yield curve. The top panel displays the benchmark model with two regimes (Model III) and the bottom panel displays the benchmark model with separate and independent real rate and inflation regimes (Model IV). For Model IV, the regime $s_t = 1$ corresponds to $(s_t^r = 1, s_t^\pi = 1)$, $s_t = 2$ to $(s_t^r = 1, s_t^\pi = 2)$, $s_t = 3$ to $(s_t^r = 2, s_t^\pi = 1)$ and $s_t = 4$ to $(s_t^r = 2, s_t^\pi = 2)$. The x-axis displays maturities in quarters of a year.

Figure 4: Time-Series of Real versus Nominal Yields

Real versus Nominal 1-Quarter Yields

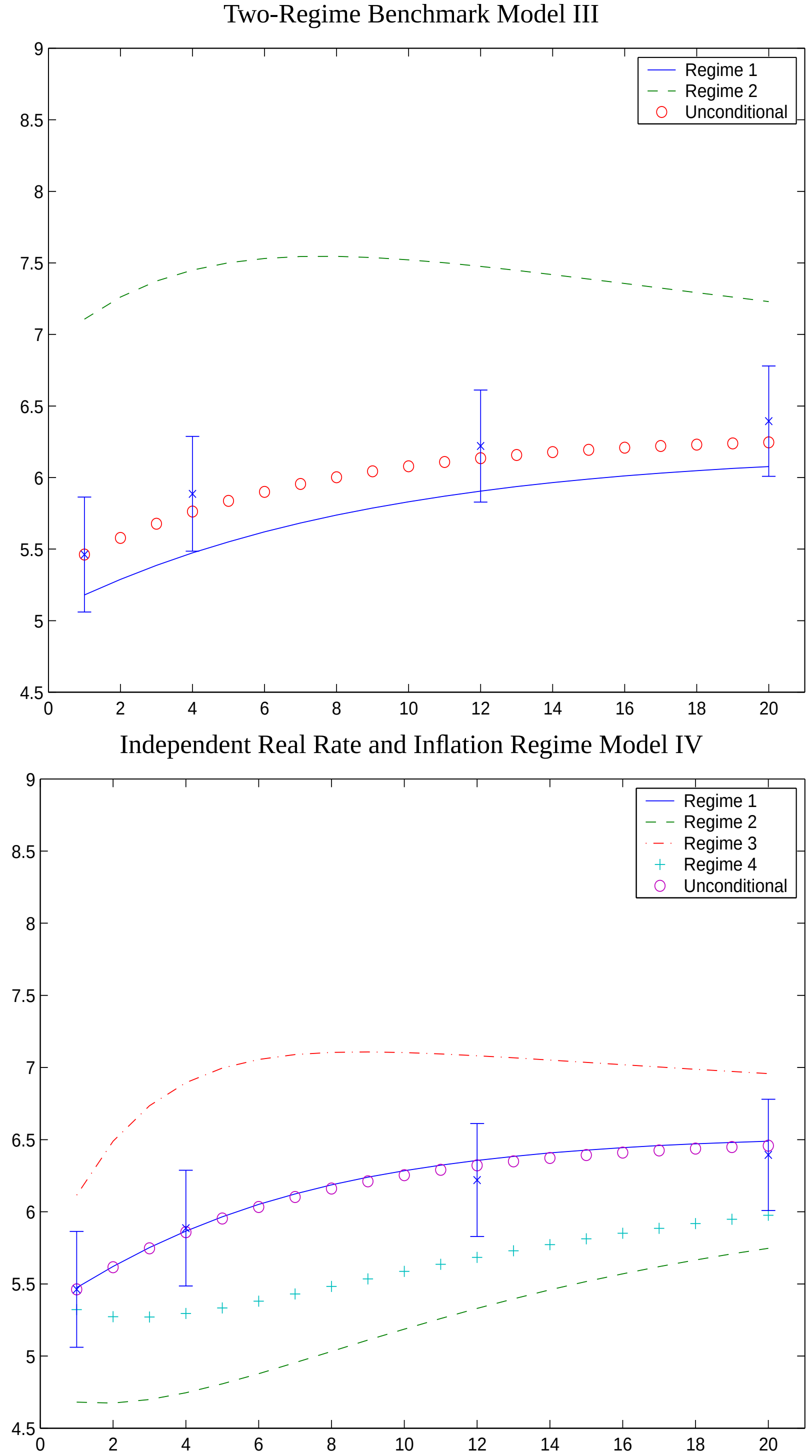


Real versus Nominal 20-Quarter Yields



The figure graphs the time-series of real and nominal 1-quarter yields (top panel) and real and nominal 20-quarter yields (bottom panel) from the benchmark model with independent real rate and inflation regimes (Model IV). NBER recessions are indicated by shaded bars.

Figure 5: Nominal Term Structure



The figure graphs the nominal yield curve, conditional on each regime and the unconditional nominal yield curve. The top panel displays the benchmark model with two regimes (Model III) and the bottom panel displays the benchmark model with separate and independent real rate and inflation regimes (Model IV). For Model IV, the regime $s_t = 1$ corresponds to $(s_t^r = 1, s_t^\pi = 1)$, $s_t = 2$ to $(s_t^r = 1, s_t^\pi = 2)$, $s_t = 3$ to $(s_t^r = 2, s_t^\pi = 1)$ and $s_t = 4$ to $(s_t^r = 2, s_t^\pi = 2)$. The x-axis displays maturities in quarters of a year. Average yields from data are represented by 'x', with 95% confidence intervals represented by vertical bars.

Appendix Tables

Table A-1: Single Regime Models

		Inflation Model I				**Expected Inflation Model II**			
		q	f	π		q	f	w	π
$\mu' \times 100$		0.000	0.000	0.158		0.000	0.000	0.579	0.000
				(0.045)				(0.089)	
		q	f	π		q	f	w	π
Φ	q	0.962	0.000	0.000	q	0.971	0.000	0.000	0.000
		(0.017)				(0.015)			
	f	0.000	0.669	0.000	f	0.000	0.841	0.000	0.000
			(0.032)				(0.012)		
	π	0.262	1.381	0.828	w	0.452	0.920	0.490	-0.065
		(0.066)	(0.655)	(0.019)		(0.081)	(0.116)	(0.083)	(0.049)
					π	0.000	0.000	1.000	0.000
		q	f	π		q	f	w	π
$\sigma \times 100$		0.116	0.095	0.493		0.119	0.165	0.371	0.577
		(0.011)	(0.027)	(0.026)		(0.008)	(0.011)	(0.027)	(0.033)
δ_0		0.010				0.010			
		(0.001)				(0.006)			
		q	f	π		q	f	w	π
δ_1'		1.000	1.000	-0.607		1.000	1.000	-0.496	-0.101
				(0.028)					(0.023)
		q	f	π		q	f	w	π
λ'			-0.314	0.000			-0.265	0.000	0.000
			(0.000)				(0.074)		
γ_1		-26.8				-19.6			
		(15.1)				(13.15)			
Std Dev $\times 100$ of Measurement Errors									
y_t^4		0.079				-			
		(0.005)							
y_t^{12}		0.039				0.028			
		(0.002)				(0.001)			

The left (right)-hand columns report parameter estimates for the single-regime equivalents of the Inflation and Expected Inflation Models, outlined in Sections 3.1 and 3.4, respectively.

Table A-2: Benchmark Model III

		Regime 1			Regime 2		
		q	f	π	q	f	π
$\mu(s_t)' \times 100$		0.000	-0.016	0.453	0.000	0.096	0.199
			(0.015)	(0.078)		(0.088)	(0.120)
		q	f	π			
Φ	q	0.962	0.000	0.000			
		(0.020)					
	f	0.000	0.784	0.000			
			(0.010)				
	π	0.459	1.034	0.555			
		(0.143)	(0.522)	(0.061)			
		q	f	π	q	f	π
$\sigma(s_t) \times 100$		0.100	0.077	0.400	0.100	0.258	0.973
		(0.011)	(0.020)	(0.018)	(0.011)	(0.099)	(0.381)
δ_0		0.009					
		(0.001)					
		q	f	π			
δ_1'		1.000	1.000	-0.516			
				(0.059)			
		q	f	π	q	f	π
$\lambda(s_t)'$			-0.482	0.000		-0.005	0.000
			(0.104)			(0.353)	
γ_1		-42.2					
		(21.2)					
		$s_{t+1} = 1$	$s_{t+1} = 2$				
Π	$s_t = 1$	0.979	0.021				
		(0.006)	(0.006)				
	$s_t = 2$	0.121	0.879				
		(0.032)	(0.032)				
Std Dev $\times 100$ of Measurement Errors							
y_t^4		0.076					
		(0.005)					
y_t^{12}		0.039					
		(0.003)					

The table reports estimates of the Benchmark Regime-Switching Inflation Model, where real rates and inflations have the same regimes. The stable probabilities of regime 1 and 2 are 0.853 and 0.146, respectively.

Table A-3: Benchmark Model with Independent Real Rate and Inflation Regimes IV

		Regime 1			Regime 2
$\mu_f(s_t^r) \times 100$		-0.006			0.039
		(0.004)			(0.023)
$\mu_\pi(s_t^\pi) \times 100$		0.435			0.219
		(0.079)			(0.095)
		q	f	π	
Φ	q	0.976	0.000	0.000	
		(0.015)			
	f	0.000	0.759	0.000	
			(0.012)		
	π	0.499	0.851	0.593	
		(0.135)	(0.479)	(0.058)	
$\sigma_q \times 100$		0.096			
		(0.010)			
$\sigma_f(s_t^r) \times 100$		0.078			0.246
		(0.020)			(0.020)
$\sigma_\pi(s_t^\pi) \times 100$		0.479			0.471
		(0.027)			(0.052)
δ_0		0.009			
		(0.001)			
		q	f	π	
δ_1'		1.000	1.000	-0.536	
				(0.052)	
$\lambda_f(s_t)$		-0.523			0.335
		(0.100)			(0.157)
γ_1		-19.0			
		(16.2)			
		$s_{t+1}=1$	$s_{t+1}=2$	$s_{t+1}=3$	$s_{t+1}=4$
Π	$s_t=1$	0.930	0.022	0.047	0.001
		(0.019)	(0.008)	(0.015)	(0.001)
	$s_t=2$	0.162	0.790	0.008	0.040
		(0.038)	(0.037)	(0.003)	(0.013)
	$s_t=3$	0.319	0.007	0.659	0.015
		(0.072)	(0.003)	(0.070)	(0.006)
	$s_t=4$	0.056	0.271	0.115	0.559
		(0.019)	(0.061)	(0.028)	(0.068)
Std Dev $\times 100$ of Measurement Errors					
y_t^4		0.053			
		(0.003)			
y_t^{12}		0.026			
		(0.001)			

The table reports estimates of the Regime-Switching Inflation Model with independent regimes in real rates and inflation. The regime $s_t = 1$ corresponds to $(s_t^r = 1, s_t^\pi = 1)$, $s_t = 2$ to $(s_t^r = 1, s_t^\pi = 2)$, $s_t = 3$ to $(s_t^r = 2, s_t^\pi = 1)$ and $s_t = 4$ to $(s_t^r = 2, s_t^\pi = 2)$. The stable probabilities of regime 1 to 4 are 0.769, 0.103, 0.113 and 0.015.

Table A-4: Testing Independent versus Correlated Real Rate and Inflation Regimes

Panel A: Transition Probabilities from Independent Real Rate and Inflation Regimes (Model IV)

	$s_{t+1}=1$	$s_{t+1}=2$	$s_{t+1}=3$	$s_{t+1}=4$
$s_t=1$	0.930	0.022	0.047	0.001
	(0.019)	(0.008)	(0.015)	(0.001)
$s_t=2$	0.162	0.790	0.008	0.040
	(0.038)	(0.037)	(0.003)	(0.013)
$s_t=3$	0.319	0.007	0.659	0.015
	(0.072)	(0.003)	(0.070)	(0.006)
$s_t=4$	0.056	0.271	0.115	0.559
	(0.019)	(0.061)	(0.028)	(0.068)

Panel B: Transition Probabilities from Correlated Real Rate and Inflation Regimes Case A

	$s_{t+1}=1$	$s_{t+1}=2$	$s_{t+1}=3$	$s_{t+1}=4$
$s_t=1$	0.927	0.023	0.050	0.000
	(0.021)	(0.008)	(0.017)	(0.003)
$s_t=2$	0.166	0.825	0.009	0.000
	(0.039)	(0.107)	(0.004)	(0.118)
$s_t=3$	0.342	0.004	0.635	0.019
	(0.077)	(0.005)	(0.075)	(0.007)
$s_t=4$	0.061	0.145	0.114	0.679
	(0.021)	(0.152)	(0.028)	(0.158)

Panel C: Transition Probabilities from Correlated Real Rate and Inflation Regimes Case B

	$s_{t+1}=1$	$s_{t+1}=2$	$s_{t+1}=3$	$s_{t+1}=4$
$s_t=1$	0.946	0.017	0.032	0.005
	(0.018)	(0.009)	(0.013)	(0.003)
$s_t=2$	0.180	0.782	0.008	0.028
	(0.043)	(0.041)	(0.005)	(0.011)
$s_t=3$	0.252	0.005	0.640	0.103
	(0.081)	(0.003)	(0.069)	(0.043)
$s_t=4$	0.048	0.208	0.167	0.577
	(0.020)	(0.067)	(0.067)	(0.101)

Likelihood ratio test for independent versus Case A p-value = 0.251
Likelihood ratio test for independent versus Case B p-value = 0.125

The table reports parameter estimates of the transition probability matrix of the Benchmark Model with independent real rate and inflation regimes (Model IV) in Panel A (equation (A-1)) and correlated real rate and inflation regimes Case A (Panel B) using the formulation (A-2). In Panel C, we report the correlated real rate and inflation regimes Case B using the formulation (A-3). The regime $s_t = 1$ corresponds to $(s_t^r = 1, s_t^\pi = 1)$, $s_t = 2$ to $(s_t^r = 1, s_t^\pi = 2)$, $s_t = 3$ to $(s_t^r = 2, s_t^\pi = 1)$ and $s_t = 4$ to $(s_t^r = 2, s_t^\pi = 2)$. Standard errors reported in parenthesis are computed using the delta-method.

Table A-5: Regime-Switching Expected Inflation Model V

		Regime 1				Regime 2			
		q	f	w	π	q	f	w	π
$\mu(s_t)' \times 100$		0.000	0.009	0.477	0.000	0.000	-0.092	1.396	0.000
			(0.009)	(0.056)			(0.008)	(0.178)	
		q	f	w	π				
Φ	q	0.968	0.000	0.000	0.000				
		(0.014)							
	f	0.000	0.850	0.000	0.000				
			(0.012)						
	w	0.390	0.756	0.518	-0.134				
		(0.064)	(0.133)	(0.075)	(0.032)				
	π	0.000	0.000	1.000	0.000				
		q	f	w	π	q	f	w	π
$\sigma(s_t) \times 100$		0.116	0.110	0.176	0.551	0.116	0.302	0.650	0.551
		(0.009)	(0.010)	(0.022)	(0.034)	(0.009)	(0.034)	(0.121)	(0.034)
δ_0		0.009							
		(0.001)							
		q	f	w	π				
δ_1'		1.000	1.000	-0.377	-0.113				
				(0.068)	(0.021)				
		q	f	w	π	q	f	w	π
$\lambda(s_t)'$			-0.195	0.000	0.000		-0.414	0.000	0.000
			(0.099)				(0.323)		
γ_1		-24.1							
		(12.7)							
		$s_{t+1} = 1$	$s_{t+1} = 2$						
Π	$s_t = 1$	0.984	0.016						
		(0.014)	(0.014)						
	$s_t = 2$	0.171	0.829						
		(0.104)	(0.104)						
Std Dev $\times 100$ of Measurement Errors									
y_t^4		0.028							
		(0.001)							

The table reports estimates of the Regime-Switching Expected Inflation Model, where real rates and inflations have the same regimes. The stable probabilities of regime 1 and 2 are 0.913 and 0.087, respectively.

Table A-6: Expected Inflation Model with Independent Real Rate and Inflation Regimes VI

		Regime 1				Regime 2
$\mu_f(s_t^r) \times 100$		-0.029				0.059
		(0.010)				(0.020)
$\mu_\pi(s_t^\pi) \times 100$		0.518				1.557
		(0.063)				(0.211)
		q	f	w	π	
Φ	q	0.987	0.000	0.000	0.000	
		(0.012)				
	f	0.000	0.834	0.000	0.000	
			(0.016)			
	w	0.429	1.141	0.499	-0.015	
		(0.094)	(0.161)	(0.047)	(0.021)	
	π	0.000	0.000	1.000	0.000	
$\sigma_q \times 100$		0.086				
		(0.008)				
$\sigma_\pi \times 100$		0.576				
		(0.031)				
$\sigma_f(s_t^r) \times 100$		0.077				0.288
		(0.009)				(0.057)
$\sigma_w(s_t^\pi) \times 100$		0.147				0.576
		(0.015)				(0.031)
δ_0		0.005				
		(0.001)				
		q	f	w	π	
δ_1'		1.000	1.000	-0.326	-0.039	
				(0.063)	(0.015)	
$\lambda_f(s_t)$		0.081				0.283
		(0.116)				(0.393)
γ_1		-19.1				
		(14.4)				
		$s_{t+1} = 1$	$s_{t+1} = 2$	$s_{t+1} = 3$	$s_{t+1} = 4$	
Π	$s_t = 1$	0.863	0.098	0.035	0.004	
		(0.023)	(0.017)	(0.010)	(0.002)	
	$s_t = 2$	0.517	0.444	0.021	0.018	
		(0.070)	(0.067)	(0.006)	(0.007)	
	$s_t = 3$	0.072	0.008	0.826	0.093	
		(0.020)	(0.003)	(0.026)	(0.017)	
	$s_t = 4$	0.043	0.037	0.495	0.425	
		(0.013)	(0.012)	(0.067)	(0.066)	
Std Dev $\times 100$ of Measurement Errors						
y_t^{12}		0.025				
		(0.001)				

The table reports estimates of the Regime-Switching Expected Inflation Model with independent regimes in real rates and inflation. The regime $s_t = 1$ corresponds to $(s_t^r = 1, s_t^\pi = 1)$, $s_t = 2$ to $(s_t^r = 1, s_t^\pi = 2)$, $s_t = 3$ to $(s_t^r = 2, s_t^\pi = 1)$ and $s_t = 4$ to $(s_t^r = 2, s_t^\pi = 2)$. The stable probabilities of regime 1 to 4 are 0.567, 0.107, 0.274 and 0.052.