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ON THE CONVERGENCE OF AN INTEGER-PROGRAMMING PROCESS*

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INTRODUCTION

The purpose of this paper is to analyze the finiteness of a procedure for integer programming as described by G. B. Dantzig[1] in a paper which left the finiteness question open. The result given here shows that the process will not be finite or even converge to the optimal integer answer x^0 unless certain necessary conditions are satisfied. In particular, the procedure will not be finite unless x^0 already lies on at least $n-1$ of the faces of the polyhedron cut out by the inequalities of the linear programming problem.

REVIEW OF THE PROCEDURE OF REF. [1]

Consider a system of inequalities

$$(1) \qquad \sum_{j=1}^{j=n} a_{ij} x_j \geq b_i \qquad i = 1, \ldots, m$$

$$x_j \geq 0 \qquad j = 1, \ldots, n ,$$

where the b_i and a_{ij} are integers. The integer programming problem is to find the integer vector x^0 that satisfies (1), and minimizes

$$(2) \qquad \sum_{j=1}^{j=n} c_j x_j .$$

We will call any integer vector satisfying (1) a solution. A solution x^0 minimizing (2) will be called optimal.

The procedure described in Ref. [1] is subsumed in the following: Choose a set of n independent inequalities from (1) and set the corresponding slack variables, $s_{i_1}, \ldots, s_{i_n}$, equal to zero. (These are the non-basic variables of a simplex-type procedure.) This gives n independent equations to be solved to obtain a point x'. If x' is not a solution (either because of being non-integer or because it does not satisfy all of (1)), then there is no solution for which $s_{i_j} = 0$ $j = 1, \ldots, n$, as these conditions determine x' uniquely. Since any solution gives the s_{i_j} integral non-negative values, we know, following Ref. [1], that every solution satisfies the new inequality.

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[1] Dantzig, G. B., "Note on Solving Linear Programs in Integers," Naval Research Logistics Quarterly, 6:75-76 (1959).

(3) $$s_{i_1} + s_{i_2} + \ldots + s_{i_n} \geq 1 .$$

Inequality (3), stated in terms of the variables x_j, can now be adjoined to (1) to form a larger set $(1)_1$ which has the same (integer) solutions as (1). This process of inequality formation can next be applied to any n independent inequalities drawn from $(1)_1$, and so on. By repeating this process, we obtain larger and larger sets of inequalities $(1)_k$ giving smaller and smaller feasible polyhedra, always however containing the same integer points. An optimal solution can be obtained only when a polyhedron P is finally obtained having the properties (P1) the optimal integer point is a vertex, and (P2) this vertex minimizes (2) over P.

The actual procedure described in Ref. [1] is much more streamlined than this in that inequalities are dropped as well as adjoined, and the selected bases succeed each other in a way that preserves dual feasibility. Nevertheless, a P satisying (P1) and (P2) must be produced to obtain the solution.

NECESSARY CONDITIONS

Let x, not necessarily integral, satisfy (1) and let $s_i(x)$, $i = 1, \ldots, m + n$ be the corresponding slacks. Designate the $n - 1$ smallest of these by $\bar{s}_j(x)$, $j = 1, \ldots, n-1$, $0 \leq \bar{s}_j(x) \leq \bar{s}_{j+1}(x)$. Let $t_p(x)$, $p = 1, 2, \ldots$ be the slacks of the first, second, third, and so on inequalities added in some particular application of the method of Ref. [1]. Then we have the following:

LEMMA: If $$\sum_{j=1}^{j=n-1} \bar{s}_j(x) \geq 1 , \qquad \text{then}$$

$$t_p(x) \geq \bar{s}_{n-1}(x) \qquad \text{for all } p.$$

PROOF: Consider the first added inequality (3). $t_1(x)$, the slack of this inequality, is given by

$$t_1(x) = s_{i_1}(x) + \ldots + s_{i_n}(x) - 1 .$$

Let $s_{i_r}(x)$ be the largest of the s_{i_j}, then $s_{i_r} \geq s_{n-1} \geq 0$. So, by using the hypothesis of the LEMMA,

$$t_1(x) = s_{i_r}(x) + \left(\sum_{\substack{q=1 \\ q \neq r}}^{q=n} s_{i_q}(x) - 1 \right) \geq s_{i_r}(x) \geq \bar{s}_{n-1}(x) .$$

Since $t_1(x)$ is now known to be $\geq \bar{s}_{n-1}(x)$, the $n - 1$ smallest slacks in $(1)_1$ can be taken to be the same set as in (1). Since all the conditions for the LEMMA now hold for the set $(1)_1$, the reasoning can be repeated to get $t_2(x) \geq \bar{s}_{n-1}(x)$, and so on.

We can now state the following:

THEOREM 1: If x^0 is an optimal integer solution to (1), then the process of Ref. [1] can converge only if the $n - 1$ smallest slacks in (1), $\bar{s}_1(x^0), \ldots, \bar{s}_{n-1}(x^0)$ are all zero.

PROOF: For the process to converge x^0 must eventually become a vertex (condition P1), so there must be at least n zero slacks in some inequality set $(1)_k$. But if at the outset

$$\sum_{i=1}^{i=n-1} \bar{s}_i\left(x^0\right) \geq 1, \text{ and hence } \bar{s}_{n-1} > 0 ,$$

then $t_p\left(x^0\right) \geq \bar{s}_{n-1}\left(x^0\right) > 0$ for all p, and n zero slacks can never be obtained. Therefore, for convergence, we must have at the start

$$\sum_{i=1}^{i=n-1} \bar{s}_i\left(x^0\right) < 1$$

which, since the slacks of an integer x are integers, implies $\bar{s}_i\left(x^0\right) = 0$, $i = 1, \ldots, n - 1$.

Thus, geometrically speaking, the process can converge only if x^0 lies on the 1 - skeleton of the original polyhedron.

This condition is, however, always met in the important class of problems in which the variables x_j are either 0 or 1. Here any solution x^0 is actually a vertex of the cube $0 \leq x_j \leq 1$ $j = 1, \ldots, n$. Nevertheless, the process does not always converge for these problems as there is an additional necessary condition expressed in

THEOREM 2: Let z be the objective function minimized by x^0. Let x be any point satisfying (1) with $z(x) < z\left(x^0\right)$, then a necessary condition for convergence is

$$\sum_{i=1}^{i=n-1} \bar{s}_i(x) < 1 .$$

PROOF: For the convergence of the process, condition P2 must be met; i.e., x^0 must minimize z over some polyhedron. For this to happen, x must have been removed from the polyhedron, so there must have been some inequality added to (1) which x does not satisfy. However, a negative slack $t_p(x)$ is not possible with

$$\sum_{i=1}^{i=n-1} \bar{s}_i(x) \geq 1 ,$$

hence the THEOREM.

To illustrate THEOREM 2, consider the following example:

$$\begin{array}{ll} \text{minimize} & z = -4x_1 - 3x_2 - 3x_3 \\ \text{subject to} & 3x_1 + 4x_2 + 4x_3 \leq 6 \\ \text{and} & 0 \leq x_1 \leq 1 \\ & 0 \leq x_2 \leq 1 \\ & 0 \leq x_3 \leq 1 . \end{array}$$

The optimal integer answer clearly gives $z = -4$, but the point $(1/2, 1/2, 1/2) = x$ gives a z of -5. x satisfies all the inequalities with slacks of $1/2$ so that, although the condition for THEOREM 1 is satisfied, the condition for THEOREM 2 is not, and the process cannot converge on this 0 - 1 problem.

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