CONTRIBUTIONS TO THE THEORY OF NONLINEAR OSCILLATIONS

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V. CRITICAL POINTS AT INFINITY AND FORCED OSCILLATION

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INTRODUCTION

The problem of oscillations, both forced and free, is central in non-linear differential equations. Poincaré [1] and Bendixson [1] in their fundamental papers discussed the existence of free oscillations in a pair of first order equations

$$\text{(a)} \qquad \frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y),$$

while the interest aroused by van der Pol's equation has caused more modern writers to turn their attention especially to the system arising from the second order equation

$$\text{(b)} \qquad \frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0.$$

The free oscillations of this equation and the forced oscillations of

$$\text{(c)} \qquad \frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = E(t)$$

have been investigated in numerous papers, especially Levinson and Smith [1], and Levinson [1].

A central condition in these investigations is that the equations be of the type called by Levinson [2] "dissipative for large displacements." This condition ensures that with increasing time all solutions tend to the interior of some circle in the phase plane. This makes it possible to apply the Brouwer fixed point theorem to (c), or, provided there is an unstable critical point at the phase plane origin, to apply the Poincaré-Bendixson theorem to (b).

The main contribution of this paper will be to drop this condition, and to prove the existence of forced periodic solutions to a variety of differential equations not dissipative for large displacements. In fact many of these equations will have, in addition to their periodic solutions, other solutions which become unbounded with increasing t, and still others which become unbounded with decreasing t.

The notion of critical point at infinity due to Poincaré will be used throughout. Poincaré extended the planar autonomous system

$$\frac{dx}{X(x,y)} = \frac{dy}{Y(x,y)}$$

to a sphere, (the doubly covering surface of the projective plane), and discussed his equations on the entire sphere including the equator and its critical points, (the doubly covered line at infinity and its critical points). Later Bendixson briefly discussed a single point at infinity, but since that time, with the exception of a recent paper by Lefschetz [1], authors have dealt mainly with the finite plane. This may be due to the fact that while Poincaré restricted himself to equations with simple critical points at infinity, the systems arising from (b) rarely have this property, and so are not as easily treated.

In Part I, systems of the Poincaré type will be treated, and their forced oscillations related to the nature of their critical points at infinity. In Part II the nature of the critical points at infinity arising from a wide class of equations (b) will be completely analyzed. This analysis gives the asymptotic behavior of trajectories that become unbounded. Geometric information, such as the existence of limit-cycles to equations of the van der Pol type, will be deduced. In Part III the results on critical points from II will be applied to obtain new criteria for the existence of oscillations in (c).

The general condition which in one form or another will be used to replace dissipativity may be stated roughly as follows. Let the plane be completed by the addition of a single point at infinity. This point is generally a critical point, and in a dissipative system it must be an unstable node or focus. Our conditions are roughly equivalent to requiring that the index of this point be $\neq 2$, and that it have no saddle sectors.

PART I: FORCED OSCILLATIONS IN A CLASS OF FIRST ORDER SYSTEMS

§1. In this section we will consider the system

$$\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y) \tag{1}$$

where $X(x, y)$, $Y(x, y)$ are both polynomials of degree $n > 0$. The theorem stated below will relate the nature of the critical points at infinity of (1) to the question of the existence of periodic solutions to

$$\frac{dx}{dt} = X(x, y) + E_1(t), \quad \frac{dy}{dt} = Y(x, y) + E_2(t) \tag{2}$$

where $E_1(t)$ and $E_2(t)$ have continuous derivatives and are periodic with period T.

We will start by extending equation (1) to the projective plane.

Adopting homogeneous coordinates (x, y, z), with $(x, y, 1)$ a designation for the point with affine coordinates (x, y), we see that (1) and

$$\frac{d}{dt}\left(\frac{x}{z}\right) = X\left(\frac{x}{z}, \frac{y}{z}\right), \quad \frac{d}{dt}\left(\frac{y}{z}\right) = Y\left(\frac{x}{z}, \frac{y}{z}\right), \tag{3}$$

have the same solutions. However we can easily extend (3) to cover points for which $z = 0$.

Using (3) and multiplying by z^{n+1} gives

$$\begin{aligned} z^{n-1}\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) &= zX^H\left(x, y, z\right) \\ z^{n-1}\left(z\frac{dy}{dt} - y\frac{dz}{dt}\right) &= zY^H\left(x, y, z\right), \end{aligned} \tag{4}$$

where X^H and Y^H are the original X and Y made homogeneous of degree n with z. Adopting a new parameter τ such that $\frac{dt}{d\tau} = z^{n-1}$ we obtain from (4) in the region covered by coordinates $(x, 1, z)$

$$\begin{aligned} \frac{dz}{d\tau} &= -zY^H(x, 1, z) \\ \frac{dx}{d\tau} &= X^H(x, 1, z) - xY^H(x, 1, z) \end{aligned} \tag{5}$$

and in the region covered by coordinates $(1, y, z)$ using the parameter σ such that $\frac{dt}{d\sigma} = z^{n-1}$:

$$\begin{aligned} \frac{dz}{d\sigma} &= -zX^H(1, y, z) \\ \frac{dy}{d\sigma} &= Y^H(1, y, z) - yX^H(1, y, z). \end{aligned} \tag{6}$$

It is easily seen that solutions to (5) and (6) with $z \neq 0$ trace out the same trajectories as those of (1), the parametrizations only being different. As (5) and (6) are defined for $z = 0$, we can consider them as extending (1), which was defined only for $(x, y, 1)$, to $(x, 1, z)$, and $(1, y, z)$.

A critical point on the line at infinity is simply a critical point of (5) or (6) on $z = 0$. An elementary or simple critical point is one for which the linear terms in the Taylor expansion about the point have a non-zero determinant.

§2. We are now in a position to state the theorem.

THEOREM: If the critical points at infinity of the extended equation (1) satisfy the following conditions:

(a) there are isolated critical points and they are simple;

(b) their index sum is $\neq 1$;

(c) there are no consecutive saddle points;

then equation (2) has a periodic solution of period T.

It is probably worth while to clarify exactly what is meant by "consecutive saddle points." As the line at infinity is topologically a circle we may trace it through in some fixed sense, encountering in succession all critical points. Condition (c) is that in this cyclic arrangement saddle points should not follow saddle points. In particular, if there is only one critical point and it is a saddle, condition (c) is not satisfied.

To facilitate application to specific equations, the conditions for the theorem will be restated entirely in terms of the original polynomials X and Y. To do this, designate by X_n, Y_n the terms of degree n of X, Y. Condition (a) is readily seen to be equivalent to

(a1) $xY_n - yX_n$ has real roots, no multiple roots, and does not vanish identically.

(a2) $xY_n - yX_n$ and X_n have no roots in common, and the same holds for $xY_n - yX_n$ and Y_n.

As the index sum of a vector distribution in the projective plane is 1, condition (b) is that the sum of the indices of the finite critical points of (1) be $\neq 0$. Condition (c) may be tested for directly by using the following observation from Poincaré ([1], p. 25).

The coordinates of critical points at infinity have ratios $\frac{y}{x} = \alpha$ which satisfy $xY_n - yX_n = 0$. If, when α increases from $\alpha - \epsilon$ to $\alpha + \epsilon$ the expression

$$\frac{Y_n}{X_n} - \frac{y}{x}$$

changes from negative to positive, the critical point is a saddle point;

if the expression changes from positive to negative, it is a node.

In terms of a single point at infinity the conditions ensure an index $\neq 2$, and the absence of saddle sectors.

§3. PROOF OF THE THEOREM. The first step in the proof will be the construction of a large closed contour J, containing all the finite critical points of (1) and with the following special property. A solution $x(t)$, $y(t)$ of (2) which at time t_o is at a point p on J, never is at p again for $t > t_o$.

We will start by extending (2). Equation (2) is merely (1) with time dependent additions to the constant terms in X and Y. Hence from (5) the extension to $(x, 1, z)$ is

(7)
$$\frac{dz}{d\tau} = -z\left\{Y^H(x, 1, z) + z^n E_2(t)\right\}$$
$$\frac{dx}{d\tau} = \left\{X^H(x, 1, z) + z^n E_1(t)\right\} - x\left\{Y^H(x, 1, z) + z^n E_2(t)\right\}$$

or, arranging in powers of z,

(8)
$$\frac{dz}{d\tau} = -z\left\{P_o + P_1 z + \dots + (P_n + E_2)z^n\right\}$$
$$\frac{dx}{d\tau} = \left\{Q_o + Q_1 z + \dots + (Q_n + E_1 - xE_2)z^n\right\}$$

where P_i and Q_i are polynomials in x. From condition (a) we have that P_o and Q_o do not vanish identically and have no common real zeros. Also Q_o has real roots, all of which are simple.

Let x^o be a zero of Q_o and consider (8) near x^o on $z = 0$. Such a point is a critical point at infinity of (1). As $P_o(x^o) \neq 0$ and E_2 is periodic, $\frac{dz}{d\tau}$ is $\neq 0$ for all t throughout some sufficiently small box B, $0 < z \leq \delta_1$, $x^o - \delta_1 \leq x \leq x^o + \delta_1$ (Fig. 1)

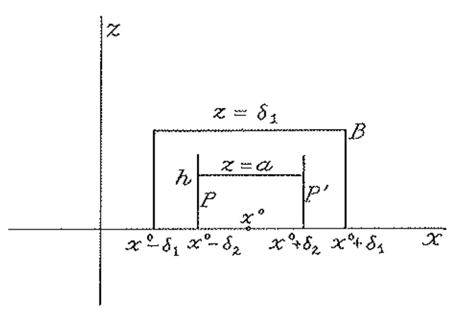

FIGURE 1

so $\frac{dz}{d\tau}$ has constant sign in B. As x^o is a simple zero of Q_o, there is a δ_2 such that $Q_o(x^o + \delta_2)$, $Q_o(x - \delta_2)$ are $\neq 0$ and have opposite signs. Hence at $x = x^o \pm \delta_2$ on $z = 0$ we may erect perpendiculars P and P' of some height $h < \delta_1$, and $\frac{dx}{d\tau}$ will have the sign of $Q_o(x^o \pm \delta_2)$ on these lines (Fig. 1). If we then connect P and P' by any line $z = a$, $0 < a < h$ we will have formed a three sided box whose sides are segments without contact, that is they are never tangent to the vector distribution $(\frac{dx}{d\tau}, \frac{dy}{d\tau})$. With increasing τ these sides may be crossed in various directions by solutions of (7). As $\frac{dx}{d\tau}$ has opposite signs on P and on P' there are exactly four possible combinations of crossing directions as is indicated by the velocity vectors in Fig. 2.

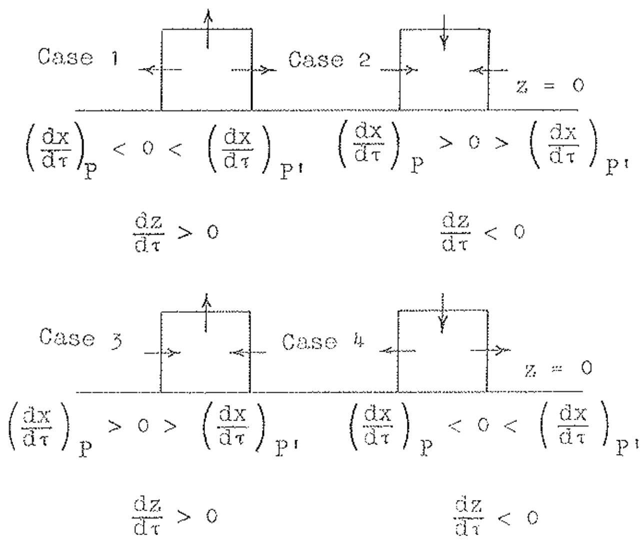

FIGURE 2

If x^o is a node of (1), then either case 1 or case 2 occurs, if x^o is a saddle then case 3 or case 4 occurs. We will still apply these names to x^o even though the vector field arising from (2) is time dependent. Thus if case 1 holds we call x^o a repulsive node with respect to the parameter τ, designating it by NR_τ, if case 2 holds, an attractive node NA_τ, if case 3, a repulsive saddle SR_τ, if case 4, an attractive saddle SA_τ.

These boxes of segments without contact, formed near critical points x^o, will make up part of the final curve J. The boxes cannot follow each other in any random order along $z = 0$, for by considering the sign of Q_o on the sides of successive boxes it appears that the only possible successive pairs are NR_τ, NA_τ; NR_τ, SR_τ; NA_τ, SA_τ; SR_τ, SA_τ; and

SR_τ, SA_τ is excluded by condition (c) of the theorem.

These boxes lie in the domain covered by the original x, y coordinates where they appear as flat ended wedges open toward infinity (Fig. 3).

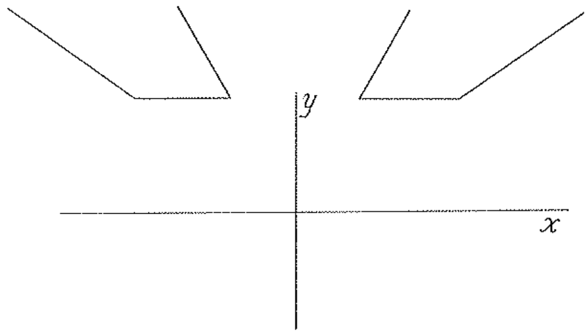

FIGURE 3

The ends are lines $y = \text{const.}$, the sides are lines $\frac{x}{y} = \text{const.}$ If $(1, 0, 0)$ is a critical point we simply adopt $(1, y, z)$ coordinates and proceed in exactly the same way to form a suitable box.

The next step is to join these boxes together using more segments without contact.

In the region covered by $(x, 1, z)$ consider a closed interval $[ab]$ on $z = 0$ which is free of critical points. Then $Q(x^0) \neq 0$ on $[ab]$. So if δ' is chosen sufficiently small $\{P_0 + P_1 z + \dots (P_n + E_2)z^n\}$ is bounded, and $\{Q_0 + Q_1 z + \dots + (Q_n + E_1 - xE_2)z^n\}$ is bounded and bounded away from zero in the box B', $0 \leq z \leq \delta'$, $a \leq x \leq b$, and for all t. From (8) then for points in B'

$$(9) \qquad - Mz < \frac{dz}{dx} < - M'z$$

for some constants M and M'. Therefore the portions of the curves $z = z(a)e^{-M(x-a)}$, $z = z(a)e^{-M'(x-a)}$ lying in B' are arcs without contact, one having slope $\frac{dz}{dx}$ greater than the slope of solutions to (8), the other having smaller slope (Fig. 4). Since $z(b) \longrightarrow 0$ as $z(a) \longrightarrow 0$, it follows immediately that one may connect any arbitrarily small perpendicular at a to an arbitrarily small perpendicular at b using arcs without contact. One arc will be crossed only by parametrized solutions which, as

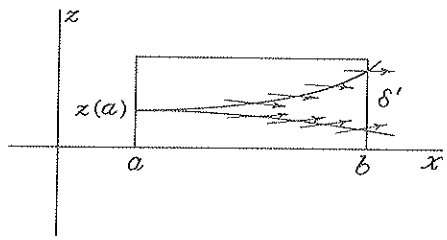

FIGURE 4

τ increases, move into the sector under it, while the other will be crossed only by solutions moving outward (Fig. 4).

Although in this construction we have assumed that a and b were in the region covered by coordinates (x, 1, z), the same construction applies to any a and b on z = 0 whose perpendiculars form a box not containing a critical point. It is only necessary to divide [ab] into subintervals lying entirely in one coordinate system or the other, and connect the separate arcs obtained for each.

We now return to x, y coordinates and the parameter t and put together boxes and arcs to form J.

For each singularity at infinity we have a flat ended wedge. We will take the solution curves as parametrized with t, thus when we refer to an attractive node, we will mean with respect to t (Fig. 5).

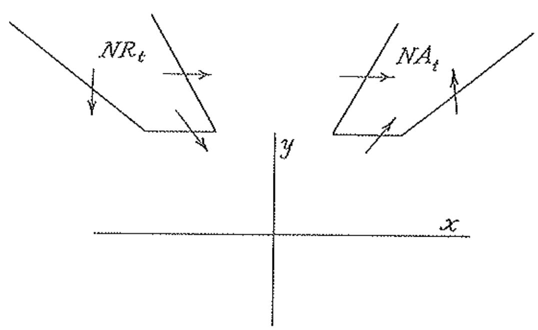

FIGURE 5

Successive boxes at infinity give rise to wedges which succeed each other angularly. It is easily seen that the only possible successive wedges are NR_t, NA_t; NR_t, SR_t; NA_t, SA_t; SR_t, SA_t. The last is excluded by condition (c).

We construct J in two steps. First, we connect any pair of neighboring wedges by an arc without contact. As the sides of the wedges are perpendiculars to $z = 0$, and the sides of successive wedges form the ends of a box without critical points, the previous construction applies.

If an NR_t is being connected to an NA_t, choose the arc without contact so that solutions cross it outward with increasing t, i.e., toward the line at infinity (Fig. 6).

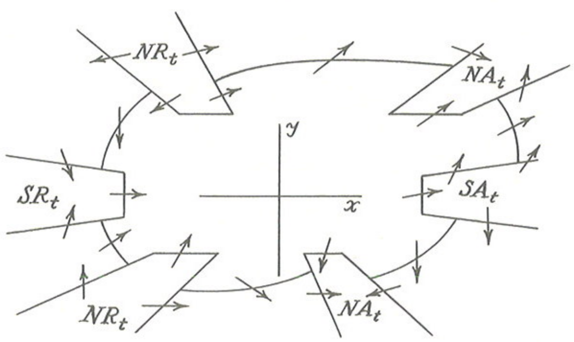

FIGURE 6

If an NR_t is being connected to an SR_t, let the connecting arc be chosen so that the solutions cross it moving away from the line at infinity. If an NA_t is connected to an SA_t let the arc be crossed toward the line at infinity (Fig. 6).

Now the wedges with their connecting arcs bound a 2-cell. With one modification the boundary of this 2-cell will be the curve J.

We remember that in constructing the boxes at critical points on $z = 0$, the top of the box was a segment of $z = a$, where a could be taken arbitrarily small. We now choose a new a' and a new segment $z = a'$ so near $z = 0$ that the segment cuts the sides of the box below the intersections with the connecting arcs (Fig. 7). This change will be made only on saddle boxes. The new boundary of the central 2-cell is the curve J.

We will now show that a solution through a point p on J never returns to p.

First, suppose p is an attractive nodal wedge (Fig. 8). Then

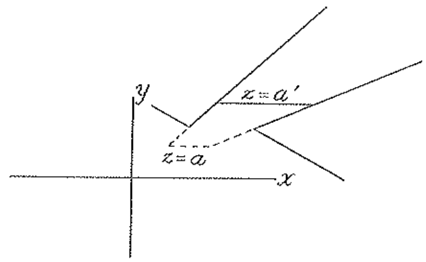

FIGURE 7

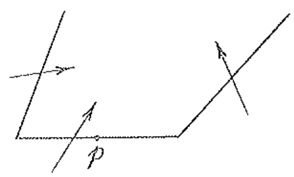

FIGURE 8

the path through p enters the wedge and can never leave it, hence cannot return to p. Similarly, if the node is repulsive, the path leaving the wedge can never re-enter it, so cannot return to p.

If now p is on an arc connecting two nodal wedges as in Fig. 9,

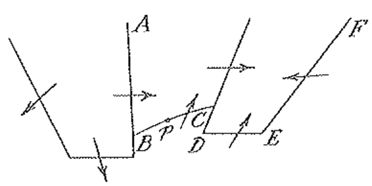

FIGURE 9

then one wedge combined with the box between the nodal wedges forms a region which is only entered and never left by solutions. See Fig. 9. — A B C D E F — Thus the solution through p cannot return to p.

Finally, suppose p is on a saddle wedge, or on an arc connecting a saddle wedge with a nodal wedge. Since there are no two consecutive saddle wedges, every saddle is flanked with nodal wedges (Fig. 10).

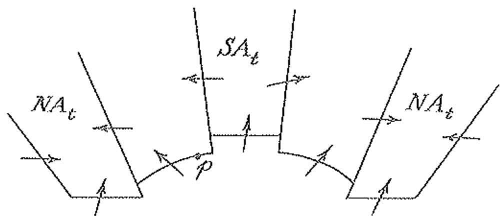

FIGURE 10

Combining all three wedges and the two enclosed boxes we form another region only entered by solutions. Thus p again has the desired property, and this last argument clearly holds whether the wedges involved are attractive or repulsive.

As we have now covered all possible locations of p we conclude that J has the desired property.

It is also clear from the mode of construction that one may select J as far out in the plane as desired. In particular, for reasons which will appear later, we will take J so large that it contains all the finite critical points of (1) and of

$$\frac{dx}{dt} = X(x, y) + E_1(0), \quad \frac{dy}{dt} = Y(x, y) + E_2(0).$$

We now consider

$$\frac{dx}{dt} = \left\{X(x, y) + E_1(t)\right\} F(x, y)$$

(10)

$$\frac{dy}{dt} = \left\{Y(x, y) + E_2(t)\right\} F(x, y),$$

where $F(x, y)$ has continuous derivatives, is always positive, and is identically 1 in some circle containing J, while as $R \longrightarrow \infty$, $F(x, y) \longrightarrow 0$ so rapidly that the right hand sides in (10) are bounded in the entire plane. An appropriate F can easily be found.

Now the curve J has its special property for the solutions of (10) as well as for those of (2). For the velocity vectors of (10) have the same direction as those of (2) and so point into or out of the same wedges. Thus, if a solution to (10) is at p on J, it never returns there at a later time.

Also equation (10) gives us a mapping of the plane into itself. For, let $u(t, u^{o})$ be the position at time t of that solution to (10) which is at u^{o} for $t = 0$. Then define the mapping φ_t by $\varphi_t u^{o} = u(t, u^{o})$. Because of the boundedness of (10) φ_t is defined for all t.

If we assign to the point u^{o} the vector $\varphi_t u^{o} - u^{o}$ we obtain a continuous vector distribution in the plane which also varies continuously with t. We will now compute the index of this distribution on J.

First of all, the index is the same for all $t > 0$, for the index is an integer and the vector distribution varies continuously with t. In this situation the index can only change with t if for some point u^{o} on J and some t, the vector $\varphi_t u^{o} - u^{o}$ vanishes. But this means $u^{o} = u(t, u^{o})$ which cannot occur on J as on J no solution may return to its starting point. Therefore, the index is the same for all t.

As we have seen before, J may be taken so large that the velocity vectors $v_1(x, y) = (X(x, y), Y(x, y))$, and $v_2(x, y) = (X(x, y) + E_1(0), Y(x, y) + E_2(0))$ vanish only inside J. Therefore, both of these are vector distributions with a well defined index on J. We will see that the two indices computed with respect to these vector distributions on J are the same, and in fact that they are both equal to the index computed with respect to the vectors $\varphi_t u^{o} - u^{o}$.

The first equality results from the fact that addition of $E_1(0)$ and $E_2(0)$ to X and Y does not affect the nature of the critical points at infinity; they remain nodes or saddles as before as is evident from (8). As the sum of all critical point indices must be 1, and in both cases all the finite critical points are in J, we have for both vector distributions

$$\text{index } (J) = 1 - \sum_i \text{index } (p_i), \tag{11}$$

the p_i being the critical points on $z = 0$. So the two indices are equal.

Now at any point $u^{o} = (x^{o}, y^{o})$ on J we may write

$$x(t,x^{o},y^{o}) = x(0,x^{o},y^{o}) + \frac{dx}{dt}(0,x^{o},y^{o})t + \frac{d^2x}{dt^2}(t',x^{o},y^{o})\frac{t^2}{2}$$

$$y(t,x^{o},y^{o}) = y(0,x^{o},y^{o}) + \frac{dy}{dt}(0,x^{o},y^{o})t + \frac{d^2y}{dt^2}(t'',x^{o},y^{o})\frac{t^2}{2}$$

where $x(t, x^{o}, y^{o})$, $y(t, x^{o}, y^{o})$ are the components of $u(t, u^{o})$ and t' and t'' are less than t. Because the right hand sides in (10) are bounded in the entire plane, $u(t, u^{o})$ lies inside some finite region R, for all u^{o} on J and all t less than any fixed t_1. But in

$R \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$ are bounded, so as $t \longrightarrow 0$,

$$\frac{u(t,u^o)-u^o}{t} \longrightarrow v_2(x^o, y^o)$$

uniformly on J.

Therefore, for some t_o sufficiently small, the angle between $v_2(x^o, y^o)$ and $\varphi_{t_o} u^o - u^o$ is less than ϵ throughout J, and the index of the vector distribution $\varphi_{t_o} u^o - u^o$ is the same as the index of v_2, namely $1 - \sum_i \text{index } (p_i)$. By condition (b) of the theorem, $1 - \sum_i \text{index } (p_i) \neq 0$.

Since the index is the same for all φ_t, it is $\neq 0$ for φ_T, T being the period of the E_i. Hence there is a fixed point under the map φ_T. That is, there exists at least one point u' inside J for which $\varphi_T u' - u' = 0$. The solution through u' must return to u' after T seconds, and so is evidently a periodic solution to (10).

However, because of its periodicity, this solution must lie inside J not only for $t = 0$, $t = T$, and so forth, but for all t. For if at any time it cut the curve J, say at p, then it could never return to p for later t, and this contradicts its periodicity. Therefore, it must be inside J for all t.

As $F(x, y)$ is identically 1 inside J, the solution through u' is in fact a periodic solution to (2). This establishes the theorem.

§4. It is not really necessary for the $E_i(t)$ to be purely time dependent. If the E_i are replaced by $E_1'(x, y, t)$, $E_2'(x, y, t)$, where the E_i' are polynomials of degree less than n in x and y with coefficients periodic in t, the proof of the theorem goes through unchanged.

PART II: CRITICAL POINTS AT INFINITY OF A CLASS OF SECOND ORDER EQUATIONS

§1. The second order equation

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0 \tag{12}$$

gives rise to the phase plane system

(13) $$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -f(x)y - g(x).$$

In this section the critical points at infinity of (13) will be completely analyzed in the case where $f(x)$ is a polynomial

$$\sum_{s=0}^{m} a_s x^s \quad \text{of degree } m,$$

and $g(x)$ a polynomial

$$\sum_{s=0}^{n} b_s x^s \quad \text{of degree } n,$$

and $m \geq n > 0$. The nature of these critical points will be shown to be completely determined by the leading coefficients a_m and b_n of the polynomials, and by the parity of m and n. From these facts geometric results will be deduced about the behavior in the large of trajectories.

As in Part I, consider in place of (13)

(14) $$\frac{d}{dt}\left(\frac{x}{z}\right) = \frac{y}{z}, \quad \frac{d}{dt}\left(\frac{y}{z}\right) = -f\left(\frac{x}{z}\right)\frac{y}{z} - g\left(\frac{x}{z}\right)$$

and proceeding as before, with $\frac{dt}{d\tau} = z^m$ for the region covered by $(x, 1, z)$ we obtain

(15) $$\begin{cases} \dfrac{dz}{d\tau} = z\left[f^H(x, z) + z^{m-n+1}g^H(x, z)\right] \\[2ex] \dfrac{dx}{d\tau} = x\left[f^H(x, z) + z^{m-n+1}g^H(x, z)\right] + z^m \end{cases}$$

and for the region with coordinates $(1, y, z)$, using $\frac{dt}{d\sigma} = z^m$,

(16) $$\begin{cases} \dfrac{dz}{d\sigma} = -z^{m+1}y \\[2ex] \dfrac{dy}{d\sigma} = -f^H(1, z)y - z^{m-n+1}g^H(1, z) - z^m y^2 \end{cases}$$

where $f^H(x, z)$, $g^H(x, z)$ denote f and g made homogeneous with z.

It is evident from (15) and (16) that $z = 0$ is a trajectory connecting critical points, and that the only critical points on $z = 0$ are $(0, 1, 0)$ and $(1, 0, 0)$. Before proceeding with an analysis of these critical points, we will mention some lemmas on trajectories near a singularity.

§2. Let C be a trajectory of an analytic system approaching an isolated critical point P. At a regular point q on C any sufficiently small circular neighborhood $N(q)$ is divided by C into cells lying on opposite sides of C. These cells will be called half-neighborhoods, $N_{1/2}(q)$. We will often refer to a curve C together with a side A of C along which all the half-neighborhoods are to be chosen. The choice of side will be indicated by a subscript A or B.

In terms of half-neighborhoods we will define what is meant by a prolongation.

Let C_A approach P with a definite limiting direction. Then any sufficiently small circle S around P will be cut only once by C, and the intersection point q will not be a tangency.

We will say that C_A has a prolongation with respect to S if there is another curve $C'_{A'}$ cutting S at q' and approaching P such that for every $N_{1/2}(q')$ there is an $N_{1/2}(q)$ such that any trajectory through $N_{1/2}(q)$ in S cuts $N_{1/2}(q')$, and lies entirely inside S between the two neighborhoods. (Fig. 11)

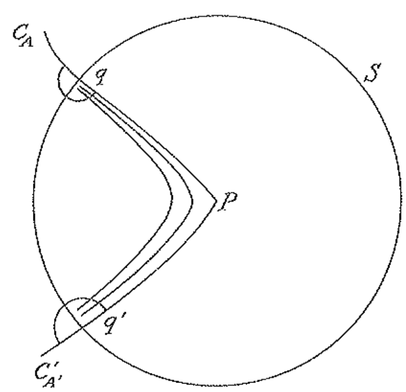

FIGURE 11

It is further stipulated that no trajectory shall tend to P between C and C'. Here between means in the cell $qPq'q$ that intersects the half-neighborhoods.

We are now in a position to state a lemma which is an immediate consequence of theorems of Bendixson [1].

LEMMA 1. Let C_A tend toward P with a definite limiting direction. Then for any sufficiently small circle S around P cut by C at q either

(a) C_A has a prolongation with respect to S, or

(b) there is a half-neighborhood $N_{1/2}(q)$, such that every trajectory through $N_{1/2}(q)$ approaches P lying always inside S. (Fig. 12)

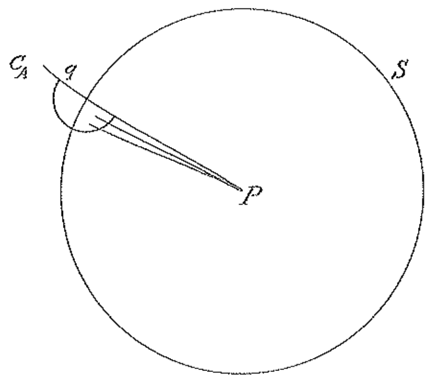

FIGURE 12

Provided we have taken S sufficiently small, it is clear that if case (a) holds, it holds also when C is referred to even smaller circles S'. Therefore, the same remark must apply when case (b) holds. So, although this is not always the case for non-analytic systems, we can use here the phrase "C_A has a prolongation" meaning (a) holds for all sufficiently small S, or "C_A finishes" meaning (b) holds, without specifying a particular circle S.

Another useful lemma is

LEMMA 2. Let S be a circle containing the isolated critical point P, and which contains no closed trajectory, no critical point other than P, and is not itself a trajectory. Then either

(a) there is a curve tending to P which has a prolongation, or

(b) there is a neighborhood N of P such that if $q \in N$, then the trajectory through q tends to P in one direction or the other, and without touching S.

PROOF: In this proof we will use half-trajectory or half-characteristic to indicate a trajectory pursued from a starting point in one direction or the other. By positive (negative) half-trajectory is meant the path traced out by the starting point as parameter values increase (decrease).

Now suppose (b) does not hold. Then there exists a sequence of points $p_n \longrightarrow P$ such that both the half-trajectories through p_n either

(1) fail to tend to P, or

(2) cut S.

In fact we may assume (2) holds, for if a trajectory C fails to tend to P it must tend to another singular point, to a closed trajectory, or to a curve with prolongations, Bendixson [1]. Under our hypotheses this implies it cuts S, or else tends in S to a curve with a prolongation, which already proves (a), therefore we may assume (2) holds.

With the trajectory C_n through P_n associate the points a_n and b_n, the first intersections with S of the positive and negative half-trajectories through p_n. Then $a_n p_n b_n$ divides the interior of S into two 2-cells. We will call the one containing P the exterior, and the other the interior. The two arcs $a_n b_n$ of S will be called exterior and interior arcs.

Because of analyticity, S has only a finite number of tangencies with the vector field, these occur at points T_m. Consider all the C_n having T_m on their interior arc. For some m there is an infinity $C_{n'}$ of these. As the $C_{n'}$ cannot cross, they may be arranged in order, with each containing its predecessor in its interior 2-cell (Fig. 13).

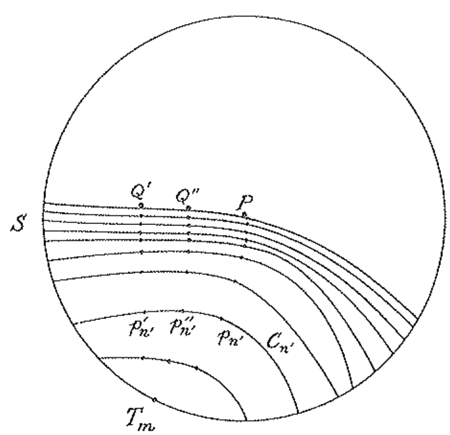

FIGURE 13

Denoting by $d(C_{n'}, P)$ the distance from $C_{n'}$ to P, it is clear that every $C_{n'}$ has points at any distance d' from P, $d(C_{n'}, P) < d' < R$, where R is the radius of S. Since $d(C_{n'},P) \longrightarrow 0$ as $n' \longrightarrow \infty$, we may always find a sequence of points $p^{\prime}_{n'}$, $p^{\prime}_{n'} \in C_{n'}$, such that the $p^{\prime}_{n'}$ have a limit point Q' at a distance d' from P, $0 < d' < R$.

Choose such a sequence with a limit point Q' at distance d' from P. It will appear that the characteristic C' through Q' must tend to P in one direction or the other, for if it merely connects two points of S we have the following consequences. All $C_{n'}$ lie in the exterior of C' for n' greater than some $N^{\prime}_{0}$, for these $C_{n'}$ will have points at a distance from $P < d(C', P)$, and therefore will have one point in the exterior of C', and so lie entirely in the exterior. Now choose a new sequence $p^{\prime\prime}_{n'}$ on the $C_{n'}$ with limit Q'' at distance d'' from P, $0 < d'' < d'$. The characteristic C'' through Q'' lies in the exterior of C', and the $C_{n'}$, $n' > N^{\prime\prime}_{0}$ lie in the exterior of C'', both by the same argument as before. Thus the $p^{\prime}_{n'}$ lie in the exterior of C'', $n' > N^{\prime\prime}_{0}$, and so Q' cannot be their limit point, a contradiction. Hence C' tends to P in one direction or the other.

We will show C' to have a prolongation. C' approaches P with a definite limiting direction. (A characteristic either approaches a critical point with a definite limiting direction or else its angle increases without limit, this latter is impossible as C' would cross some $C_{n'}$.) Thus there is a sufficiently small circle S^* such that C' cuts S^* at Q^* and then always lies in the interior of S^*.

As C' is the characteristic through Q', by continuity there are characteristics $C_{n'}$ arbitrarily close to Q^* on one side of C', these $C_{n'}$ of course, do not tend to P. Therefore, the conditions for case (b), Lemma 1, are not fulfillable, and case (a), Lemma 1, must hold. There is a prolongation. This establishes Lemma 2.

§3. We will now analyze the critical points at infinity, starting with the point $(0, 1, 0)$, or the $x = 0$, $z = 0$ of equation (15).

Let $\theta = \arctan\left(-\frac{z}{x}\right)$ and $R^2 = x^2 + z^2$, then from (15)

$$\frac{d\theta}{d\tau} = \frac{z^{m+1}}{R^2} \tag{17}$$

so that if the axes are taken as in Fig. 14, trajectories below the line $z = 0$ move counterclockwise, those above counterclockwise or clockwise accordingly as m is odd or even. It is also clear that $z = 0$ $x > 0$, and $z = 0$ $x < 0$ are trajectories tending to $(0, 0)$ with directions of

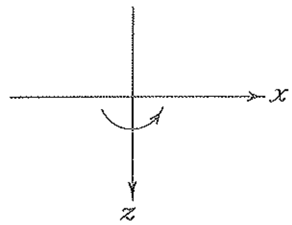

FIGURE 14

motion given by

$$\left(\frac{dx}{d\tau}\right)_{z=0} = a_m x^{m+1}. \tag{18}$$

Since these are trajectories tending to $P = (0, 0)$ with a definite limiting direction, it follows by Bendixson [1] that any characteristic approaching P will have a limiting direction for its tangent. The possible limiting directions are not arbitrary but are those that satisfy

$$0 = z\left(\frac{dx}{d\tau}\right)_L - x\left(\frac{dz}{d\tau}\right)_L = z^{m+1} \tag{19}$$

where the subscript L indicates that only the terms of degree

$$L = \min\left(\text{degree } \frac{dx}{d\tau}, \text{ degree } \frac{dz}{d\tau}\right)$$

are taken. So any trajectory approaching P must approach tangency with $z = 0$, either from above or below.

It will also be useful to distinguish the two sides of a half-trajectory $z = 0$ near $(0, 0)$. A side will be called positive if it borders on a quadrant in which the sign of $\frac{d\theta}{d\tau}$ is such that the acute angle φ between $z = 0$ and a radius vector to a point p in the quadrant constantly decreases in absolute value. It will be called negative if $|\varphi|$ increases. As $\frac{d\theta}{d\tau}$ has constant sign throughout each quadrant, there is no ambiguity.

We have at once this simple lemma.

LEMMA 3. Let C be a positive (negative) half-trajectory tending to $(0, 0)$. Then C approaches tangency with one of the half-trajectories $z = 0$ on a positive (negative) side.

PROOF: As C, a positive half-trajectory, approaches P, it must approach tangency with one of the half-characteristics $z = 0$. Since C cannot reach or cross $z = 0$ it will always lie in the same quadrant for $\tau >$ some τ_0. At time τ_0, $|\varphi| = b > 0$, while for τ_2 sufficiently large $|\varphi| < b$. Hence for some τ', $\tau_0 < \tau' < \tau_2$, $\frac{d|\varphi|}{dt} < 0$, so the quadrant is such that the side approached is positive. A similar proof applies in the negative case.

We will now proceed with a case by case analysis of the singularity at $P = (0, 0)$.

Case (1a). m even, $a_m > 0$.

Equation (18) shows that $z = 0$ consists of two negative half-trajectories approaching P : C', $z = 0$ $x > 0$, and C'', $z = 0$ $x < 0$. If we adopt A and B for their lower and upper edges, we see from equation (17) that C'_A and C'_B are positive sides, and C''_A and C''_B negative (Fig. 15).

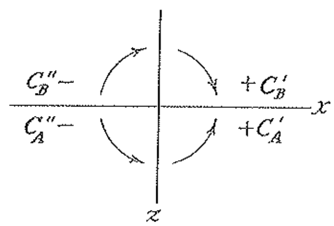

FIGURE 15

We next notice that there are in fact no positive half-trajectories approaching P. For if we consider the curve α

$$f^H(x, z) + z^{m+1-n}g^H(x, z) = 0 \tag{20}$$

the tangents to the various branches of the curve at $(0, 0)$ are the directions satisfying

$$f^H(x, z) = 0 \tag{21}$$

for $f^H(x, z)$ is homogeneous of degree m, while $z^{m+1-n}g^H(x, z)$ is of degree $m + 1$. Since z does not divide $f^H(x, z)$, $z = 0$ is not a tangent to any of the curves, so there will be an ϵ_1, and an ϵ_2 such that the region D, $0 < |x| < \epsilon_1$, $|\frac{z}{x}| < \epsilon_2$ is not entered by any of the branches of α (Fig. 16).

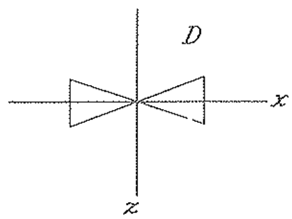

FIGURE 16

In each of the two open triangles of D $f^H(x, z) + z^{m+1-n}g^H(x, z)$ cannot change sign, and therefore has the same sign as for $z = 0$, that is sign $(a_m x^m)$. Hence from (15) in these triangles

$$\text{sign } \frac{dz}{d\tau} = \text{sign}\left(za_m x^m\right) = \text{sign } (z). \tag{22}$$

Since any positive half-trajectories must approach tangency to $z = 0$, they must lie inside the triangles for all τ greater than some τ_0. If they are not the trajectories C' or C'' themselves, then $|z(\tau_0)| > 0$, and $|z(\tau)| \longrightarrow 0$ as $\tau \longrightarrow \infty$. But (22) states that for these trajectories in D, $\frac{d|z|}{d\tau} > 0$, a contradiction. As C' and C'' themselves are negative half-trajectories approaching P, no positive half-trajectories tend to P.

Since any prolongation obviously involves both a positive and a negative trajectory, there are no curves through P with a prolongation. Hence, using Lemma 2, all curves inside a sufficiently small circle S' tend to P as $\tau \longrightarrow \infty$ or as $\tau \longrightarrow -\infty$ without leaving a second small circle S. By the above we know that in fact curves can tend to P only as $\tau \longrightarrow -\infty$, therefore they are all negative trajectories and by Lemma 3 can tend only to the negative sides of C' and C''. Since $z = 0$ cannot be crossed, we see that for $z > 0$ the trajectories tend with decreasing τ toward tangency with C''_A, and for $z < 0$ toward tangency with C''_B (Fig. 17).

Case (1a)

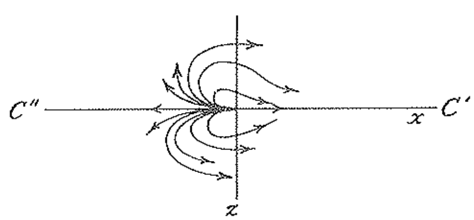

FIGURE 17

So for m even, $a_m > 0$ P is a (non-simple) unstable node.

Case (1b). m even, $a_m < 0$.

From equation (18) we see that C' and C" are now positive half-trajectories approaching P. As (17) shows, $\frac{d\theta}{d\tau}$ is unaffected by the sign of a_m so that the sides C'_A, C'_B, etc. are positive or negative as before. The argument immediately preceding equation (22) applies again only (22) now becomes

$$\text{(23)} \qquad \text{sign } \frac{dz}{d\tau} = \text{sign } \left(za_m x^m\right) = - \text{ sign } (z)$$

and the argument which previously showed that no positive half-trajectory approaches P, now shows that no negative ones do. Again there are no prolongations, and now inside some small circle all paths are positive half-trajectories and tend to P. Only now they must tend to positive sides, and so tend to C'_A and C'_B (Fig. 18).

Case (1b)

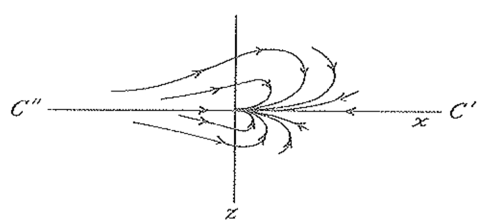

FIGURE 18

So P is a (non-simple) stable node.

Case (2a). m odd, $a_m > 0$.

Equation (18) shows that C' is a negative, C" a positive half-trajectory. Equation (17) shows $\frac{d\theta}{d\tau}$ to be always non-negative, so C'_A and C''_B are positive sides, and C'_B and C''_A negative. Also using once more the argument based on $\frac{dz}{d\tau}$ we find

$$\text{(24)} \qquad \text{sign } \frac{dz}{d\tau} = \text{sign } \left(za_m x^m\right) = \text{sign } (xz)$$

for the sign of $\frac{dz}{d\tau}$ inside the small triangles. This implies that negative trajectories cannot approach tangency to C''_A and C''_B, and that positive ones cannot approach C'_A and C'_B. As C''_A is a negative side, this means that no characteristics at all approach tangency to it, and as C'_A is positive, no characteristics approach it. So no characteristics other than C' and C" enter P from the lower half plane. Therefore C''_A

must have a prolongation, for if it finished at P, by Lemma 1 there would be a half-neighborhood on the A side, through which passed a characteristic tending to P, and clearly tending to P from the lower half plane. As this is impossible, case (a) of the lemma must hold, so C_A'' has a prolongation. The prolongation cannot lie in the upper half plane, for then characteristics passing near a point on C'' and a point on its prolongation, would cut C'. As there are no characteristics other than C' and C'' tending to P from the lower half plane, the prolongation can only be C'.

Thus the lower half plane near P is a saddle sector as shown in Fig. 19.

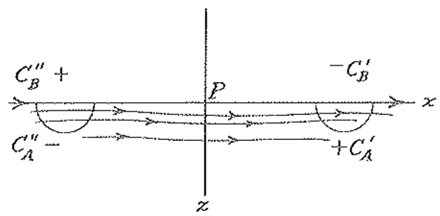

FIGURE 19

We will next show that there are no curves with prolongations tending to P from the upper half plane. For suppose there existed a C_1, a positive semi-characteristic with a prolongation on some side A. C_1 must tend to tangency with C_B'', and its prolongation C_2, being negative, must tend to C_B'. By Lemma 1 there exists a trajectory T passing arbitrarily close to points of C_1 and then proceeding with increasing τ arbitrarily close to points of C_2 while lying always inside a small circle around P. Now when near C_1, T is near C_B'' and hence has a θ coordinate greater than say $\frac{3}{4}\pi$. At a later time, near C_2, it is near C_B' and therefore has a θ coordinate less than $\frac{1}{4}\pi$. So θ has decreased with time. However, in the region in which (17) applies, and in which T always lies during the time under discussion, $\frac{d\theta}{d\tau}$ is non-negative, a contradiction. Hence such a prolongation is impossible.

Denoting points $z \leq 0$ by U, and repeating the arguments of Lemma 2 but restricting ourselves to points in U, we conclude that as there is no prolongation, given any circle S, there is another circle S' such that if p belongs to $S' \cap U$, then the positive or the negative half-trajectory through p tends to P inside $S \cap U$. From Lemma 1 it follows that as there are no prolongations, the set of points inside S' whose positive half-trajectories tend to P in S is open, and of course the same applies to the set of points whose negative trajectories tend to

P. As the union of these two sets is the interior of the connected set $S' \cap U$, their intersection is not void, so there is a point p whose positive and negative half-trajectories tend to P. Taking a new semi-circle S_1 through p, we may find a smaller semi-circle S_2 which is divided into three sectors by the positive and negative half-trajectories $C^+(p)$ and $C^-(p)$ through p (Fig. 20). S_2 may also be taken to have the property that if q is in S_2, either $C^+(q)$ or $C^-(q)$ tends to P inside S_1.

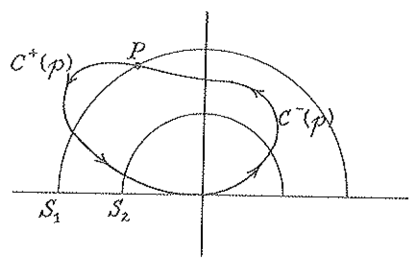

FIGURE 20

If we consider q in the central sector, $C^+(q)$ and $C^-(q)$ lie always inside C(p), so both must tend to P. If we consider q in either of the other sectors, then either $C^+(q)$ or $C^-(q)$ tends to P inside S_1 and outside C(p). In fact as C_B'' is a positive, and C_B' a negative side, if q is in a sector bordering C_B'', then $C^+(q)$ tends to P in S_1 and $C^-(q)$ does not, and if q is in a sector bordering C_B', $C^-(q)$ tends to P in S_1, and $C^+(q)$ does not.

Thus the configuration at P is as in Fig. 21.

Case (2a)

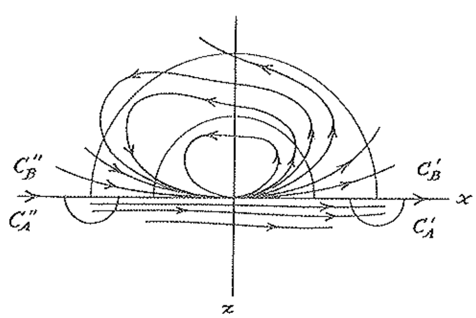

FIGURE 21

Case (2b). m odd, $a_m < 0$.

Equation (18) shows C' to be a positive, C'' to be a negative half-characteristic. The sides of the characteristics are positive and negative as before, as $\frac{d\theta}{d\tau}$ is unchanged. With obvious modifications all arguments go through as in case (2a) and the singular point again has one prolongation and one closed nodal region, although with positions reversed as indicated in Fig. 22.

Case (2b)

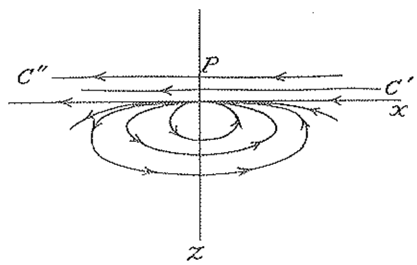

FIGURE 22

We have thus determined the nature of the singularity $x = 0$, $z = 0$, which is the only singularity on the line at infinity for $y \neq 0$. The nature of this singularity has turned out to be completely determined by the parity of the degree m of $f(x)$, and the sign of its leading coefficient a_m.

We next turn to the critical point $Q = (1, 0, 0)$. Using y, z coordinates and equation (16) our differential equations are of the form

$$\frac{dz}{d\sigma} = Z(y, z), \quad \frac{dy}{d\sigma} = cy + dz + Y(y, z), \tag{25}$$

where Y and Z consist of terms of degree ≥ 2. In (16) $c = -a_m \neq 0$, while $d = -b_n$ if $m = n$, and is zero otherwise.

The possible directions of approach are given by

$$0 = z\left(\frac{dy}{d\sigma}\right)_L - y\left(\frac{dz}{d\sigma}\right)_L = z\left(cy + dz\right), \tag{26}$$

and are $z = 0$ and the line $L : cy + dz = 0$. Of course $z = 0\ y > 0$, and $z = 0\ y < 0$, are actually trajectories of (16). Now for equations of the form (25) we may use results due to Bendixson. First there are two and only two characteristics tending to Q tangent to $z = 0$. In (16) then these are the two trajectories $z = 0$. Secondly, considering the curves tending to Q tangent to L, we may divide them into two groups, those near Q with $z > 0$, and those near Q with $z < 0$.

If the index of Q is 1, both of these will form nodes, i.e., there will be an infinity of trajectories tending to Q in each group, and if C tends to Q, so do all trajectories through a neighborhood of C.

Index 1

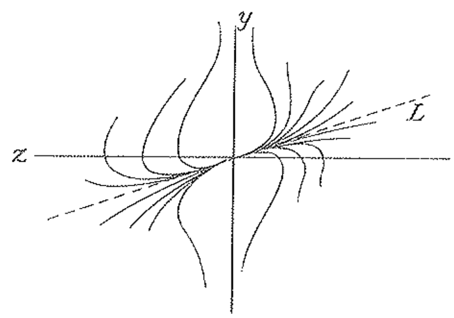

FIGURE 23

If the index is -1, there will be a unique curve in each group tending to Q, and it will have as prolongation one of the trajectories tending to $z = 0$. That is, we will have a saddle.

Index -1

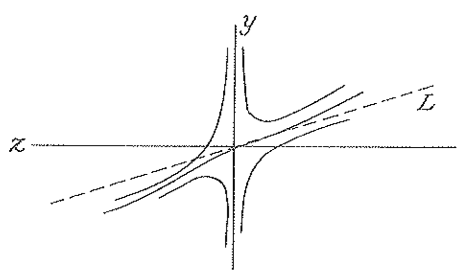

FIGURE 24

If the index is zero we will have one nodal and one saddle side. In this last case there are two possible arrangements depending on which side is nodal and which saddle. However, aside from this we see, following Bendixson, that the nature of the singularity Q is largely determined by the index. Now the index of Q is given by

$$\text{index}(Q) + \text{index}(P) + \sum_i \text{index}(p_i) = 1 \tag{27}$$

Index 0

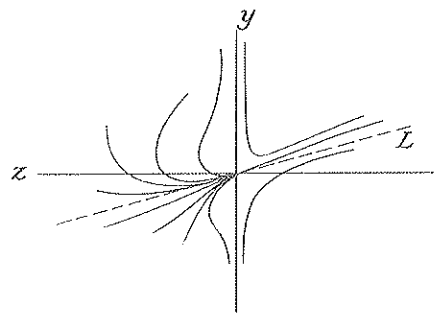

FIGURE 25

where the p_i are the critical points not on the line at infinity. The index of P may be computed from a formula of Bendixson

$$2(-\text{ index }(P) + 1) = c - n_f,$$

c being the number of curves having prolongations, n_f the number of closed nodal regions. Referring to cases (1a), (1b), (2a), (2b) we find always

(28) $$\text{index }(P) = 1.$$

The p_i are, of course, the points where $\frac{dx}{dt}$ and $\frac{dy}{dt}$ vanish simultaneously. Hence, referring to (13), all the p_i must lie on the x-axis and have x-coordinates a_i such that $g(a_i) = 0$. If we assume for the moment that $g(x)$ has only simple roots, a routine calculation shows that the points $(a_i, 0)$ are simple critical points, and that they have index $+1$ if $g'(a_i) > 0$, and index -1 if $g'(a_i) < 0$.

At successive intersections of the curve $g(x) = 0$ with the x-axis, $g'(x)$ will change its sign, hence at successive critical points the index alternates from $+1$ to -1. If $g(x)$ is of even degree, i.e., n even, there will be equal numbers of both types so (27) gives

(29) $$n \text{ even}, \quad \text{ind}(Q) = 0,$$

while if $g(x)$ is of odd degree there are two cases

(30) $$\begin{aligned} &n \text{ odd}, \quad b_m < 0 \quad \text{ind}(Q) = 1 \\ &n \text{ odd}, \quad b_m > 0 \quad \text{ind}(Q) = -1. \end{aligned}$$

Hence the index of Q is determined by $g(x)$ alone.

Equations (29) and (30) hold even when g has multiple roots, for these multiplicities can be removed by small changes in g which, as they change the vector field only slightly, cannot affect the index sum.

In the cases covered by equation (30) the index completely determines the nature of Q except for the direction in which the trajectories are to be pursued. This last is directly determined from (16) and depends on the sign of a_m.

In the case n even, it remains to determine which side of $z = 0$ at Q is a saddle and which side a node. Remembering that $z = 0$ is a trajectory, we observe that if the side $z > 0$ is a saddle side the vector field should rotate through $-\pi$ along any curve connecting a point on the positive y-axis with a point on the negative y-axis through the half-plane $z > 0$. Similarly, if the side is to be nodal, the rotation should be $+\pi$. We will actually obtain this rotation along a simple path.

Let the path consist of the sides formed in the half-plane $z > 0$ by the lines $y = \pm a$, $z = \epsilon > 0$ (Fig. 26).

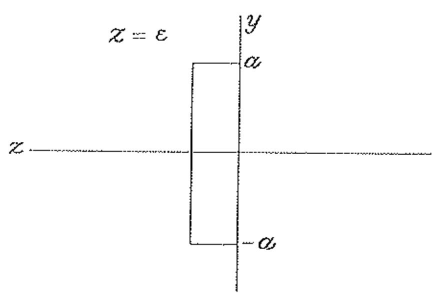

FIGURE 26

From (16) it appears that at $(a, 0)$ and $(-a, 0)$ the vectors $(\frac{dy}{d\sigma}, \frac{dz}{d\sigma})$ are vertical and oppositely directed. On $y = a$ $z > 0$, and on $z = \epsilon$ $y > 0$, we have $\frac{dz}{d\sigma} < 0$, so the vectors point toward the right half-plane $z < 0$. At $y = 0$, $\frac{dy}{d\sigma} = -\epsilon^{m-n+1} g^{H}(1, \epsilon)$ which for ϵ sufficiently small has the sign of $-b_n$. So at $z = \epsilon$, $y = 0$, the vector points toward the half-plane $-yb_n > 0$. On $z = \epsilon$ $y < 0$, and on $y = -a$ $z > 0$, $\frac{dz}{d\sigma} > 0$ and the vector points into the left half-plane, $z > 0$. This gives a rotation of $-\pi$ if $b_n > 0$, and $+\pi$ if $b_n < 0$.

So if n is even and $b_n > 0$, $z > 0$ is a saddle side (Fig. 27), and if $b_n < 0$, $z > 0$ is a nodal side (Fig. 28). We find that that the form of the trajectories near Q, that is, whether they are saddle or nodal sectors, is determined by the parity of n and the sign of b_n. So f(x) controls P and g(x) controls Q.

n even, $b_n > 0$

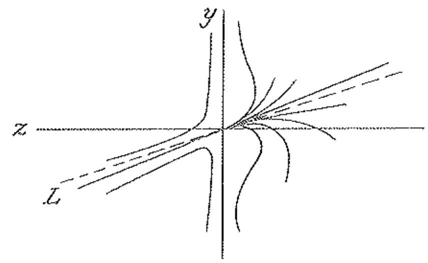

FIGURE 27

n even, $b_n < 0$

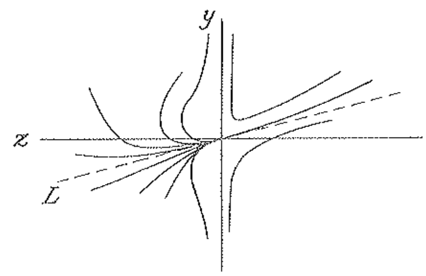

FIGURE 28

We can now give the various possible modes of behavior in the projective plane near the line at infinity, that is, the asymptotic behavior of solutions to (13) that become unbounded either for increasing or decreasing t (Fig. 29). In these figures the trajectories in the finite plane are given parametrized by t. The projective plane is represented as a circle and its interior, with opposite points on the circumference identified. The circumference itself consists of the trajectories $z = 0$ and the critical points P and Q.

Next consider $a_m < 0$. For m even the following changes should be made. First the direction of the parametrization is reversed, and secondly, the curves approaching P are now tangent at the other side. For example, from 1 comes 1A (Fig. 30). Similarly from 2, 3, 4 come 2A, 3A, 4A. For m odd the role of the upper and lower sides of P is interchanged. Thus from 5, 6, 7, 8 come 5A, 6A, 7A, 8A (Fig. 31).

The information obtained about behavior at infinity can be used to deduce results about the behavior in the large of the trajectories. For

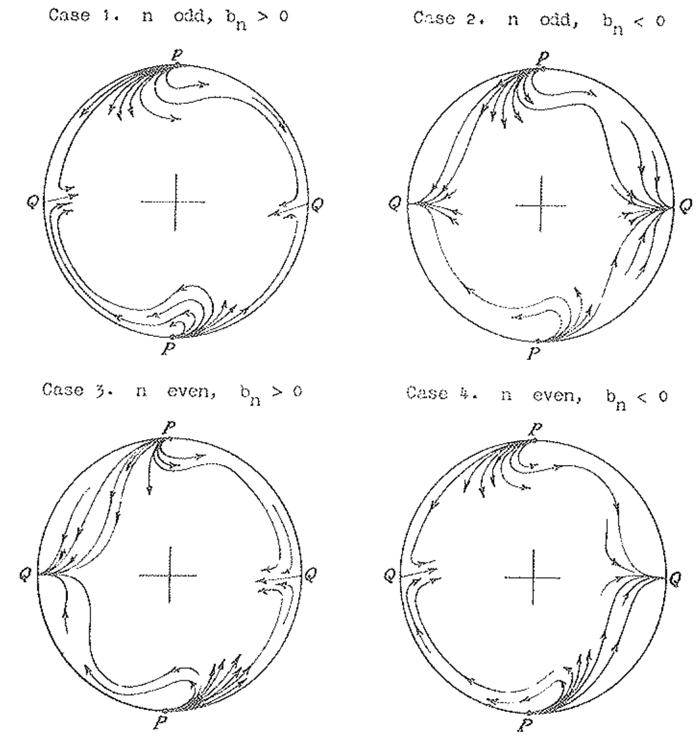

FIGURE 29a. m even, $a_m > 0$

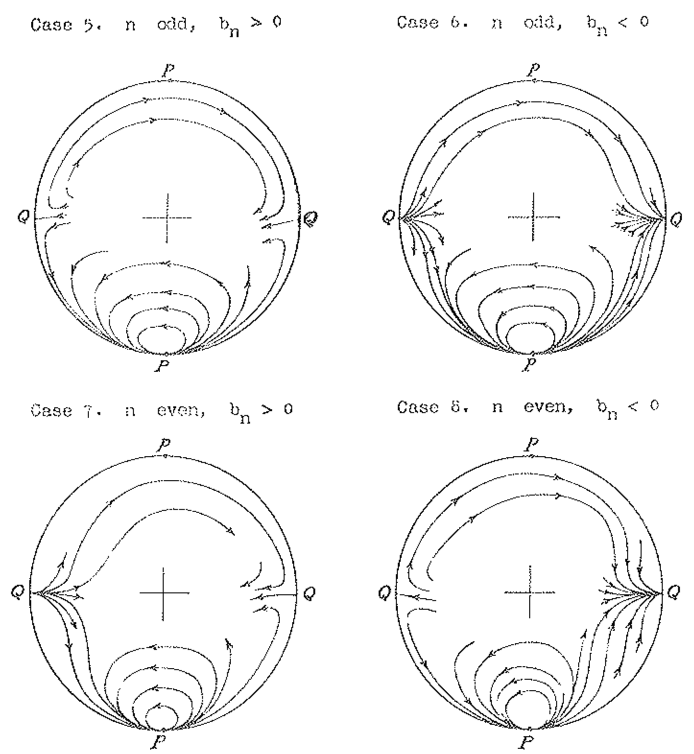

FIGURE 29b m odd, $a_m > 0$

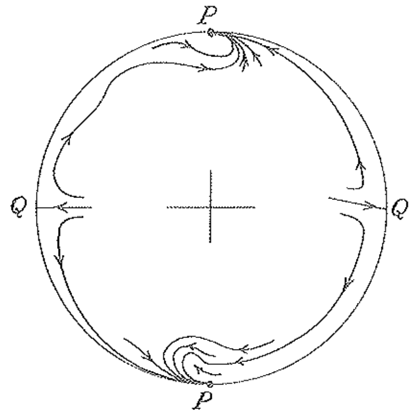

FIGURE 30

example, a glance at Fig. 29 shows that the only equations dissipative for large displacements are those of case (1), n even, m odd, a_m and b_n positive. Furthermore, if we take one of this class which, like van der Pol's equation, has a single unstable singularity in the finite plane, we can readily deduce the existence of limit cycles. For the trajectories emanating from the finite singularity must have as their limit sets either a limit-cycle, a critical point, or a graph made up of separatrices and critical points. As no trajectories tend to P or Q with increasing time, the last two possibilities are ruled out and limit cycles exist, in fact every point outside the outermost limit-cycle must tend to it with increasing time.

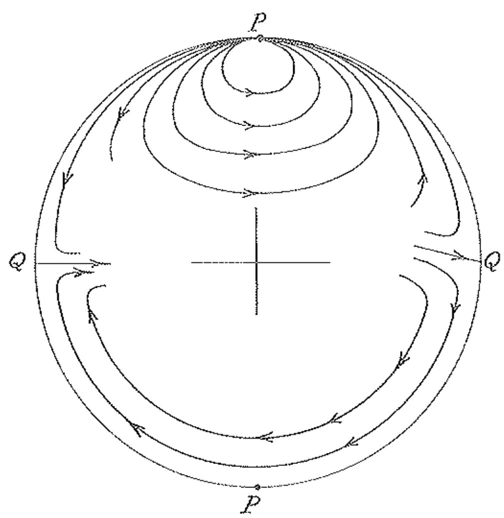

FIGURE 31

A similar remark applies to 1A with a single stable singularity.

If we take case (6) with a single simple singularity R in the finite plane, R will necessarily be a saddle. Consider the limit sets of the two positive separatrices C_1 and C_2 emerging from R. There are no limit-cycles since a limit-cycle must contain singularities with an index sum of 1. For the same reason neither separatrix can return to R, so a loop of linked separatrices is impossible. Hence, each must tend either to P or to Q along a possible direction of approach for positive trajectories. They cannot both tend to P (or Q) for if they did, RC_1PC_2 would enclose a certain region A free of singularities and every positive and negative trajectory in A would have to tend to P between C_1 and C_2. But this is a direction of approach for positive trajectories only. Hence, C_1 tends to Q and C_2 to P (Fig. 32). The paths of the other trajectories are then completely determined. C_2 and another separatrix bound the closed nodal region of P.

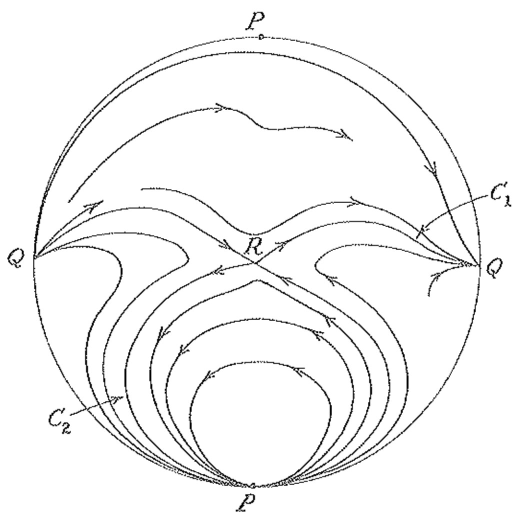

FIGURE 32

A single singularity in case (2) again gives an easily determined configuration, while in (5) there are many possibilities.

In (3), (4), (7), and (8) the simplest case is $g(x)$ never zero. With no finite singularities the destination of the unique separatrix issuing from Q is uniquely determined, and in turn it determines the destinations of the other paths. In case (3) it tends from Q to Q, in case (4) from Q to Q, in (7) from Q to P, in (8) from P to Q, so that the paths in the finite plane are as shown in Fig. 33.

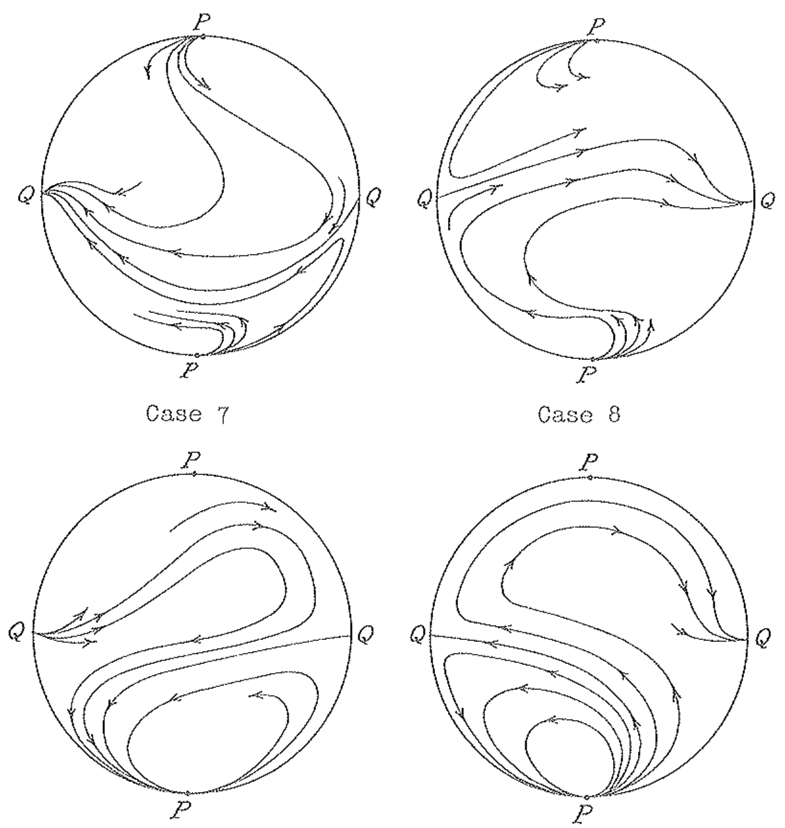

FIGURE 33

We see that in all cases (1) to (8) and (1A) to (8A) trajectories tend to infinity with a definite asymptotic direction. They are asymptotic to the line at infinity at P, and to a finite line at Q. If $m > n$, they are in fact asymptotic to the x-axis as they tend to Q. In no case is there a spiralling toward infinity.

PART III: FORCED OSCILLATIONS IN A CLASS OF SECOND ORDER EQUATIONS

§1. In this section the method of Part I and the knowledge of critical points obtained in Part II will be used to prove a forced oscillation theorem about

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = E(t). \tag{31}$$

Here, as before,

$$f(x) = \sum_{s=0}^{m} a_s x^s, \quad g(x) = \sum_{s=0}^{n} b_s x^s,$$

and $m \geq n > 0$, while the forcing term $E(t)$ has a continuous derivative and is periodic of period T.

THEOREM. If n is odd, and if we do not have both $b_n > 0$ and m odd, then equation (31) has a periodic solution of period T.

§2. As a first step toward establishing the theorem we will introduce a pair of equations closely related to the phase plane form of (31).

Choose any E' greater than $\sup_t |E(t)|$, and let

$$h_1(y) = \begin{Bmatrix} -E', & y \geq 0 \\ +E', & y < 0 \end{Bmatrix}$$

and

$$h_2(y) = \begin{Bmatrix} +E', & y \geq 0 \\ -E', & y < 0 \end{Bmatrix}$$

then introduce

$$\frac{dx}{y} = \frac{dy}{-f(x)y-g(x)+h_1(y)} \tag*{$(32)_1$}$$

and

$$\frac{dx}{y} = \frac{dy}{-f(x)y-g(x)+h_2(y)} \quad . \tag*{$(32)_2$}$$

In all the following we will only consider these equations in the exterior of a circle S which surrounds all the zeros of all the polynomials $-g(x) + c$, $|c| \leq E'$.

At every point of this region the equations $(32)_i$ satisfy Lipschitz conditions either in x or in y, so that we have unique trajectories, and in fact, in a sense to be made more precise below, these trajectories will have the same behavior at infinity as the trajectories of the corresponding equation (13).

Consider first the singularity $P = (0, 1, 0)$. Near P and on one side of the trajectory $z = 0$ the trajectories of $(32)_1$ coincide with those of

$$(33)_1 \qquad \frac{dx}{dt} = y \quad \frac{dy}{dt} = -f(x)y - g(x) - E'$$

while on the other side they coincide with the trajectories of

$$(33)_2 \qquad \frac{dx}{dt} = y \quad \frac{dy}{dt} = -f(x)y - g(x) + E'.$$

The arguments used in Part II to determine the behavior of the trajectories of (13) at P can be applied separately to each of the equations $(33)_j$ to find the behavior of the trajectories of $(32)_1$ on each side of $z = 0$. As the $(33)_j$ differ from (13) only by a constant, and the type of singularity obtained at P is determined by m and a_m, we will have only the same types of sides and the same combinations of sides as in the corresponding equation (13).

The same remark applies to $(32)_2$.

Near the critical point $Q = (1, 0, 0)$ consider the side $z > 0$, ($x > 0$ in x, y coordinates), and notice that $y = 0$ is a segment without contact. Suppose $z > 0$ is an attractive node or saddle side for (13) and hence for $(33)_1$ and $(33)_2$. Equation (16) shows that we have $\frac{dz}{d\sigma} = -z^{m+1}y$ for all three equations. Therefore all the trajectories, (or the unique trajectory), tending to Q do so in the half-plane $y > 0$. This behavior then is not affected when the trajectories of the $(33)_j$ are pieced together along $y = 0$ to obtain the trajectories of the $(32)_i$. Similar remarks apply to repulsive nodes and saddles and to $z < 0$. The following more precise statements may be verified at once.

Let the trajectories of the $(32)_i$ be given the parametrization obtained naturally from the appropriate $(33)_j$. Suppose first that the side $z > 0$ of Q is an attractive nodal side for (13). Then, given any semicircle S_1 around Q, there will exist an S_2, such that if x^0 is in S_2, the trajectory of $(32)_i$ through x^0 tends to Q inside S_1. Now suppose that $z > 0$ is a saddle side of Q for (13). Then for each equation $(32)_i$ there is a unique trajectory tending to Q, and its prolongations are the parts $y > 0$ and $y < 0$ of the line $z = 0$, exactly as is the case for trajectories of (13).

These statements may be repeated for a repulsive node or saddle, and for the side $z < 0$.

§3. We will now introduce some notations needed for the main lemma of this section. By C_i^+, C_i^- will be meant positive and negative half-trajectories of the $(32)_i$, the parametrization being obtained from the appropriate equation (33). The region $x^2 + y^2 \geq R^2$ will be denoted by $K(R)$. Also, given some $K(R)$, the set of points x^o in $K(R)$ such that $C_i^+(x^o)$ tends to infinity in $K(R)$ will be denoted by $A_i^+(R)$. Similarly from the C_i^- we obtain the sets $A_i^-(R)$.

A large part of the proof of the theorem consists of establishing the following lemma.

> LEMMA. If the $f(x)$ and $g(x)$ in (32) satisfy the conditions for the theorem, then, given any sufficiently large R there is an R_o depending on R such that
>
> $$A_1^+(R) \cup A_2^-(R) \supset K(R_o).$$

PROOF. The proof will proceed by a case by case analysis. The conditions for the theorem mean that we deal only with the cases 1, 2, and 6, 1A, 2A, and 6A of Fig. 29. We will give the proof of the Lemma only for cases 1, 2, and 6, the modifications for the other cases being obvious.

We take a fixed R, and consider all the following constructions as being carried out in $K(R)$.

Case 1, m even, n odd, $a_m > 0$, $b_n > 0$. P is an unstable node for both of the equations $(32)_i$. Considering especially $(32)_2$, we see that given a circle S_1 around P, there is an S_2 such that if x^o is inside S_2, $C_2^-(x^o)$ tends to P inside S_1. So the interior of $S_2 \subset A_2^-(R)$. Also for $x > 0$ and near Q choose another x^o so that $C_2^-(x^o)$ is the unique trajectory tending toward Q. As the line at infinity forms the two prolongations of this trajectory, it follows immediately from the definition of prolongation that we may find a short segment without contact, ax^ob, orthogonal to $C_2^-(x^o)$ and such that $C_2^-(a)$ and $C_2^-(b)$ cut S_2 (Fig. 34). Repeating the construction for $x < 0$ with an $x^{o\prime}$ and a segment $a'x^{o\prime}b'$ and trajectories $C_2^-(a')$ and $C_2^-(b')$ we see that S_2, $C_2^-(a)$, $C_2^-(b)$, $C_2^-(a')$, and $C_2^-(b')$ bound a region A which clearly has the property that if $x^o \in A$, $C_2^-(x^o)$ lies entirely in the region bounded by the same four trajectories and S_2. Thus $A \subset A_2^-(R)$, and any $K(R_o) \subset A$ is a $K(R_o)$ fulfilling the condition of the Lemma.

Case 2, m even, n odd, $a_m > 0$, $b_n < 0$. As P is an unstable, Q a stable node for the $(32)_i$, just as in case 1 we may find

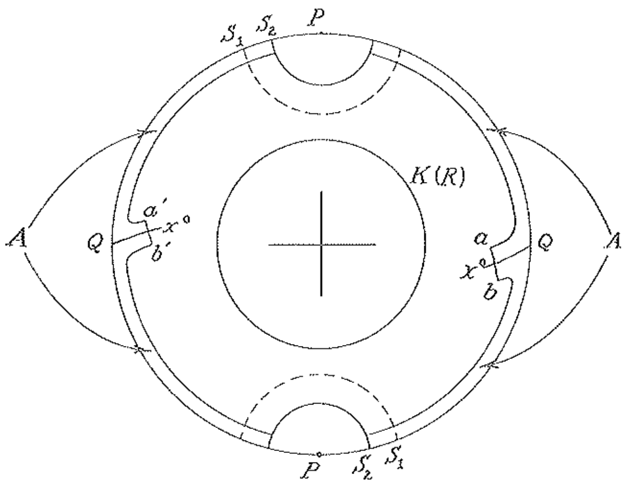

FIGURE 34

circles, S_2 around P, S_2' around Q, such that

$$S_2 \subset A_2^-(R)$$

$$S_2' \subset A_1^+(R) \ .$$

Also, since $z = 0$ is a trajectory, using continuity with respect to initial conditions, we may find an x^O on S_2 with z co-ordinate so small that $C_1^+(x^O)$ cuts S_2', and an $x^{O'}$ on S_2' with z so small that $C_2^-(x^{O'})$ cuts S_2 (see Fig. 35). So if p lies under both

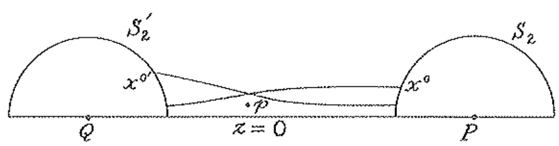

FIGURE 35

these trajectories $C_2^-(p)$ will enter S_2, and $C_1^+(p)$ will enter S_1. Thus p lies in $A_1^+(R)$ and in $A_2^-(R)$. In this way the circles may be connected by long pipes along $z = 0$, the pipes lying in $A_1^+(R)$ and $A_2^-(R)$. Thus combining pipes and circles we obtain a region A lying in $A_1^+(R) \cup A_2^-(R)$, and any R_o such that $K(R_o) \subset A$ is a $K(R_o)$ as required in the Lemma (Fig. 36).

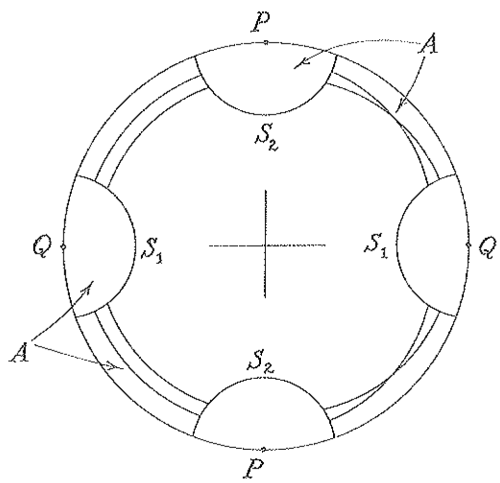

FIGURE 36

Case 6, m odd, n odd, $a_m > 0$, $b_n < 0$. In dealing with the critical point Q we construct a circle S as before. The part of the circle near Q with $x > 0$ will belong to $A_1^+(R)$, the part with $x < 0$ to $A_2^-(R)$. But in dealing with P on the side $y < 0$ we have a more complicated singularity and must slightly modify the procedure.

In Part II in discussing this critical point we saw that the interior of any sufficiently small circle S_2 around P, $y < 0$, would be divided into three sectors by a trajectory of (13) there referred to as C(p) (Fig. 20). Turning to equations $(32)_1$ and $(32)_2$ we have different circles and different divisions, but choosing a circle sufficiently small to serve for both equations, we have it divided into the three closed sectors E_1, E_2, E_3, by a suitable trajectory $C_1(a)$ of $(32)_1$, and into sectors E_1', E_2', E_3', by a suitable $C_2(a')$ of $(32)_2$. The properties ascribed to the sectors in Part II show that $E_1 \cup E_2 \subset A_1^+(R)$, and $E_2' \cup E_3' \subset A_2^-(R)$. Also, as $C_1(A)$ and $C_2(a')$ both approach P tangent to $z = 0$, we have for a sufficiently small circle S_2, $E_1 \cap E_3' = E_1' \cap E_3 = \varphi$ (the null set).

Using this circle then we may write

$$(E_1 \cup E_2 \cup E_3) \cap (E_1' \cup E_2' \cup E_3') = S_2 \text{ and interior}$$

$= (E_1 \cup E_2) \cup (E_3 \cap E_2') \cup (E_3 \cap E_3') \subset (E_1 \cup E_2) \cup (E_2' \cup E_3') \subset A_1^+(R) \cup A_2^-(R)$. So we have a circle around P, $y < 0$, of the desired type.

We may now construct pipes to the circle around Q by linking S (of Q, $x > 0$) with S_2 by a C_1^+ which starts on S_2 and cuts S with $x > 0$, and a C_2^- starting on S_2 and cutting S, $x < 0$ (Fig. 37). The points under this C_1^+ will belong to $A_1^+(R)$, those under C_2^- to $A_2^-(R)$.

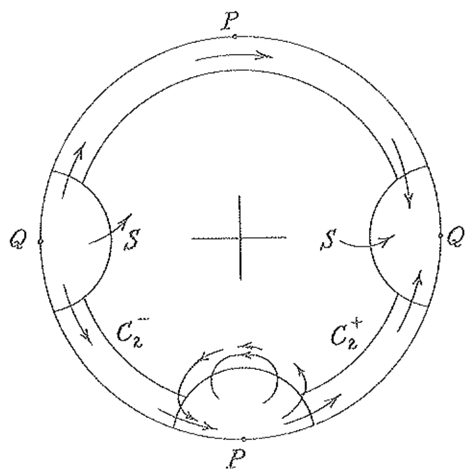

FIGURE 37

Since the line $z = 0$ is its own prolongation at $P(y > 0)$, we may construct a tube linking $Q(x < 0)$ with $Q(x > 0)$ past $P(y > 0)$. The points in the tube will have the property of belonging to $A_1^+(R)$ or $A_2^-(R)$ or to both. Together the tubes and circles form a region $A \subset A_1^+(R) \cup A_2^-(R)$. Thus any $K(R_0)$ in A is a suitable $K(R_0)$.

As we have now dealt with all the cases, the Lemma is established.

§4. Now take any specific $f(x)$ and $g(x)$ satisfying the conditions for the theorem, and consider the corresponding equations $(32)_1$ and $(32)_2$ together with an R and R_0 such that

$$A_1^+(R) \cup A_2^-(R) \supset K(R_0).$$

Choose any point $p = (a, b)$ lying in $K(R_0)$. We will associate with p a half-trajectory and a ray in the following manner. If $p \in A_1^+(R)$, the trajectory is $C_1^+(p)$, if $p \in A_2^-(R)$ the trajectory is $C_2^-(p)$, if b is positive, the ray is the ray $x = a$, $y \geq b$, if b is ≤ 0, the ray is

$x = a,\ y \leq b$.

The chosen trajectory through p may behave in one of two possible ways (Fig. 38 is an example), either it cuts the chosen ray again after leaving p, or it does not. If it does cut again at p', then the trajectory and the segment pp' divide the plane into two regions, and a path

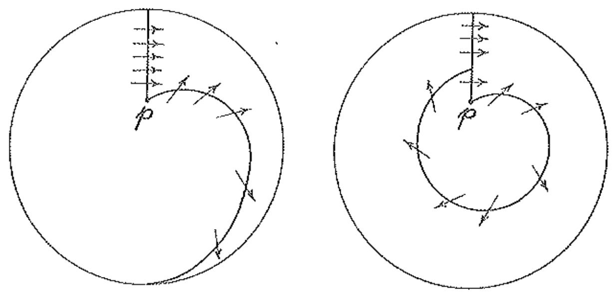

FIGURE 38

proceeding from one region to the other must cut pp' or the trajectory. If the trajectory does not cut the ray again, then by the definition of $A_1^+(R)$ or $A_2^-(R)$ the trajectory must tend to infinity inside $K(R)$. Thus the ray and trajectory combined again separate the plane into two regions. We will always call this separating curve through p, $J(p)$.

Now consider the phase plane form of (31).

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -f(x)y - g(x) + E(t). \tag{34}$$

Comparing the slopes of trajectories of $(32)_1$, (34), and $(32)_2$, we have for any point (x, y), $y \neq 0$,

$$\frac{-f(x)-g(x)+h_1(y)}{y} < \frac{-f(x)y-g(x)+E(t)}{y} < \frac{-f(x)y-g(x)+h_2(y)}{y} \quad .$$

This shows that if $C_1^+(p)$ is pursued in the direction of increasing time, the motions of (34) will cross it from right to left except possibly at points where $y = 0$. Similarly if $C_2^-(p)$ is pursued in the direction of decreasing time, the motions of (34) cross it from right to left. Also, as $\frac{dx}{dt} = y$, the same statement holds for the ray through p, provided it is traversed in the direction of decreasing $|y|$.

We may run through $J(p)$ by first traversing the ray with $|y|$ decreasing, and then running through the assigned trajectory after reaching

p. All along this path, with the possible exception of points $y = 0$, the trajectories of (34) cross from right to left. Since the points $y = 0$ on $J(p)$ are isolated, it follows from continuity that even at these points the motions of (34) cross $J(p)$ from right to left. Hence, all solutions starting on $J(p)$ and moving with increasing t will enter only one of the two regions into which $J(p)$ divides the plane, while with decreasing t they all enter the other.

We conclude that any trajectory of (34) cuts $J(p)$ at most once. In particular, the solution through p itself never returns to p, where p is any point in $K(R_o)$. We are now in a position to duplicate the argument of Part I.

§5. Consider in place of (34)

$$\frac{dx}{dt} = y\,F(x, y)$$

(35)

$$\frac{dy}{dt} = [-f(x)y - g(x) + E(t)]F(x, y)$$

where $F(x, y)$ has continuous derivatives, is always positive, is identically 1 inside some circle of radius $R' > R_o$, and tends to zero so rapidly that the right hand sides of (35) are bounded in the entire plane.

As the special property of the separating curves $J(p)$ depended only on the direction, not the length of the velocity vectors of (34), the curves $J(p)$ have this property again with respect to solutions of (35). Thus a solution to (35) passing through a point p in $K(R_o)$ never returns to p.

Just as in Part I, equation (35) gives us a mapping $\varphi_t : u^o \longrightarrow u(t, u^o)$ of the plane into itself. If we consider the index of this mapping on any circle S lying in $K(R_o)$ with radius R'', $R_o < R'' < R'$, we find, just as before, that since solutions cannot return to their starting point the index of φ_t on S is independent of t. In fact, by the argument of I, it is equal to the index on S of the velocity vectors taken at $t = 0$.

But, as the theorem assumes that $g(x)$ is of odd degree, this last index, obtained from

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -f(x)y - g(x) + E(0)$$

must be $\neq 0$. Hence even the index of φ_T is $\neq 0$, and there is a point

u^0 inside S with $\varphi_T u^0 = u^0$. The solution through u^0 is a periodic solution to (35). However, this solution passes through no point of $K(R_0)$, for a trajectory through p in $K(R_0)$ cannot return to p. So the solution lies always inside S. Here however, $F(x, y)$ is identically 1, so we have a periodic solution to (34) and hence to (31).

This establishes the theorem.

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