Some Polyhedra Related to Combinatorial Problems

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ABSTRACT

This paper first describes a theory and algorithms for asymptotic integer programs. Next, a class of polyhedra is introduced. The vertices of these polyhedra provide solutions to the asymptotic integer programming problem; their faces are cutting planes for the general integer programming problem and, to some extent, the polyhedra coincide with the convex hull of the integer points satisfying a linear programming problem. These polyhedra are next shown to be cross sections of more symmetric higher dimensional polyhedra whose properties are then studied. Some algorithms for integer programming, based on a knowledge of the polyhedra, are outlined.

INTRODUCTION

It is well known that a great variety of combinatorial problems can be written as integer programming problems, that is, as systems of inequalities:

$$A'x' \leqslant b, \qquad x' \geqslant 0, \qquad x' \text{ integer,} \tag{1}$$

together with a linear function $c' \cdot x'$ to be maximized. In (1), A' is an $m \times n$ integer matrix, x' an integer n-vector, and b an integer m-vector.

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Alternatively, the integer programming problem can be written as

$$\max c \cdot x \tag{2}$$

subject to

$$Ax = b, \qquad x \geqslant 0, \qquad x \text{ integer}.$$

In this formulation, A is an $m \times (m + n)$ integer matrix, x is an $(m + n)$ integer vector (including the slacks of the previous formulation), and b is an integer m-vector. For simplicity in what follows, we assume that A contains an $m \times m$ unit matrix.

One difficulty with the integer programming approach to combinatorial problems is that the formulation of (1) or (2) often provides neither an effective algorithm nor insight into the form or other properties of the solution. This contrasts with the ordinary linear programming problem, which is (1) or (2) without the integer restriction. There the simplex method provides what is empirically known to be an effective computational method, and there is also information about the form of solution. For example, we know that there are at most m positive components in x.

The differences between linear and integer programming are not easily removed, for to be able to use the simplex method on (1) we would first have to obtain the faces of the polyhedron P which is the convex hull of the lattice points (integer points) satisfying (1). With P or, more precisely, the faces of P available, the problem would become an ordinary linear programming problem over P, although of an as yet unknown size and degree of degeneracy.

However, and this is the essential point, the dependence of P on A or A' can be complicated indeed. Although the algorithms of [4] and [5] obtain faces or vertices of P from the inequalities of (1), they are unpredictable in length and have so far shed little light on the structure of the polyhedron P.

Since any algorithm for the integer programming problem, whether related to linear programming, branch and bound, exhaustive search, or whatever, must end up finding a vertex of P, information on P seems relevant to any approach to the integer programming problem. Yet information about P is very difficult to obtain.

In this paper we attack the problem by introducing a family of polyhedra closely related to P but having simpler and more accessible properties.

These polyhedra are obtained by replacing the columns of the linear programming problem by elements of a finite Abelian group and examining the solutions. The polyhedra so obtained are related to the original integer programming problem in two ways. First, the vertices of the polyhedra give the complete list of solutions to the asymptotic integer programming problem—the problem that results when b becomes large. Second, the faces of the polyhedra give inequalities that are in a certain sense the strongest possible that can be used for a "cutting plane" [4] approach to the original problem. Third, under conditions that will be explained, the new polyhedra actually coincide with that part of P which is near one vertex of the linear programming polyhedron.

It will be shown that many different polyhedra of this sort can be obtained by intersecting various subspaces with a single family of higher dimensional polyhedra. Thus we shall see that many seemingly different combinatorial problems can actually correspond to different cross sections of the same large polyhedron.

The paper is divided into three main parts: Sections 1, 2, and 3.

Section 1A introduces the group equations for a first version of the new polyhedra, the corner polyhedra, and gives a geometrical interpretation. Section 1B introduces asymptotic integer programming and connects it with the corner polyhedra. Results are then given showing the periodicity of asymptotic integer solutions, as well as results on their form, domain of applicability, and methods of calculation. Appendix 1, which relates to this part of the paper, gives a numerical example of an asymptotic integer programming problem.

Section 2 is devoted to the corner polyhedra. Section 2A connects the faces of the corner polyhedra with cutting plane methods for the general integer programming problem. It then gives a theorem showing that all faces of the corner polyhedra can be computed by linear programming. Appendix 3 gives a dynamic programming calculation for producing one face. Section 2B develops special properties of the faces of the corner polyhedra. Section 2C introduces the family of higher dimensional polyhedra, the master polyhedra $P(\mathscr{G}, g_0)$, of which the various corner polyhedra are cross sections and explains the connection between the $P(\mathscr{G}, g_0)$ and corner polyhedra.

Section 3 is devoted to the $P(\mathscr{G}, g_0)$. Section 3A deals with the effect of group automorphisms. We see here that there is one master polyhedron $P(\mathscr{G}, g_0)$ for each automorphism class of each finite Abelian group. Section 3B develops properties of faces of the $P(\mathscr{G}, g_0)$. In particular it includes the proof of a theorem which connects the faces of the master polyhedra with the vertices of a special, highly structured linear programming problem. This theorem, which is useful in many ways, is then used to compute the first 34 master polyhedra, which are tabulated in Appendix 5. It is apparent from these tables that faces can be obtained by more special methods. One such method, for the polyhedra belonging to cyclic groups, is given in Section 3C. Connections between faces and subgroups are given in Section 3D. These theorems enable us to produce special faces for any group. Section 3E discusses group characters. It is shown that the characters enable us to produce cutting planes while doing a linear programming calculation but without knowing what group is involved. They explain precisely the relation between the inequalities of [4] and the present families of inequalities. Section 3F describes special properties of the groups $\mathscr{G}_{2n}$ and $\mathscr{G}_{3n}$ and their associated master polyhedra. The number of vertices is obtained as $n \to \infty$. Section 3G summarizes some of the more obvious algorithmic possibilities and some directions for further research.

This paper* follows up the P.N.A.S. notes [7] and [8] in which many of the results of Sections 1 and 2 were first outlined. The group notion introduced in [7] has since been developed in interesting directions by Glover [3] and White [12]. Some of the ideas of Section 1 can be found in primitive form in Gilmore and Gomory [2], and some of the results of Section 2 were anticipated in a very interesting unpublished paper by Taylor [10] which he has recently brought to my attention.

I would like to thank Alan Konheim for his contributions to Section 3E, Alan Hoffman for many useful remarks and suggestions, and Carol Shanesy for carrying out the programming and computation of the faces, vertices, and incidence matrices of the polyhedra.

1. CORNER POLYHEDRA AND ASYMPTOTIC INTEGER PROGRAMMING

A. *Equations for Corner Polyhedra*

In (2) let B be any nonsingular submatrix formed from m columns of A. Without loss of generality we can assume that B consists of the first m columns; so

* Most of the results of this paper were presented in a series of seminars sponsored by the American Mathematical Society at Stanford University in July 1967 and at the International Symposium on Mathematical Programming at Princeton University in August 1967.

$$A = (B, N),$$

where B is $m \times m$, and N is $m \times n$. In terms of B and N we can write (2) as

$$Bx_B + Nx_N = b, \tag{2a}$$

where x_B is an m-vector and x_N an n-vector. Once x_N is chosen, $x_B = B^{-1}(b - Nx_N)$ is uniquely determined by (2a). If $x = (x_B, x_N)$ is to satisfy all the conditions of (2), a nonnegative integer x_N must be chosen such that the resulting x_B is both:

(i) integer and
(ii) nonnegative.

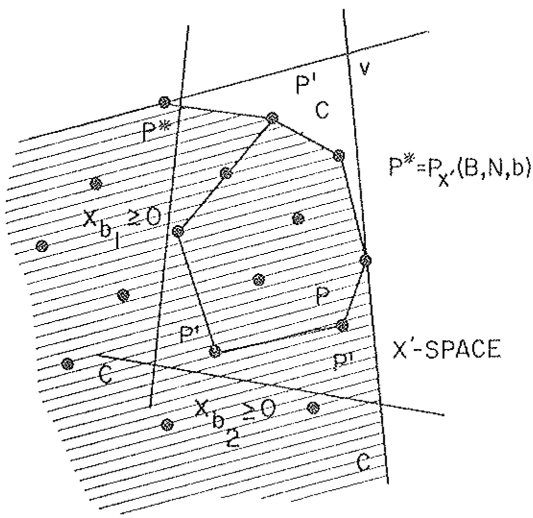


FIG. 1a. P^* is shaded region.

These are the conditions that give a feasible solution to the integer programming problem. However, in what follows we shall examine instead the problem that results when condition (i) is maintained but condition (ii) is dropped, i.e.,

$$Bx_B + Nx_N = b, \quad x_N \geqslant 0, \quad (x_N, x_B) \text{ integer}. \tag{2b}$$

Whenever B is a feasible basis the conditions (2b) have a geometrical interpretation which motivates much of what follows. The basis B can be thought of as determining a vertex v of the linear programming polyhedron P' (see Fig. 1a). Inside P' is the integer programming polyhedron P. If we relax the conditions $x_B \geqslant 0$ as we have done in (2b), the new linear programming problem becomes the one corresponding to an unbounded polyhedral cone we will refer to as C, and the integer programming problem, given by conditions (2b), corresponds to the polyhedron P^*, which is the convex hull of the integer points in C. It is P^*, the striped area in Fig. 1a, which we refer to as the corner polyhedron. As we will see, and as Fig. 1a suggests, P^* is often closely connected to P.

We now turn to the conditions on x_N that ensure an integral, but not necessarily nonnegative x_B.

Let f be a fixed homomorphism sending $M(I)$, the space of all integer m-vectors, onto $\mathscr{G} = M(I)/M(B)$. $M(B)$ is the module generated over the integers by the columns of B, i.e., the lattice in m-space of all integer combinations of these columns, and $\mathscr{G}$ is the (finite) factor group in which all elements of $M(B)$ are treated as zero (are mapped by f into the zero, $\bar{0}$, of $\mathscr{G}$). An f can be calculated explicitly from B by standard methods.[†] See, for example, Van der Waerden [11].

Applying f to (2a),

$$f(Bx_B) + f(Nx_N) = fb.$$

$f(Bx_B) = \bar{0}$ if and only if x_B is integer, so

$$f(Nx_N) = fb \tag{3}$$

is a necessary and sufficient condition on x_N to produce integer x_B.

If f maps the column N_i of N into the element g_i of $\mathscr{G}$, and sends b into g_0, then (3) gives

$$\sum_{i=1}^{i=n} f(N_i x_{m+i}) = fb = \sum_{i=1}^{i=n} (fN_i)x_{m+i}$$

or

$$\sum_{i=1}^{i=n} g_i x_{m+i} = g_0. \tag{4}$$

[†] This is done for a numerical example in Appendix 1.

So the group equation (4) together with the conditions $x_{m+i} \geqslant 0$ and x_{m+i} integer ensures nonnegative integer x_N and integer x_B.

We next introduce variables that have advantages in dealing with the group equation (4). With these variables there will be at most one variable for each group element.

Let $\mathcal{N}$ be the set of nonzero group elements fN_i, $i = 1, \ldots, n$, $fN_i \neq \bar{0}$. Let $n' = |\mathcal{N}|$, $|n'| \leqslant n$. Introduce n' variables $t(g)$, one for each $g \in \mathcal{N}$; let T be the n'-vector with components $t(g)$. There is then a natural correspondence, which in general is many-one, between points in x-space that solve (2a) and points in T-space. The correspondence F is

$$F: (x_B, x_N) \rightarrow (x_N) \rightarrow T,$$

where T is given by

$$t(g) = \sum_{\{i \mid f(N_i) = g\}} x_{m+i}.$$

Of course, if there is no duplication, i.e., different columns are mapped by f into different nonzero g, then T has exactly the same components as x_N.

The points x satisfying (2b), or equivalently the group equation (4), correspond under F to those nonnegative integer points of T-space which satisfy the group equation

$$\sum_{g \in \mathcal{N}} t(g) \cdot g = g_0. \tag{5}$$

The vertex $x = (x_B = B^{-1}b,\ x_N = 0)$ corresponds to $x_N = 0$ and to the origin in n'-dimensional T-space. The portion of x-space, in which the nonbasic variables remain $\geqslant 0$, corresponds to the first (or nonnegative) orthant in T-space (Fig. 1b). Alternatively, in terms of the inequalities of (1) and the original n-vector x', the nonnegative orthant of T-space corresponds to the cone C in Fig. 1a. If T is a nonnegative integer n-vector and also solves (5), it is shown in Fig. 1b as one of the circled integer points. x' is an integer point of C in Fig. 1a if, and only if, the x of which it is the nonbasic part corresponds to one of these circled points in T-space.

The corner polyhedron is the convex hull, in x-space, of the nonnegative integer solutions to (2b). We shall refer to this as $P_x(B, N, b)$. The convex hull, in T-space, of the nonnegative integer solutions to (5) will be the polyhedron $P(\mathscr{G}, \mathscr{N}, g_0)$. The corner polyhedron in x'-space is $P_{x'}(B, N, b)$. $P_{x'}(B, N, b)$ and $P(\mathscr{G}, \mathscr{N}, g_0)$ are the objects pictured in Figs. 1a and 1b.

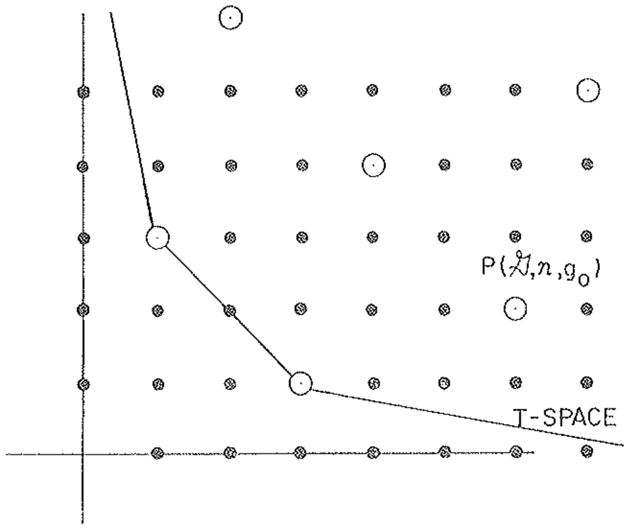


Fig. 1b.

These polyhedra are, of course, essentially the same, and we shall work throughout with $P(\mathscr{G}, \mathscr{N}, g_0)$. The following easily verified remarks give the connection between $P_x(B, N, b)$ and $P(\mathscr{G}, \mathscr{N}, g_0)$.

Remark 1. (x_B, x_N) is a vertex of $P_x(B, N, b)$ if and only if

(i) $t = F(x_B, x_N)$ is a vertex of $P(\mathscr{G}, \mathscr{N}, g_0)$, and
(ii) whenever $fN_i = fN_j$, $i \neq j$; then either $x_{m+i} = 0$ or $x_{m+j} = 0$;
(iii) whenever $fN_i = \bar{0}$, $x_i = 0$.

In other words, to produce one of the x-space vertices corresponding to vertex t, use the component value $t(g)$ as exactly one of its corresponding nonbasic x_{m+i}, and use the component value 0 in the others. Doing this for each $t(g)$ and setting $x_i = 0$ if $fN_i = \bar{0}$ produces x_N. For x_B use $x_B = B^{-1}(b - Nx_N)$.

Turning now to faces, we can denote the T-space inequality $\sum_{g\in\mathcal{N}} \pi(g)t(g) \geqslant \pi_0$ by (π, π_0), where π is a n'-vector and π_0 a scalar. In x-space we denote an inequality by (π_B, π_N, π_0). Using $x_B = B^{-1}(b - Nx_N)$ gives an equivalent inequality on the nonbasic variables alone, that is, an inequality of the form $(0, \bar{\pi}_N, \bar{\pi}_0)$.

Remark 2. $(0, \bar{\pi}_N, \bar{\pi}_0)$ is an $(n-1)$-dimensional face of $P_x(B, N, b)$ if, and only if, (π, π_0), where $\pi(|N_j) = \bar{\pi}_j$, is an $(n'-1)$-dimensional face of $P(\mathcal{G}, \mathcal{N}, g_0)$.

Thus, to obtain a face of $P_x(B, N, b)$ from a face of $P(\mathcal{G}; N, g_0)$ we merely write the component value $\pi(g)$ in *all* the corresponding places in $\bar{\pi}_N$.

B. Asymptotic Integer Programming

We will say that an integer point $t(g)$ of $P(\mathcal{G}, \mathcal{N}, g_0)$ is irreducible if for any set of integers $s(g)$ and $r(g)$ the conditions $0 \leqslant s(g) \leqslant t(g)$, $0 \leqslant r(g) \leqslant t(g)$, and $\sum_{g\in\mathcal{N}} s(g) \cdot g = \sum_{g\in\mathcal{N}} r(g) \cdot g$ imply $r(g) = s(g)$ for all $g \in \mathcal{N}$.

We shall see that the integer points of $P(\mathcal{G}, \mathcal{N}, g_0)$ that occur in problems of linear maximization and minimization are points with the property of irreducibility.

THEOREM 1. *If t, with nonnegative integer components $t(g)$, is irreducible, then the $t(g)$ satisfy*

$$\prod_{g\in\mathcal{N}} (1 + t(g)) \leqslant |\mathcal{G}|,$$

where $|\mathcal{G}|$ is the number of elements in the group.

Proof. Let $t' = (t'(g))$ with $0 \leqslant t'(g) \leqslant t(g)$ and integer. Since there are $t(g) + 1$ different possible entries in each component, there are $\prod_{g\in\mathcal{N}} (1 + t(g))$ possible different vectors t' satisfying these inequalities. If t is irreducible, the sum $\sum_{g\in\mathcal{N}} t'(g) \cdot g = g(t')$ must be a different group element $g(t')$ for each different t'. However, there are only $|\mathcal{G}|$ different group elements; hence the inequality.

It is a standard result about these groups that, if $\mathcal{G} = M(I)/M(B)$, then $|\mathcal{G}| = |\det B|$. Thus $|\mathcal{G}|$ can be replaced by $|\det B|$ in the above.

THEOREM 2. *Every vertex of $P(\mathscr{G}, \mathscr{N}, g_0)$ is irreducible.*

Proof. Suppose there is a vertex $v = (t(g))_{g \in \mathscr{N}}$ and it is reducible, i.e., there are integers $r(g)$ and $s(g)$, $0 \leqslant r(g) \leqslant t(g)$, $0 \leqslant s(g) \leqslant t(g)$, $r(g) \neq s(g)$ for some g, and $\sum_{g \in \mathscr{N}} s(g) \cdot g = \sum_{g \in \mathscr{N}} r(g) \cdot g$. So

$$g_0 = \sum_{g \in \mathscr{N}} t(g) \cdot g = \sum_{g \in \mathscr{N}} t(g) \cdot g - \sum_{g \in \mathscr{N}} r(g) \cdot g + \sum_{g \in \mathscr{N}} s(g) \cdot g$$
$$= \sum_{g \in \mathscr{N}} (t(g) - r(g) + s(g))g$$

and

$$g_0 = \sum_{g \in \mathscr{N}} t(g) \cdot g = \sum_{g \in \mathscr{N}} t(g) \cdot g - \sum_{g \in \mathscr{N}} s(g) \cdot g + \sum_{g \in \mathscr{N}} r(g) \cdot g$$
$$= \sum_{g \in \mathscr{N}} (t(g) - s(g) + r(g))g.$$

Since $t(g) - r(g) \geqslant 0$ and $t(g) - s(g) \geqslant 0$, the vectors $v_1 = (t(g) - r(g) + s(g))_{g \in \mathscr{N}}$ and $v_2 = (t(g) - s(g) + r(g))_{g \in \mathscr{N}}$ have nonnegative integer components and solve (5); hence they are in $P(\mathscr{G}, \mathscr{N}, g_0)$. But $v = (v_1 + v_2)/2$, so v is not a vertex.

It is not true in general that the vertices are the only irreducible integer points of $P(\mathscr{G}, \mathscr{N}, g_0)$. Generally, there are many irreducible points that are not vertices. However, for the special case of groups $\mathscr{G}$ that are direct sums of cyclic groups of order 2 or of groups of order 3, all the irreducible points are in fact vertices. This is shown in Section 3F.

We now return to the original integer programming problem (2) to connect it with $P(\mathscr{G}, \mathscr{N}, g_0)$.

Let B be chosen as an optimal basis of the linear programming problem, i.e., (2) without the integer restriction. Let $\bar{x} = (\bar{x}_B, \bar{x}_N)$ be an optimal solution to the integer programming problem (2). Since $\bar{x}$ satisfies

$$B\bar{x}_B + N\bar{x}_N = b,$$

the cost $c \cdot x$ can be expressed entirely in terms of the $\bar{x}_N$:

$$c \cdot \bar{x} = c_B\bar{x}_B + c_N\bar{x}_N = c_B B^{-1}(b - N\bar{x}_N) + c_N\bar{x}_N$$
$$= c_B B^{-1} b - c_N{}^* \bar{x}_N, \tag{6}$$

where $c_N^* = -c_N + c_B B^{-1}N$. Since B is the optimal linear programming basis, $c_B B^{-1}b$, which we will denote by $z_L(b)$, is the value of the linear programming solution, and the components c_{m+i}^* of c_N^* are the relative prices of linear programming, and are all $\geqslant 0$. Denoting the value $c \cdot \bar{x}$ of the optimal integer solution by $z_I(b)$, (6) becomes

$$z_I(b) = z_L(b) - c_N^* \bar{x}_N. \tag{7}$$

Let $F\bar{x} = \bar{t}$ and let c^* be the n'-vector with components $c^*(g) = \min_{\{i|fN_i=g\}} c_{m+i}^*$. Clearly, $\bar{x}_N \cdot c_N^* \geqslant \bar{t}c^*$.

Let us consider the problem of minimizing c^*t over $P(\mathscr{G}, \mathscr{N}, g_0)$ or, equivalently,

$$\min \sum_{g\in\mathscr{N}} c^*(g)t(g), \qquad \sum_{g\in\mathscr{N}} g \cdot t(g) = g_0, \tag{8}$$

where $t(g) \geqslant 0$ and integer. $\bar{t}$ is a feasible, i.e., not necessarily minimal, solution to the conditions of (8). Let t^* be a minimizing vertex solution to (8). Let x^* be a vertex of $P_x(B, N, b)$ corresponding to t^*, and using only least cost columns. That is, $x_{m+i} > 0$ only if $c_{m+i}^* = c^*(fN_i)$. x^* satisfies the conditions (2b) and the value $c \cdot x^*$ can be computed from

$$c \cdot x^* = c_B B^{-1}b - c_N^* x_N^* = z_L(b) - c_N^* x_N^*.$$

But

$$c_N^* x_N^* = c^*t^*,$$

and, since t^* minimizes in (8),

$$c^*t^* \leqslant c^*\bar{t} \leqslant c_N^*\bar{x}_N,$$

so

$$c \cdot x^* \geqslant z_L(b) - c_N^*\bar{x}_N = z_I(b). \tag{9}$$

If $x_B^* \geqslant 0$, x^* is a feasible solution to the integer programming problem. By (9) it is also maximizing, so we have the following theorem:

THEOREM 3. *If t^* is a vertex of $P(\mathscr{G}, \mathscr{N}, g_0)$ minimizing (8), then any corresponding vertex x^* of $P_x(B, N, b)$, $x^* = (B^{-1}(b - Nx_N^*), x_N^*)$ with $Fx^* = t^*$, and $x_{m+i} > 0$ only if $c_{m+i}^* = c^*(fN_i)$, is an optimal solution to the integer programming problem* (2) *provided* $B^{-1}(b - Nx_N^*) \geqslant 0$.

We now state some conditions that ensure the nonnegativity of $B^{-1}(b - Nx_N^*)$.

The points y in m-space for which B is a feasible basis form a cone given by the condition $B^{-1}y \geqslant 0$. We denote this cone by K_B. The cone of points in K_B at a euclidean distance of d or more from the frontier of K_B we denote by $K_B(d)$. So $K_B = K_B(0)$.

We can now state the following theorem:

Theorem 4. *If $b \in K_B(l_{\max}(D-1))$, where $D = |\det B|$ and $l_{\max}$ is the (euclidean) length of the longest nonbasic column, then the x^* of Theorem 3 is an optimal integer solution to* (2).

Proof. For the $t^*(g)$ of t^*, $\prod_{g \in \mathcal{N}} (1 + t^*(g)) \leqslant |\mathcal{G}| = D$. Expanding shows

$$\sum_{g \in \mathcal{N}} t^*(g) \leqslant |\mathcal{G}| - 1;$$

so

$$\|Nx_N^*\| = \left\| \sum_{i=1}^{i=n} N_i x_{m+i}^* \right\| \leqslant l_{\max} \sum_{i=1}^{i=n} x_{m+i}^*$$

$$= l_{\max} \sum_{g \in \mathcal{N}} t(g) \leqslant l_{\max}(D-1).$$

So, when $b \in K_B(l_{\max}(D-1))$, $(b - Nx_N^*) \in K_B$; so $B^{-1}(b - Nx_N^*) \geqslant 0$ and Theorem 3 applies. This proves Theorem 4.

Now t^* is the same for all right-hand sides b which are equivalent mod B, since the group equation is unchanged. We can, therefore, put together the preceding theorems in the following statement:

Theorem 5. *If $b \in K_B(l_{\max}(D-1))$, then there is an optimal solution to* (2) *of the form $(B^{-1}(b - Nx_N^*(b)), x_N^*(b))$ where the n-vector x_N^* is periodic in the columns B_i of B, i.e., $x_N^*(b + B_i) = x_N^*(b)$, for any B_i. x_N^* is part of the vector $x^* = (x_B^*, x_N^*)$, where $Fx^* = t^*$ minimizes in the group problem:*

$$\min \sum_{g \in \mathcal{N}} c^*(g)t(g), \qquad \sum_{g \in \mathcal{N}} t(g) \cdot g = g_0,$$

and $x_{m+i} > 0$ only if $c_{m+i}^* = c^*(fN_i)$.

Some remarks help in understanding this theorem. First of all, let us consider the cones and the domain of applicability of the solution. As a preliminary, consider the ordinary (noninteger) linear programming problem with fixed A and c but with varying right-hand side b. We get the picture illustrated in Fig. 2a. The space of possible right-hand

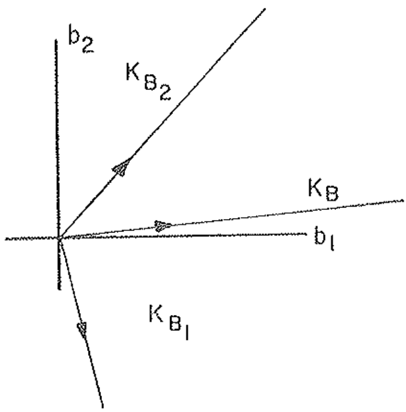


FIG. 2a.

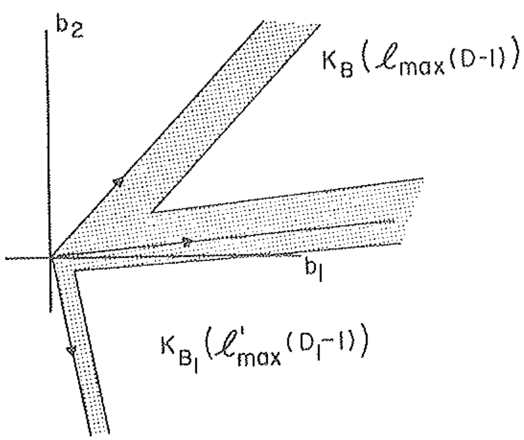


FIG. 2b.

side m-vectors b splits up into cones corresponding to the different optimal bases B, B_1, ..., etc. If B is optimal for right-hand side b, and b' is another right-hand side such that $B^{-1}b'$ is $\geqslant 0$, then B is also optimal for b' and, in fact, $B^{-1}y \geqslant 0$ is the necessary and sufficient condition that y is in the cone K_B. Theorem 5 shows that each cone has a band along the edge of fixed-width $l_{\max}(D-1)$ depending on the cone, and any point *not* lying on these bands is definitely a right-hand side where the solution obtained from the group applies. See Fig. 2b. Within the domain of applicability, the solutions have a periodic character. More precisely, the solution can be written as the sum of two terms:

$$x^*(b) = (x_B^*(b), x_N^*(b)) = (B^{-1}b, 0) + (-B^{-1}Nx_N^*(b), x_N^*(b)),$$

where the first is the linear programming solution and the second is a periodic correction. Two right-hand sides b and b' lying in the same position relative to the lattice of columns of B will correspond to the same group element, and their solutions will differ only by the linear programming term $B^{-1}(b-b')$.

The solution to the group minimization problem need be computed only once for each of the possible g_0, i.e., for one period. There are D possible g_0. When this has been done (and it can be done by a single dynamic programming calculation as explained below), the solution to the integer programming problem has been obtained for *all* right-hand sides in the domain of applicability.[†]

Every b for which the linear programming problem can be solved at all belongs to some K_B and so has a representation $b = \sum_{i=1}^{i=m} \lambda_i B_i$ in terms of the columns B_i of B with $\lambda_i \geqslant 0$. Unless some $\lambda_i = 0$, the multiple kb will, for large enough k, lie in $K_B(l_{\max}(D-1))$. Hence, except for the vectors b lying on a lower than m-dimensional surface, the integer programming problem $Ax = kb$ for k large enough always lies in the domain of applicability. Hence the name asymptotic theorem.

Let us remark that, for a vector b in the domain of applicability, the restriction on the form of the solution to the group equation, which says that the components $t(g)$ must satisfy

$$\prod_{g \in \mathcal{N}} (1 + t(g)) \leqslant D,$$

carries over to the corresponding vertex x of $P(B, N, b)$. The reason is that each nonzero x_{m+i} is numerically equal to some $t(g)$. Hence there are optimal integer programming solutions x to (2) with

$$\prod_{i=1}^{i=n} (1 + x_{m+i}) \leqslant D. \tag{10}$$

[†] Appendix 1 contains a numerical example of an asymptotic integer programming problem.

When $D = 1$, we have strip width $l_{\max}(D - 1) = 0$ and so, from (10), all $x_{m+i} = 0$. This is the unimodular case. As D increases, we gradually move away from this linear programming form (all $x_{m+i} = 0$), and the strip widths become larger. At all times the number of positive nonbasic variables in the solution x is, again from (10), limited by $\log_2 D$.

Thus we see that there is a gradual transition away from the linear programming form of solution toward the most general integer programming form as D increases.

Turning now to the actual computation of t, we see that the group minimization problem can be formulated as a dynamic programming problem with D states, one for each element of G.

For any set $\mathscr{S} \subset \mathscr{N}$ and element $h \in \mathscr{G}$, we define $\phi(\mathscr{S}, h)$ as

$$\phi(\mathscr{S}, h) = \min \sum_{g \in \mathscr{S}} c^*(g)t(g), \qquad \sum_{g \in \mathscr{S}} t(g) \cdot g = h,$$

where, as usual, the $t(g)$ are required to be nonnegative integers. ϕ satisfies a simple recursion relation for, if $g' \in \mathscr{S}$, then either $t(g') = 0$ or $t(g') \geqslant 1$ in the minimizing solution. In the first case, $\phi(\mathscr{S}, h) = \phi((\mathscr{S} - g'), h)$; in the second case, $\phi(\mathscr{S}, h) = c^*(g') + \phi(\mathscr{S}, h - g')$; so in every case

$$\phi(\mathscr{S}, h) = \min_{g'}\{\phi((\mathscr{S} - g'), h), c^*(g') + \phi(\mathscr{S}, h - g')\}.$$

Although this is not quite a simple dynamic programming recursion, it is very close, and the dynamic programming approach is easily adapted to the situation. The calculation is given in detail in Appendix 2. The arithmetic work involved is proportional to nD.

It is often useful, both for the group minimization problem and for Sections 2 and 3, to introduce the graph $H(\mathscr{G}, \mathscr{N}, \pi)$ which consists of:

(i) a vertex $\mathscr{V}(g)$ for each $g \in \mathscr{G}$,

(ii) directed arcs $e(g, g + g')$ from $\mathscr{V}(g)$ to $\mathscr{V}(g + g')$ for all $g \in \mathscr{G}^+$ and all $g' \in \mathscr{N}^+$,

(iii) an arc length $\pi(g')$ assigned to each arc $e(g, g + g')$ for all g.

The graph for a cyclic group of four elements $\bar{0}, g_1, g_2, g_3$, with $g_2 = 2g_1$ and $g_3 = 3g_1$, is shown in Fig. 3. $\mathscr{N} = \{g_1, g_2\}$ and $\pi = \{4, 5\}$.

The minimization problem is exactly the problem of finding the shortest path from $\mathscr{V}(\bar{0})$ to the vertex $\mathscr{V}(g_0)$ corresponding to the right-hand side element g_0 when π is taken to be c^*.

Any of the standard shortest-path methods can be applied to this perfectly ordinary graph $H(\mathscr{G}, \mathscr{N}, \pi)$ and provide methods for finding the shortest path to g_0 or for finding the shortest path from $\bar{0}$ to g for all g and, hence, giving $t^*(g)$ for all g.

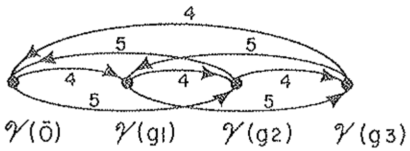


FIG. 3.

Finally, let us turn to the actual width of the bands in which the solution does *not* apply. Theorem 4 shows that this width is $\leqslant l_{\max}(D - 1)$, and this formula can easily be converted into a calculation that shows

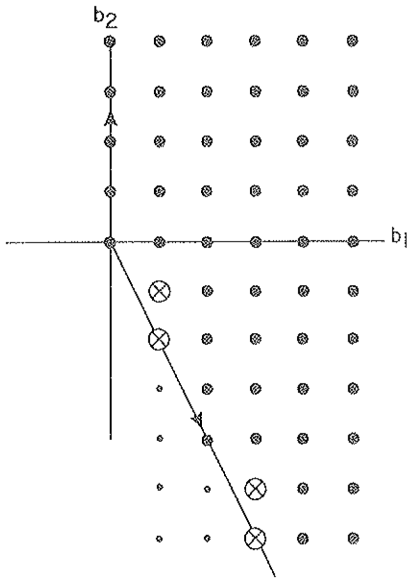


FIG. 4. K_B is shown for $B = \begin{pmatrix} 0 & \\ 3 & -4 \end{pmatrix}$. For circled dots (b_1, b_2), the asymptotic solution applies. For dots with crosses inside circles, it fails.

whether or not a given b is in $K_B(l_{\max}(D - 1))$. However, numerical examples and the actual vertices of the polyhedra $P(\mathscr{G}, \mathscr{N}, g_0)$ computed so far indicate that this width, although it can be attained for just the right combination of g_0, and of the column attaining l_{max}, etc., is usually a gross overestimate in the sense that most of the points in the band are points where the asymptotic solution does in fact apply. See, for example, Fig. 4 which gives the areas of applicability of the example in Appendix 1. This lack of precision in the bound is not surprising, as the bound is based on irreducibility rather than on properties peculiar to the vertices themselves.

2. THE CORNER POLYHEDRA

A. Properties and Faces

We now turn to the study of the polyhedra $P(\mathscr{G}, \mathscr{N}, g_0)$ themselves, independent of group minimization problems. These polyhedra have the following useful properties:

(i) their vertices provide solutions to the asymptotic problem and all extreme point solutions to the asymptotic problem correspond to these vertices.

(ii) their faces provide valid inequalities for the original integer programming problem;

(iii) for some right-hand sides b, the $P_x(B, N, b)$ actually coincide with portions of the sought-after polyhedron P.

Property (i) has already essentially emerged from our discussion. To state it more precisely: if c is some objective function for which B is optimal (the condition is $c_B(B^{-1}N) \geqslant c_N$), then the asymptotic solution x to (2) is obtained from some vertex of $P_x(B, N, b)$ and, hence, from a vertex of $P(\mathscr{G}, \mathscr{N}, g_0)$. Also, if v is any vertex of $P(\mathscr{G}, \mathscr{N}, g_0)$, there is a c, with optimal basis B, whose asymptotic solution is obtained from that vertex.

Property (ii) follows from the fact that the original convex hull P of feasible integer solutions to (2) is certainly contained in $P_x(B, N, b)$ and, therefore, all its points lie on one side of the faces of $P_x(B, N, b)$. These faces, as we observed earlier, are almost the same as, and are easily obtained from, the faces of $P(\mathscr{G}, \mathscr{N}, g_0)$.

Property (iii) can be stated more precisely. Consider a vertex v of $P_x(B, N, b)$. If v satisfies the inequalities $x_B \geqslant 0$, it will also be a vertex of P. If $b \in K_B((D-1)l_{max})$, it follows from Theorem 5 that all vertices of $P_x(B, N, b)$ satisfy $x_B \geqslant 0$. Hence all vertices of $P_x(B, N, b)$ are vertices of P. Similarly, all bounded faces of $P_x(B, N, b)$, being determined by a set of linearly independent vertices, are also faces of P. So, except for its unbounded faces, $P_x(B, N, b)$ coincides with a portion of P for problems with $b \in K_B((D-1)l_{\max})$.

It is easily seen that the part of P which coincides with $P_x(B, N, b)$ consists of (i) all vertices of P which maximize some linear objective function c whose linear programming maximum is at the linear programming vertex determined by the basis B; (ii) the faces of P determined by these vertices.

Turning now to $P(\mathscr{G}, \mathscr{N}, g_0)$, we see at once that it is either empty or n'-dimensional. For, if g_0 does not lie in $\mathscr{G}_n$, the subgroup generated by $\mathscr{N}$, then no solution to the group equations is possible. On the other hand, if $g_0 \in \mathscr{G}_n$, then

$$g_0 = \sum_{g \in \mathscr{N}} t(g) \cdot g$$

with the $t(g)$ taken $0 \leqslant t(g) < s(g)$. $s(g)$ is the order of the element g, i.e., $s(g) \cdot g = \bar{0}$. These $t(g)$ provide one point t of $P(\mathscr{G}, \mathscr{N}, g_0)$. If we use $u(g)$ to denote the unit vector with $t(g) = 1$ and all other components 0, then clearly $t + s(g)u(g)$ is also a solution to (5) for each of the n' possible $g \in \mathscr{N}$; so $P(\mathscr{G}, \mathscr{N}, g_0)$ is n'-dimensional.

Of course, if $P(\mathscr{G}, \mathscr{N}, g_0)$ is empty, so is $P_x(B, N, b)$ and the original integer programming problem has no solution.

We next consider faces. By a face of $P(\mathscr{G}, \mathscr{N}, g_0)$ we will always mean an $(n'-1)$-dimensional face or, more precisely, an $(n'-1)$-dimensional hyperplane (i) with all points of $P(\mathscr{G}, \mathscr{N}, g_0)$ on one side and (ii) generated by the points of $P(\mathscr{G}, \mathscr{N}, g_0)$ lying on it. (Here generated means that all points of the hyperplane are weighted sums of the generating points with total weight 1.)

Every face corresponds to some inequality, and we denote the coefficients in an inequality such as

$$\sum_{g \in \mathscr{N}} \pi(g)t(g) \geqslant \pi_0 \tag{11}$$

by (π, π_0), where π is n'-vector different from 0 and π_0 a scalar $\geqslant 0$. For (π, π_0) to provide a face, the vectors t satisfying (11) and the equality $\pi t = \pi_0$ must meet the conditions outlined in (i) and (ii) above.

Now let us call the set of nonnegative integer solutions to (5) the set T. Since $P(\mathscr{G}, \mathscr{N}, g_0)$ is the convex hull of the points of T we can easily prove that (11) provides a face of $P(\mathscr{G}, \mathscr{N}, g_0)$ if and only if

(i)′ for every $t \in T$, $\pi \cdot t \geqslant \pi_0$, and
(ii)′ there are $t^i \in T$ which generate the hyperplane $\pi t = \pi_0$.

The first fact about faces is:

Theorem 6. *If* (π, π_0) *is a face of* $P(\mathscr{G}, \mathscr{N}, g_0)$, *then* $\pi(g) \geqslant 0$ *for all* $g \in \mathscr{N}$, *and* $\pi_0 \geqslant 0$.

Proof. If $t(g)$ solves (5) so that $t(g) \in T$, then so does $t(g) + n(g)s(g)$ when the $n(g)$ are any nonnegative integers. If (π, π_0) is a face, we must, therefore, have

$$\sum_{g \in \mathscr{N}} (t(g) + n(g)s(g))\pi(g) \geqslant \pi_0$$

for any choice whatsoever of the $n(g)$. But, if $\pi(g')$ were < 0 for some g', then for $n(g')$ sufficiently large $t(g) + n(g)s(g)$ would not satisfy the inequality. So $\pi(g) \geqslant 0$ for all g. Also, since the $t(g)$ and $\pi(g)$ are $\geqslant 0$, and equality must be obtained for some $t(g)$ if (π, π_0) is a face, it follows that $\pi_0 \geqslant 0$.

Next we have a theorem which connects the faces of $P(\mathscr{G}, \mathscr{N}, g_0)$ with a linear programming problem:

Theorem 7. *The inequality* $\pi t \geqslant \pi_0 > 0$ *provides a face of* $P(\mathscr{G}, \mathscr{N}, g_0)$ *if, and only if,* π *is a basic feasible solution of the system of inequalities*

$$\pi t \geqslant \pi_0, \qquad \textit{all} \quad t \in T.$$

This system involves one inequality for each $t \in T$. A basic feasible solution is one which satisfies all the inequalities and produces equality on a set of rows of rank n'.

Proof. If (π, π_0) provides a face, then, by (i)′, $\pi \cdot t \geqslant \pi_0$ for all $t \in T$, and, by (ii)′, there are n' $t^i \in T$ which satisfy $\pi \cdot t^i = \pi_0$ and generate the hyperplane $\pi \cdot t = \pi_0$. Since the t^i generate the hyperplane and since $\pi_0 > 0$, the hyperplane does not pass through the origin, the t^i must be linearly independent. Therefore π is a basic solution.

Now, if π is a basic feasible solution to the system of inequalities, we have $\pi \cdot t \geqslant \pi_0$ all $t \in T$ and, hence, (i)′ is satisfied. Since π is basic, there are n' independent rows which are satisfied as equalities; we may take these rows as the vectors t^i. Since they are linearly independent, they must generate the entire hyperplane $\pi \cdot t = \pi_0$; so (ii)′ is satisfied and (π, π_0) is a face.

There are a number of remarks to be made about this theorem:

(i) We can perfectly well fix π_0 at 1 in Theorem 7, since positive multiples of (π, π_0) yield the same face.

(ii) Although there is an infinity of $t \in T$, all t with $t(g) > s(g)$ are superfluous. Hence the number of inequalities is trivially reducible to the finite number $\prod_{g \in \mathcal{N}} (1 + s(g))$.

(iii) The large finite number of rows remaining in $\pi t \geqslant \pi_0$ could be dealt with in a computation by using row generating methods like those of [6] and [9]. Basically, one uses either the primal or the dual simplex method but produces rows only when needed.

In the primal method, after selecting a pivot column, one needs to know the row which represents the inequality that is violated first if the variable of the pivot column is increased. This leads to an extremalization problem over all possible rows. The row selected is the pivot row for the pivot step, and this is repeated.

In the dual method, we look for the row whose inequality is most violated and select it for each pivot step.

In our case these extremalization problems over all rows become extremalization problems over all $t \in T$ or over all solutions to the group equations; so any of the shortest-path or dynamic programming methods for group minimization apply.

(iv) To get started with a primal method, a primal feasible solution is desirable. That is, one face of $P(\mathcal{G}, \mathcal{N}, g_0)$ is wanted. Now (at least) one face can be obtained by variants of the dynamic programming or shortest-path schemes for group maximization. The idea of the variant is this: choose the $\pi(g)$ so that the shortest-path problem, say, has ties in its solution, i.e., more than one shortest path. By changing the $\pi(g)$ to produce more ties, one eventually produces a π for which there are $n' - 1$ independent shortest paths (independent solutions t^i), each costing π_0 (i.e., $\pi \cdot t^i = \pi_0$), and all other paths are longer, i.e., for all other t, $\pi \cdot t \geqslant \pi_0$. This calculation is given in Appendix 3.

B. Faces: Special Properties

Let us turn now to some more of the properties of faces of $P(\mathscr{G}, \mathscr{N}, g_0)$.

In Theorem 7 we discussed only faces (π, π_0) with $\pi_0 > 0$. Since any face can be given by an inequality (π, π_0) with $\pi_0 \geqslant 0$, there still remains the possibility $\pi_0 = 0$.

THEOREM 8. *The only possible faces (π, π_0) of $P(\mathscr{G}, \mathscr{N}, g_0)$ with $\pi_0 = 0$ are the n' hyperplanes $\pi = u(g)$ or, equivalently, $t(g) = 0$.*

Proof. Suppose $(\pi, 0)$ is a face of $P(\mathscr{G}, \mathscr{N}, g_0)$. Since the origin lies on this hyperplane, it is a $(n' - 1)$-dimensional subspace, and, as it is generated by elements $t \in T$, there must be a set of $n' - 1$ linearly independent $t \in T$ on the hyperplane. So, for each of these t^i, $i = 1, \ldots, n' - 1$, we have $\pi \cdot t^i = 0$. Since the $\pi(g)$ are $\geqslant 0$, we have $\pi(g) = 0$ unless $t^i(g) = 0$ for all vectors t^i. But, if $t^i(g) = 0$, all i, for more than one element g, the rank of the $n' - 1$ vectors t^i would be $n' - 2$ or less, a contradiction. So the only possibility aside from $\pi(g) = 0$, all $g \in \mathscr{N}$, is $\pi(h) > 0$, and $\pi(g) = 0$, $g \neq h$, which yields the face of the theorem.

We have shown that the conditions $t(h) \geqslant 0$ give the only possible faces with right-hand side $\pi_0 = 0$. It is also easy to say when these conditions actually do give faces.

THEOREM 9. *$t(h) \geqslant 0$ is a face of $P(\mathscr{G}, \mathscr{N}, g_0)$ if, and only if, the element g_0 lies in the subgroup $\mathscr{G}_{\mathscr{N} - h}$ of $\mathscr{G}$ generated by the elements of $\mathscr{N} - h$. If $g_0 \notin \mathscr{G}_{\mathscr{N} - h}$, then $t(h) \geqslant p > 0$ is a face. Here p is the smallest positive integer defining the coset $ph + \mathscr{G}_{\mathscr{N} - h}$ in which g_0 lies.*

Proof. If $g_0 \notin \mathscr{G}_{\mathscr{N} - h}$, then there are no solutions to the group equations with $t(h) = 0$; hence $t(h) \geqslant 0$ is not a face. If $g_0 \in \mathscr{G}_{\mathscr{N} - h}$, then we must have a representation

$$g_0 = \sum_{g \in \mathscr{N} - h} t(g) \cdot g, \qquad 0 \leqslant t(g) < s(g),$$

and the unit vectors $s(g)u(g)$ added to t for all $g \in \mathscr{N} - h$ form an $(n' - 1)$-dimensional array of solutions to $t(h) = 0$; hence $t(h) \geqslant 0$ is a face.

Now, if $g_0 \notin \mathscr{G}_{\mathscr{N}-h}$, since $\mathscr{G}_{\mathscr{N}-h}$ splits $\mathscr{G}_{\mathscr{N}}$ into cosets and, if $P(\mathscr{G}, \mathscr{N}, g_0)$ is not empty, g_0 is in one of these cosets. The coset must be of the form $ph + \mathscr{G}_{\mathscr{N}-h}$, and we can choose for each coset the smallest possible p. Again, this gives one solution to the equality $t(h) \geqslant p$, and the others follow by adding unit vectors $u(g)$, $g \in \mathscr{N} - h$. Since there is no representation for g_0 with $t(h) < p$, the inequality $t(h) \geqslant p$ is satisfied for all $t \in T$.

In what follows it is particularly useful to remember the shortest-path interpretation based on the graph $H(\mathscr{G}, \mathscr{N}, \pi)$. In this graph a shortest path P from $\mathscr{V}(\bar{0})$ to $\mathscr{V}(g_0)$, which contains $t(g)$ arcs corresponding to g, gives a solution $t = t(g)$, $g \in \mathscr{N}$ to the group minimization problem. The objective function is $\pi = \pi(g)$, $g \in \mathscr{N}$. There are, of course, many different paths corresponding to the same solution $t(g)$ in which the corresponding arcs are taken in different orders.

LEMMA 1. *Suppose P is a shortest path from $\mathscr{V}(\bar{0})$ to $\mathscr{V}(g_0)$ and t the corresponding solution. Then, if t' is any nonnegative integer vector with $t'(g) \leqslant t(g)$, and $\sum_{g \in \mathscr{N}} t'(g) \cdot g = h$, then any path starting at $\mathscr{V}(g)$ and corresponding to the vector $t'(g)$ is a shortest path from $\mathscr{V}(g)$ to $\mathscr{V}(g + h)$.*

Proof. It is clear from the definitions that any path corresponding to t' and starting at $\mathscr{V}(g)$ must go from $\mathscr{V}(g)$ to $\mathscr{V}(g + h)$. If there were a shorter path P'' from $\mathscr{V}(g)$ to $\mathscr{V}(g + h)$, then the corresponding t'' would have $t'' \cdot \pi < t' \cdot \pi$; and, since $\sum_{g \in \mathscr{N}} t''(g)g = h$, $(t'' + (t - t'))$ would give a path from $\mathscr{V}(0)$ to g_0 which would be shorter than the shortest path since $(t'' + (t - t')) \cdot \pi < t \cdot \pi$. This is a contradiction.

LEMMA 2. *If (π, π_0), $\pi_0 > 0$, is a face of $P(\mathscr{G}, \mathscr{N}, g_0)$, and $g \in \mathscr{N}$, then there is a shortest path from $\mathscr{V}(\bar{0})$ to $\mathscr{V}(g_0)$ with $t(g) > 0$. This implies that there is a shortest path from $\mathscr{V}(0)$ to $\mathscr{V}(g_0)$ passing through $\mathscr{V}(g)$.*

Proof. Since (π, π_0) is a face with $\pi_0 > 0$, there exist n' linearly independent t^i with $t^i\pi = \pi_0$. For all other t satisfying (5), $t \cdot \pi \geqslant \pi_0$. Each of these t^i yields shortest paths. If all $t^i(g) = 0$, for some g, the t^i would be linearly dependent; so at least one has $t^i(g) \geqslant 1$.

THEOREM 10. *If* (π, π_0), $\pi_0 > 0$ *is a face,* $\pi(g)$ *is the length of the shortest path from* $\mathscr{V}(0)$ *to* $\mathscr{V}(g)$.

Proof. By Lemma 2, there is a shortest path to $\mathscr{V}(g_0)$ with $t(g) \geq 1$. Using this solution as t and $u(g)$ as t', by Lemma 1 the one arc path corresponding to $u(g)$ is a shortest path. The length of the path is $\pi(g)$.

COROLLARY 1. *If* g_1 *and* $g_2 \in \mathscr{N}$ *and* $g = g_1 + g_2$ *with* $g \in \mathscr{N}$, *then* $\pi(g) \leq \pi(g_1) + \pi(g_2)$.

Proof. $t(g_1) = 1$, $t(g_2) = 1$ provide a path to $\mathscr{V}(g)$ of length $\pi(g_1) + \pi(g_2)$, but $\pi(g)$ is the length of the shortest path to $\mathscr{V}(g)$.

COROLLARY 2. *If* g_1 *and* $g_2 \in \mathscr{N}$, *and* $g_1 + g_2 = g_0$, *then* $\pi(g_1) + \pi(g_2) = \pi_0$.

Proof. By Lemma 2, there is a shortest path from $\mathscr{V}(0)$ to $\mathscr{V}(g_0)$ with $t(g_1) > 0$. Let the corresponding solution be t. Then both $u(g_1)$ and $t - u(g_1)$ are vectors t' satisfying Lemma 1, and hence they correspond to shortest paths from $\mathscr{V}(\bar{0})$ to $\mathscr{V}(g_1)$ and from $\mathscr{V}(0)$ to $\mathscr{V}(g_2)$ $(g_2 = g_0 - g_1)$, respectively; so

$$\pi \cdot t = \pi_0$$

$$\pi(u(g_1)) + \pi(t - u(g)) = \pi_0,$$

and therefore, by Theorem 10,

$$\pi(g_1) + \pi(g_2) = \pi_0.$$

Leaving for the moment the properties of the $\pi(g)$ associated with a fixed face of a fixed polyhedron $P(\mathscr{G}, \mathscr{N}, g_0)$, we ask instead the following question. When does a face persist from one polyhedron to the next, i.e., when, if (π, π_0) provides a face of $P(\mathscr{G}, \mathscr{N}, g_0)$, does it also provide a face of $P(\mathscr{G}, \mathscr{N}, h)$?

It is easily seen that persistence of a face in this sense is too strong a demand. For, if (π, π_0) is to be a face for both polyhedra, there would have to be at least n' shortest paths from $\mathscr{V}(\bar{0})$ to $\mathscr{V}(g_0)$ and n' more from $\mathscr{V}(\bar{0})$ to $\mathscr{V}(h)$, all of the same length, and this seems to be a very special situation. However, if we allow parallel displacement of a face, we see that persistence of a face in this sense is common and occurs under the following simple sufficient condition:

THEOREM 11. *Consider the graph* $H(\mathscr{G}, \mathscr{N}, \pi)$ *with arc lengths* $\pi(g)$ *based on a face* (π, π_0), $\pi_0 > 0$, *of* $P(\mathscr{G}, \mathscr{N}, g_0)$. *Let* $\mathscr{V}(h)$ *be any of the vertices that can be reached by a shortest path passing through* $\mathscr{V}(g_0)$. *Then for each such* $\mathscr{V}(h)$ *there is a constant* $\pi_0(h)$ *such that* $(\pi, \pi_0(h))$ *provides a face for* $P(\mathscr{G}, \mathscr{N}, h)$.

Proof. Let $t(h)$ be the vector corresponding to the shortest path from $\mathscr{V}(g_0)$ to $\mathscr{V}(h)$. Then, if the t^i, $i = 1, \ldots, n'$, give the n' linearly independent shortest paths from $\mathscr{V}(\bar{0})$ to $\mathscr{V}(g_0)$, the n' vectors $t^i + t(h)$ provide n' paths from $\mathscr{V}(\bar{0})$ to $\mathscr{V}(h)$. Because of our assumption about the existence of a shortest path through $\mathscr{V}(g_0)$ to $\mathscr{V}(h)$, these are all shortest paths. What must be shown is that these vectors are still linearly independent. Now suppose there is a dependence with weights w_i. We can assume the w_i sum to 1; so, using the weights to form a zero vector, we would have

$$\pi \cdot \left(\sum w_i(t^i + t(h))\right) = 0.$$

But $\pi \cdot t^i = \pi_0$; hence

$$\pi_0 + \pi \cdot t(h) = 0.$$

But $\pi_0 > 0$ and all components of π and of $t(h)$ are nonnegative; therefore this is a contradiction. Thus these paths are linearly independent and $(\pi, \pi_0(h))$ is a face of $P(\mathscr{G}, \mathscr{N}, h)$ with $\pi_0(h)$ the shortest-path distance from $\mathscr{V}(\bar{0})$ to $\mathscr{V}(h)$.

We now turn away from the $P(\mathscr{G}, \mathscr{N}, g_0)$ to introduce the larger polyhedra $P(\mathscr{G}, g_0)$, which are investigated in more detail in the next section.

C. The Polyhedra $P(\mathscr{G}, g_0)$

We define $P(\mathscr{G}, g_0)$ as $P(\mathscr{G}, \mathscr{G} - \bar{0}, g_0)$. The set $\mathscr{N}$ is taken to be all of $\mathscr{G}$ except $\bar{0}$ and, adopting for $\mathscr{G} - \bar{0}$ the symbol $\mathscr{G}^*$, we therefore define $P(\mathscr{G}, g_0)$ as the convex hull of the nonnegative* integer vectors

* For $g_0 = \bar{0}$, we define $P(\mathscr{G}, (0))$ as the convex hull of the nonzero nonnegative vectors satisfying (12). The solution $t(g) = 0$ is excluded.

t in $D - 1 = |\mathscr{G}| - 1$ dimensional space E_{D-1} satisfying

$$\sum_{g \in \mathscr{G}^+} t(g) \cdot g = g_0. \tag{12}$$

The relation between $P(\mathscr{G}, g_0)$ and $P(\mathscr{G}, \mathscr{N}, g_0)$ is given by the following theorem. If we let $E(\mathscr{N})$ be the n'-dimensional subspace in $(D - 1)$-dimensional space in which $t(g) = 0$, $g \notin \mathscr{N}$, we can identify our previous n'-dimensional space with this subspace and consider $P(\mathscr{G}, \mathscr{N}, g_0)$ as lying in this space. All vectors t in our previous n'-dimensional space are extended by adding components $t(g) = 0$ for all $g \notin \mathscr{N}$. Then we have the theorem.

THEOREM 12. $P(\mathscr{G}, \mathscr{N}, g_0) = P(\mathscr{G}, g_0) \cap E(\mathscr{N})$.

The theorem asserts that the various possible $P(\mathscr{G}, \mathscr{N}, g_0)$ are obtained from the master polyhedron $P(\mathscr{G}, g_0)$ by setting some variables to zero. Since $P(\mathscr{G}, \mathscr{N}, g_0)$ is the convex hull of certain lattice points in $P(\mathscr{G}, g_0) \cap E(\mathscr{N})$, the content of the theorem is that $P(\mathscr{G}, \mathscr{N}, g_0)$ is in fact the whole intersection. Of course, in general, the intersection of $P(\mathscr{G}, g_0)$ with some subspace would be completely different from the convex hull of the lattice points in that intersection. (There could be no lattice points in the intersection, for example.) They coincide here because the intersection lies entirely on one side of $P(\mathscr{G}, g_0)$. This is the only property used in the following proof.

Proof. Clearly, any nonnegative integer $t \in P(\mathscr{G}, \mathscr{N}, g_0)$ lies in $E(\mathscr{N})$ and, since it satisfies $\sum_{g \in \mathscr{N}} t(g) \cdot g = g_0$, it satisfies (12) and so is in $P(\mathscr{G}, g_0)$. Therefore $P(\mathscr{G}, \mathscr{N}, g_0) \subset P(\mathscr{G}, g_0) \cap E(\mathscr{N})$.

Now suppose a point $p \in P(\mathscr{G}, g_0) \cap E(\mathscr{N})$. Since it is in $P(\mathscr{G}, g_0)$, it is a convex combination of integer points $t^i \in P(\mathscr{G}, g_0)$ which satisfy (12). Selecting those t^i with positive, i.e., nonzero, weight λ_i only, we have $p = \sum_i \lambda_i t^i$. Since $p \in E(\mathscr{N})$, its gth component $p(g)$ must $= 0$ for $g \notin \mathscr{N}$. So each $t^i \in E(\mathscr{N})$. Since t^i satisfies (12) and lies in $E(\mathscr{N})$, it also satisfies the group equation defining $P(\mathscr{G}, \mathscr{N}, g_0)$ and so is in $P(\mathscr{G}, \mathscr{N}, g_0)$. Thus p is a convex combination of points of $P(\mathscr{G}, \mathscr{N}, g_0)$. So $p \in P(\mathscr{G}, \mathscr{N}, g_0)$ and, therefore, $P(\mathscr{G}, \mathscr{N}, g_0) \supset P(\mathscr{G}, g_0) \cap E(\mathscr{N})$, which concludes the proof.

With respect to vertices and faces the connection between $P(\mathscr{G}, g_0)$ and $P(\mathscr{G}, \mathscr{N}, g_0)$ is quite direct and is summarized as follows:

THEOREM 13. (i) *An inequality* (π, π_0) *with* π *an* n'*-vector provides an* $(n'-1)$*-dimensional face of* $P(\mathscr{G}, \mathscr{N}, g_0)$ *if and only if there is a* $(D-1)$*-dimensional face* (π', π_0) *of* $P(\mathscr{G}, g_0)$ *with* $\pi'(g) = \pi(g)$ *all* $g \in \mathscr{N}$.

(ii) *Every vertex of* $P(\mathscr{G}, \mathscr{N}, g_0)$ *is a vertex of* $P(\mathscr{G}, g_0)$. *A vertex* $t = t(g)$ *of* $P(\mathscr{G}, g_0)$ *is a vertex of* $P(\mathscr{G}, \mathscr{N}, g_0)$ *if and only if* $t \in E(\mathscr{N})$.

TABLE I

FACES

$P(\mathscr{G}_6, (3))$						$P(\mathscr{G}_6, \{(0), (1), (2), (3), (5)\}, (3))$				
π_1	π_2	π_3	π_4	π_5	π_0	π_1	π_2	π_3	π_5	π_0
1	0	1	0	1	1	1	0	1	1	1
2	1	3	2	1	3	2	1	3	1	3
1	2	3	2	1	3	1	2	3	1	3
1	2	3	1	2	3					
1	0	0	0	0	0	1	0	0	0	0
0	1	0	0	0	0	0	1	0	0	0
0	0	1	0	0	0	0	0	1	0	0
0	0	0	1	0	0					
0	0	0	0	1	0	0	0	0	1	0

VERTICES

$P(\mathscr{G}_6, (3))$	$P(\mathscr{G}_6, \{(0), (1), (2), (3), (5)\}, (3))$
$(t_1, t_2, t_3, t_4, t_5)$	(t_1, t_2, t_3, t_5)
(3, 0, 0, 0, 0)	(3, 0, 0, 0)
(1, 1, 0, 0, 0)	(1, 1, 0, 0)
(0, 0, 1, 0, 0)	(0, 0, 1, 0)
(1, 0, 0, 2, 0)	
(0, 2, 0, 0, 1)	(0, 2, 0, 1)
(0, 0, 0, 1, 1)	
(0, 0, 0, 0, 3)	(0, 0, 0, 3)

Essentially, (i) states that every face of $P(\mathscr{G}, \mathscr{N}, g_0)$ is obtained by taking some face of $P(\mathscr{G}, g_0)$ and simply omitting the components of $\pi' \notin \mathscr{N}$. If this is done for all faces of $P(\mathscr{G}, g_0)$, all faces of $P(\mathscr{G}, \mathscr{N}, g_0)$ will be obtained plus some valid but superfluous inequalities. To prove (i) we merely note that, in view of Theorem 12, $P(\mathscr{G}, \mathscr{N}, g_0)$ is the set of points satisfying

$$t(g) = 0, \qquad g \notin \mathscr{N},$$

and

$$\pi t \geq \pi_0$$

for all (π, π_0) that are faces of $P(\mathscr{G}, g_0)$. Now, if a polyhedron is given by a finite set of inequalities, each face corresponds to some one of these inequalities; so each face of $P(\mathscr{G}, \mathscr{N}, g_0)$ corresponds to some (π, π_0).

For (ii) we merely note that, if t is a vertex of $P(\mathscr{G}, g_0)$ with all $t(g) = 0$, $g \notin \mathscr{N}$, then it is in $E(\mathscr{N})$ and certainly, as a vertex of $P(\mathscr{G}, g_0)$, it is a vertex of $P(\mathscr{G}, \mathscr{N}, g_0)$. If t were a vertex of $P(\mathscr{G}, \mathscr{N}, g_0)$ and *not* a vertex of $P(\mathscr{G}, g_0)$, it would have to be a positive convex combination of points of $P(\mathscr{G}, g_0)$. But, just as in the proof of Theorem 12, these points must lie in $P(\mathscr{G}, \mathscr{N}, g_0)$; this would contradict the assumption that t is a vertex of $P(\mathscr{G}, \mathscr{N}, g_0)$.

Table I illustrates the relationships (i) and (ii) for the polyhedra $P(\mathscr{G}_6, (3))$ and $P(\mathscr{G}_6, \{(0), (1), (2), (3), (5)\}, (3))$.

Thus the faces and vertices of $P(\mathscr{G}, g_0)$ contain the faces and vertices of $P(\mathscr{G}, \mathscr{N}, g_0)$ for all possible $\mathscr{N}$, and so contain information about many different problems.

We turn next to the study of the $P(\mathscr{G}, g_0)$.

3. PROPERTIES OF THE $P(\mathscr{G}, g_0)$

A. Automorphisms

In dealing with $P(\mathscr{G}, g_0)$ there is much to be gained from the use of symmetry as expressed in the group automorphisms. In what follows we shall see that, in dealing with $P(\mathscr{G}, g_0)$, one face leads to other faces, one vertex leads to other vertices, and knowledge about $P(\mathscr{G}, g_0)$ leads to knowledge about other polyhedra $P(\mathscr{G}, h)$.

We start by describing the effect of an automorphism on a face of $P(\mathscr{G}, g_0)$.

THEOREM 14. *If (π, π_0) is a face of $P(\mathscr{G}, g_0)$ with components $\pi(g)$ and $\phi\colon \mathscr{G} \to \mathscr{G}$ is any automorphism of $\mathscr{G}$, then $(\bar{\pi}, \pi_0)$ with components $\bar{\pi}(g) = \pi(\phi^{-1}g)$ is a face of $P(\mathscr{G}, \phi(g_0))$.*

Proof. We make use of the graph $H(\mathscr{G}, \mathscr{G}^*, \pi)$ which we now refer to as $H(\mathscr{G}, \pi)$. In $H(\mathscr{G}, \pi)$, let P be any path from $\mathscr{V}(\bar{0})$ to $\mathscr{V}(g_0)$, i.e., any vector $t = t(g)$, $g \in \mathscr{G}^*$, satisfying

$$\sum_{g \in \mathscr{G}^+} t(g) \cdot g = g_0.$$

Applying the automorphism ϕ gives

$$\sum_{g \in \mathscr{G}^+} t(g) \cdot \phi(g) = \phi(g_0) = \sum_{g \in \mathscr{G}^+} t(\phi^{-1}g) \cdot g;$$

so the vector $\bar{t} = \bar{t}(g) = t(\phi^{-1}g)$ gives a path $\bar{P}$ to $\mathscr{V}(\phi(g_0))$. Now we introduce new arc lengths $\bar{\pi}(g) = \pi(\phi^{-1}g)$. The length of $\bar{P}$ in terms of the $\bar{\pi}(g)$ is

$$\sum_{g \in \mathscr{G}^+} \bar{\pi}(g)\bar{t}(g) = \sum_{g \in \mathscr{G}^+} \pi(\phi^{-1}g)t(\phi^{-1}g) = \sum_{g \in \mathscr{G}^+} \pi(g)t(g) = l(P);$$

therefore under the automorphism ϕ the path P in $H(\mathscr{G}, \pi)$ goes into a path $\bar{P}$ in $H(\mathscr{G}, \bar{\pi})$ of equal length. Thus, as ϕ has an inverse ϕ^{-1}, this sets up a one-to-one length preserving correspondence between paths in $H(\mathscr{G}, \pi)$ and paths in $H(\mathscr{G}, \bar{\pi})$. In particular, shortest paths go into shortest paths. Also, since $\bar{t}$ is merely a rearrangement of the components of t, a set of linearly independent t go into a set of linearly independent $\bar{t}$. If t^i are the independent set of shortest paths in $H(\mathscr{G}, \pi)$, the $\bar{t}^i$ are an independent set in $H(\mathscr{G}, \bar{\pi})$; thus $\bar{\pi}$ satisfies the conditions for a face.

Since the faces completely determine the polyhedra, this means that the polyhedra $P(\mathscr{G}, g_0)$ and $P(\mathscr{G}, \phi(g_0))$ are identical after the rearrangement of coordinates induced by ϕ. If ϕ leaves g_0 fixed, this is a symmetry of $P(\mathscr{G}, g_0)$. If $\phi(g_0) = h \neq g_0$, this means that the polyhedron $P(\mathscr{G}, h)$ can be obtained by simply rearranging the coordinates, once $P(\mathscr{G}, g_0)$ is obtained. For example, if $\mathscr{G}$ is cyclic of prime order, there is an automorphism ϕ mapping g_0 onto every nonzero h, so there is essentially only one polyhedron $P(\mathscr{G}, g_0)$ to be obtained. All of the various other $P(\mathscr{G}, h)$ have the same number of vertices, faces, etc., which can be explicitly exhibited, once $P(\mathscr{G}, g_0)$ has been obtained, simply by applying the automorphism ϕ that sends g_0 onto h.

In general, then, there is essentially only one polyhedron $P(\mathscr{G}, g_0)$ for each automorphism class in $\mathscr{G}$.

For vertices we have the following corollary:

COROLLARY. *If $t = t(g)$ is a vertex of $P(\mathscr{G}, g_0)$, then $\bar{t} = \bar{t}(g) = t(\phi^{-1}g)$ is a vertex of $P(\mathscr{G}, \phi(g_0))$.*

Vertices, however, have an additional property: they can be produced by taking subsets.

THEOREM 15. *Let $t = t(g)$ be a vertex of $P(\mathscr{G}, g_0)$. Let $s = s(g)$ with $0 \leqslant s(g) \leqslant t(g)$ for all $g \in \mathscr{G}^+$. Then, if*

$$h = \sum_{g \in \mathscr{G}^+} s(g) \cdot g,$$

s is a vertex of $P(\mathscr{G}, h)$.

Proof. Lemma 1, specialized to our situation, shows that, if t is a shortest path from $\mathscr{V}(0)$ to $\mathscr{V}(g_0)$, then s is a shortest path from $\mathscr{V}(0)$ to $\mathscr{V}(h)$. Lemma 1 can be applied to any π used as an objective function in this minimization problem, not only to faces π; thus any π minimized over $P(\mathscr{G}, g_0)$ at t is a π minimized over $P(\mathscr{G}, h)$ at s. Since t is a vertex, there are certainly $D - 1$ independent vectors π minimized at t; since these are minimized at s, it must be a vertex also.

COROLLARY. *Let t and s be as in Theorem 15. If there is a ϕ such that $\phi h = g_0$, then $\bar{s} = \bar{s}(g) = s(\phi^{-1} g)$ is a vertex of $P(\mathscr{G}, g_0)$.*

This corollary merely combines Theorems 14 and 15, but it allows additional vertices to be produced for the same polyhedron even when no automorphisms leave g_0 fixed. For example, consider the cyclic group of order 11. Denoting it by $\mathscr{G}_{11}$, and the generator by 1, we find from Appendix 5 that $P(\mathscr{G}_{11}, (10))$ has a vertex $t_1 = 3$, $t_7 = 1$; $t_m = 0$, $m \neq 7$, $m \neq 1$. According to Theorem 15, each t given by:

t_1	t_7	t_m $(m \neq 1, 7)$	h
3	0	0	3
2	1	0	9
2	0	0	2
1	1	0	8
1	0	0	1
0	1	0	7

is a vertex of the polyhedron $P(\mathscr{G}_{11}, (h))$ whose h appears in the last column. Applying the corollary and the automorphisms by multiplying through by 7, by 6, by 5, by 4, by 10, and by 3, respectively ($7 \times 3g = 21g = 10g$; $6 \times 9g = 54g = 10g, \ldots$, etc.), we obtain six vertices of $P(\mathscr{G}_{11}, 10g)$, namely,

$$
\begin{array}{lll}
t_{10} = 1, & & \text{all other components } 0, \\
t_6 = 2, & t_9 = 1, & \text{all other components } 0, \\
t_5 = 2, & & \text{all other components } 0, \\
t_4 = 1, & t_6 = 1, & \text{all other components } 0, \\
t_{10} = 1, & & \text{all other components } 0, \\
t_{10} = 1, & & \text{all other components } 0,
\end{array}
$$

of which four are distinct.

B. Faces

We turn next to the properties of the $\pi(g)$ making up the faces of $P(\mathscr{G}, g_0)$. We have two types of faces: those given by inequalities (π, π_0) with $\pi_0 > 0$ and those with $\pi_0 = 0$. First, for those with $\pi_0 = 0$, we have the simple result:

THEOREM 16. *The condition $t(h) \geqslant 0$, for a fixed $h \in \mathscr{G}$, yields a face of $P(\mathscr{G}, g_0)$ unless $\mathscr{G}$ is cyclic of order 2.*

Proof. Applying Theorem 9, we need only show that g_0 lies in the subgroup of elements generated by $\mathscr{G}^+ - h$ to establish the theorem. First, if h is of any order $s(h)$ other than 2, then $\mathscr{G}^+ - h$ contains $-h$ and, hence, contains h in the group it generates. If h is of order 2 but $\mathscr{G}^+$ contains some other element h', $h - h'$ must be $\neq \bar{0}$ and $\neq h$, so it is some other element $h'' \in \mathscr{G}^+$. Thus $\mathscr{G}^+$ contains h'' and h' with $h'' + h' = h$, and thus generates h. This leaves only the case $\mathscr{G}^+ = h$, h of order 2. If $\mathscr{G}$ is cyclic of order 2 and $g_0 = h$ is the generator, then $1 \cdot t(h) = 0$ is never attained as an equality. Instead, $1 \cdot t(h) = 1$ is a face because of the solution $t(h) = 1$.

Next, turning to the faces (π, π_0), $\pi_0 > 0$, we have:

THEOREM 17. *If (π, π_0) is a face of $P(\mathscr{G}, g_0)$, $g_0 \neq \bar{0}$,*

(i) $\pi(g) + \pi(g_0 - g) = \pi_0$, *all* $g \in \mathscr{G}^+$,

(ii) $\pi(g) + \pi(g') \geqslant \pi(g + g'), \quad \textit{all } g, g' \in \mathscr{G}^+,$

and

(iii)* $\pi(g_0) = \pi_0.$

Proof. (i) and (ii) are direct applications of Theorem 10, Corollaries 1 and 2, to the case $\mathscr{N} = \mathscr{G}^+$. However, (i) has more significance now, for it means that the coefficients $\pi(g)$ occur in pairs and, if $\pi(g)$ is known, then (if, say, π_0 is normalized to 1) so is its complement $\pi(g_0 - g)$. (iii) follows from the application of Theorem 10 with the g of the theorem replaced by g_0. Since π_0 is the length of the shortest path from $\mathscr{V}(\bar{0})$ to $\mathscr{V}(g_0)$, Theorem 10 asserts that $\pi(g_0) = \pi_0$.

Theorem 17 is mainly a restatement of previous lemmas with wider usefulness in the case of $P(\mathscr{G}, g_0)$. However, with its aid we can now replace Theorem 6 with the following strong result:

THEOREM 18. *(π, π_0), $\pi_0 > 0$, is a face of the polyhedron $P(\mathscr{G}, g_0)$, $g_0 \neq \bar{0}$, if and only if it is a basic feasible solution to the system of equations and inequalities:*

$$\begin{aligned} &{}^{\dagger}\pi(g_0) = \pi_0, && \\ &\pi(g) + \pi(g_0 - g) = \pi_0, && g \in \mathscr{G}^+, \quad g \neq g_0, \\ &\pi(g) + \pi(g') \geqslant \pi(g + g'), && g, \quad g' \in \mathscr{G}^+, \\ &\pi(g) \geqslant 0, && g \in \mathscr{G}^+. \end{aligned} \tag{13}$$

In contrast with Theorem 7 we can easily write down conditions (13) explicitly, and we have a highly structured matrix with at most three nonzero entries in each row and reflecting the group structure of $\mathscr{G}$ very closely.

If the equalities are used to eliminate variables, we will have roughly $D/2$ variables and, eliminating duplications among the inequalities after the equations are taken into account, about $D^2/6$ inequalities aside from the nonnegativity conditions on the variables.

* If $g_0 = 0$, (i) and (ii) hold and (iii) is dropped.

† If $g_0 = \bar{0}$, then $\pi(g_0) = \pi_0$ is omitted and the modified theorem holds.

*Proof.** We show first that any basic feasible solution to (13) is a face.

If π is a basic feasible solution of (13), it satisfies equations whose rows are of rank $D - 1$. It certainly satisfies the first of equations (13), $\pi(g_0) = \pi_0$, and the second, $\pi(g) + \pi(g_0 - g) = \pi_0$, for all $g \in \mathscr{G}^+$, $g \neq g_0$, and these provide a linearly independent set of rows. This set can then be augmented by other equations until a set of $D - 1$ independent equations, which includes the ones just listed, is obtained. These additional equations can be of two forms: either

$$\text{(i)} \qquad \pi(g) + \pi(h) - \pi(g + h) = 0, \qquad g + h \neq g_0, \neq 0;$$

or

$$\text{(ii)} \qquad \pi(g) = 0, \qquad g \neq g_0.$$

If an equation is of form (i), it is added to the equation $\pi(g + h) + \pi(g_0 - (g + h)) = \pi_0$ to form the new, and still independent, equation

$$\pi(g) + \pi(h) + \pi(g_0 - (g + h)) = \pi_0.$$

If it is of type (ii), multiply by the order of g, $s(g)$, and add the equation $\pi(g_0) = \pi_0$ to obtain

$$s(g)\pi(g) + \pi(g_0) = \pi_0,$$

which is, of course, still independent.

If at this point we consider all the rows, we see that the entries $t^i(g)$ of row t^i are nonnegative integers satisfying

$$\sum_{g \in \mathscr{G}^+} t^i(g)g = g_0,$$

and that $t^i \cdot \pi = \pi_0$. Since the t^i are linearly independent, this is a set of $n' = (D - 1)$-vectors belonging to T and generating $\pi \cdot t = \pi_0$; thus (π, π_0) is a face. Now we need only show that every face is a basic feasible solution of (13).

By Theorem 17, any face (π, π_0) satisfies (13). Since the solutions to (13) form a bounded polyhedron, (π, π_0) is either (i) a basic feasible solution to (13) (vertex) or (ii) a nontrivial convex combination (sum with

* This proof has been considerably shortened as a result of a suggestion from Ellis Johnson.

nonnegative weights adding to 1) of basic feasible solutions (π_i, π_0). But, since (π, π_0) and the (π_i, π_0) are faces, they are basic feasible solutions (vertices) of the system of inequalities appearing in the statement of Theorem 7. Since one vertex of that system cannot be a convex combination of others, (π, π_0) cannot be a convex combination of the (π_i, π_0). Thus possibility (ii) is eliminated and (i) holds. This ends the proof.

All faces of the 36 polyhedra listed in Appendix 5 have been computed using this theorem. A computer code of Balinski and Wolfe [13] was used to list all basic feasible solutions to the system (13). This list provided all vertices of the polyhedron given by (13). Generally, many (degenerate) bases gave the same vertex.

The faces listed in Appendix 5 show obvious symmetries and patterns depending on the group structure. This suggests the possibility of constructing at least some of the faces of $P(\mathscr{G}, g_0)$ without the use of Theorem 18. We shall see that this is the case and that many of the faces of $P(\mathscr{G}, g_0)$ can be produced essentially by formula.

We start with cyclic groups and then move on to some theorems that enable us to produce faces of noncyclic groups from faces of their cyclic subgroups.

C. *Some Faces for Cyclic Groups*

We shall see that it is quite easy to construct a family of faces for any cyclic group $\mathscr{G}$ and right-hand side g_0. We first consider the case $g_0 \neq \bar{0}$.

Let us examine the graph $H(\mathscr{G}, \pi)$ for a cyclic group. We designate a generator by g_1 and represent the other elements as multiples. Let $g_0 = mg_1$. In Fig. 5 we put in only a few of the arcs of the graph. Let

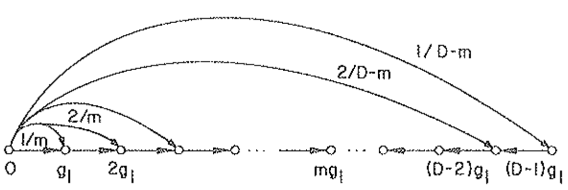


FIG. 5.

$D = |\mathscr{G}|$ be the order of the group. We can form $D - 1$ independent paths T_p by using the group element pg_1 once and then completing the path to g_0 by iterating g_1 if $p < m$, or iterating $-g_1 = (D-1)g_1$ if $p > m$. This set of paths is clearly independent. Now, if we set $\pi_m(pg_1) = p/m$, $p \leqslant m$, all the T_p, $p \leqslant m$, have a total length 1. To achieve the same result for the remaining paths we set $\pi_m(pg) = (D-p)/(D-m)$, for $p > m$.

Now it is easily seen that the length of any path in this $H(\mathscr{G}, \pi_m)$ from $\bar{0}$ to g_0 is at least 1, for in any path the occurrence of any element pg_1 can be replaced by pg's or the appropriate number of $-g_1$'s, if $p > m$, without changing the path length. It follows that the T_p are minimal paths, and so this assignment of π values gives a face with $\pi_0 = 1$.

We can produce a face this way for each possible right-hand side mg_1. If we are interested in a particular right-hand side, say $g_0 = m_0 g_1$, then, using automorphisms, we can convert the other faces that are in the same automorphism class into faces of $P(\mathscr{G}, g_0)$. For example, taking $\mathscr{G}$ to be the cyclic group of order 7, we have for $P(\mathscr{G}_7, (6))$ the face $(\pi_m, 1)$ with $m = 6$:

$$\left(\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, 1, 1\right);$$

or

$$1t_1 + 2t_2 + 3t_3 + 4t_4 + 5t_5 + 6t_6 \geqslant 6,$$

in which the fractions $\pi(g_1) = \frac{1}{6}$, etc., have been converted into integers. Similarly, for the other g_0, the $\pi(g)$ are:

	$\pi(g_1)$	$\pi(2g_1)$	$\pi(3g_1)$	$\pi(4g_1)$	$\pi(5g_1)$	$\pi(6g_1)$	π_0
$P(\mathscr{G}_7, (5))$	2	4	6	8	10	5	10
$P(\mathscr{G}_7, (4))$	3	6	9	12	8	4	12
$P(\mathscr{G}_7, (3))$	4	8	12	9	6	3	12
$P(\mathscr{G}_7, (2))$	5	10	8	6	4	2	10
$P(\mathscr{G}_7, (1))$	6	5	4	3	2	1	6

Now to produce faces for $P(\mathscr{G}_7, (6))$ we apply the automorphism sending $5g_1$ into $6g_1$ (multiplication by 4), obtaining from the top row of the table the new face of $P(\mathscr{G}_7, (6))$:

$$4t_1 + 8t_2 + 5t_3 + 2t_4 + 6t_5 + 10t_6 \geqslant 10;$$

similarly, from the next row, multiplying the group elements by 5 yields as a face of $P(\mathscr{G}_7, (6))$:

$$9t_1 + 4t_2 + 6t_3 + 8t_4 + 3t_5 + 12t_6 \geqslant 12.$$

Applying multiplication by 2 to the next row merely produces this last face once again, and the remaining faces obtained by applying automorphisms are duplicates of those already obtained.

It is easily proved that the general situation is this: If m, m', and m'' are in the same automorphism class (which simply means, for cyclic groups, that the g.c.d.'s (m, D), (m', D), and (m'', D) are equal), then the face obtained by using the face $(\pi_{m'}, 1)$ of $P(\mathscr{G}, m'g_1)$ and applying the automorphism ϕ, $\phi(m'g_1) = (mg_1)$ yields the same face of $P(\mathscr{G}, mg_1)$ as using $\pi_{m''}$ and applying ϕ'', $\phi''(m''g_1) = mg_1$ if, and only if, $m''g = -m'g$.

Thus, in general, this procedure produces, for a right-hand side mg_1, about half as many different faces as there are elements in the automorphism class of mg_1.

So far we have discussed the case $g_0 \neq \bar{0}$. If $g_0 = \bar{0}$, $P(\mathscr{G}, (0))$ consists of the convex hull of the nontrivial solutions to the group equation, and it is easily verified that the values $\pi(mg_1) = m/D$ allow, almost exactly as above, $D - 1$ independent nontrivial minimal paths from $\bar{0}$ to $\bar{0}$ in $H(\mathscr{G}, \pi)$. Hence this π is a face. The automorphism situation is, however, a little different. Every automorphism of $\mathscr{G}$ sends $\bar{0}$ into $\bar{0}$ and, hence, sends $(\pi, 1)$ into another face of the same polyhedron $P(\mathscr{G}, (0))$. Since $\pi(g) \neq \pi(g')$ whenever $g \neq g'$, it follows that all of these faces are different. Thus, in the case $g_0 = \bar{0}$, we get as many different faces from this one construction as there are automorphisms of $\mathscr{G}$. Taking again the cyclic group of order 7, we construct the face

$$1t_1 + 2t_2 + 3t_3 + 4t_4 + 5t_5 + 6t_6 = 7$$

of $P(\mathscr{G}_7, \bar{0})$ and then have by automorphisms (multiplying by 2, 3, 4, 5, and 6) the faces

$$4t_1 + 1t_2 + 5t_3 + 2t_4 + 6t_5 + 3t_6 \geqslant 7,$$

$$5t_1 + 3t_2 + 1t_3 + 6t_4 + 4t_5 + 2t_6 \geqslant 7,$$

$$2t_1 + 4t_2 + 6t_3 + 1t_4 + 3t_5 + 5t_6 \geqslant 7,$$

$$3t_1 + 6t_2 + 2t_3 + 5t_4 + 1t_5 + 4t_6 \geqslant 7,$$

$$6t_1 + 5t_2 + 4t_3 + 3t_4 + 2t_5 + 1t_6 \geqslant 7.$$

These are faces of $P(\mathscr{G}_7, (0))$, and, in fact, a comparison with the list of faces in Appendix 5 shows that these are all the faces.

We turn next to the problem of connecting faces of larger or more complicated groups with the faces of smaller ones.

D. *Lifting up Faces*

THEOREM 19. *Let ψ be a homomorphism of $\mathscr{G}$ onto $\mathscr{H}$ with kernel $\mathscr{K}$ and with $g_0 \notin \mathscr{K}$. Then, if (π', π_0) is a face of $P(\mathscr{H}, \psi g_0)$, (π, π_0) is a face of $P(\mathscr{G}, g_0)$ when $\pi(g)$ is given by $\pi(g) = \pi'(\psi g)$. (We take $\pi'(\bar{0}) = 0$; so $\pi(g) = 0$, $g \in \mathscr{K}$.)*

Proof. We proceed to construct $D - 1$ independent minimal paths in $H(\mathscr{G}, \pi)$ from $\bar{0}$ to g_0.

First we note that the value of such a minimal path is $\geqslant \pi_0$, for, if there were a path of value $\pi_1 < \pi_0$ with g occurring $t(g)$ times, we would have

$$\pi_0 > \pi_1 = \sum_{g \in \mathscr{G}^+} t(g) \cdot \pi(g) = \sum_{h \in H} \sum_{g \in \psi^{-1} h} t(g) \cdot \pi'(\psi g)$$

$$= \sum_{h \in H} \Big(\sum_{g \in \psi^{-1} h} t(g) \Big) \cdot \pi'(h);$$

then the path in $H(\mathscr{H}, \pi')$ from $\bar{0}$ to $h_0 = \psi g_0$ with components $\tau(h) = \sum_{g \in \psi^{-1} h} t(g)$ would have cost $\pi_1 < \pi_0$, and this is a contradiction.

Next we see that there are many paths in $H(\mathscr{G}, \pi)$ which actually attain the value π_0 and hence are minimal. To obtain them we select elements from each coset to imitate a path in $H(\mathscr{H}, h_0)$. If the total of these elements is not g_0, we add in an element of K (which has cost 0) to make the path go to g_0.

More precisely, let ϕ be any function from $\mathscr{H}$ into $\mathscr{G}$ which selects coset representatives, i.e., $\psi\phi(h) = h$. Any such ϕ provides a unique representation of each element g in the form

$$g = \phi(h) + k', \qquad h \in \mathscr{H}, \qquad k' \in \mathscr{K}.$$

Then for each path $\tau(h)$ in $H(\mathscr{H}, h_0)$ we obtain $|\mathscr{H}|$ paths in $H(\mathscr{G}, g_0)$ by

$$t_k(g) = 0, \qquad \text{if } g = \phi(h) + k' \quad \text{with } h \neq \bar{0} \text{ and } k' \neq k,$$

$$t_k(g) = \tau(h), \qquad \text{if } g = \phi(h) + k' \quad \text{with } h \neq \bar{0} \text{ and } k' = k,$$

$$t_k(g) = 1, \qquad \text{if } g = \phi(h) + k' \quad \text{with } h = \bar{0} \text{ and } k' = g_0 - \sum_{g \notin \mathscr{H}} t_k(g) \cdot g$$

$$t_k(g) = 0, \qquad \text{if } g = \phi(h) + k' \quad \text{with } h = \bar{0} \text{ and } k' \neq g_0 - \sum_{g \notin \mathscr{H}} t_k(g) \cdot g.$$

If we call this path $T_k(\tau)$, we obtain, using the $|\mathscr{H}| - 1$ independent minimal paths in $H(\mathscr{H} g_0)$, $(|\mathscr{H}| - 1)|\mathscr{H}|$ minimal paths in $H(\mathscr{G}, g_0)$. These paths can be arranged as rows with a component for each $g \in \mathscr{G}^+$.

If we put together rows $T_k(\tau)$ with the same k but different τ, we obtain $|\mathscr{H}|$ blocks of rows, each block containing $|\mathscr{H}| - 1$ rows. Next we put together the columns with $g \in \mathscr{H}$ and then each set of columns with $g \in \phi^{-1}(\mathscr{H}^+) + k$ for each k. We then obtain an $(|\mathscr{H}| - 1)|\mathscr{H}| \times (d - 1)$ matrix (see Fig. 6).

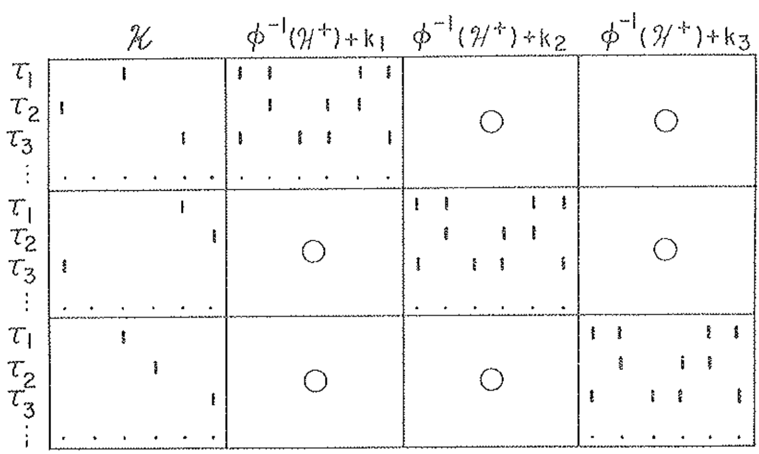


FIG. 6.

The columns belonging to each k consist of blocks which are replicas of the independent minimal paths in $H(\mathscr{H}, \pi')$. No element g appears in more than one block since the representation of g involves a unique k. The matrix is of maximal rank because the blocks are all of maximal rank. Now, if we augment this matrix (see Fig. 7) by adding for each $k \in \mathscr{K}$ a row whose only nonzero entry is the integer $p(k)$ in the column belonging to the group element k, we have a matrix which is $(D - 1) \times (D - 1)$ and of rank $D - 1$. If we choose the $p(k)$ to be the order of k,

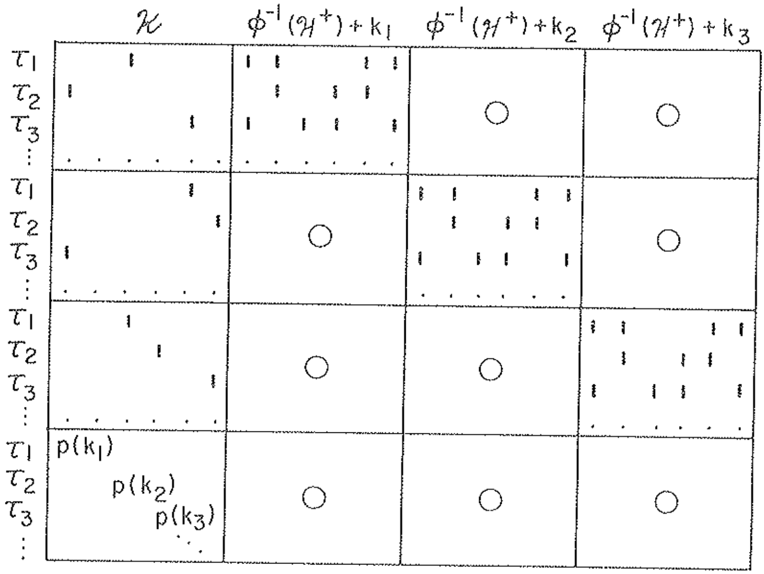


FIG. 7.

so that $p(k) \cdot k = \bar{0}$, we need only add any row from the upper part of the matrix to each row of the newly added section to have a matrix of $D - 1$ independent minimal paths.

This establishes the theorem.

A simple example of this type of face is given by $P(\mathscr{G}_6, (5))$. If we take $\mathscr{K}$ in g_6 to be the even elements, we get a mapping onto $\mathscr{G}_2$, with the face of $P(\mathscr{G}_2, (1))$ given by $\pi'(\bar{0}) = 0$, $\pi'(h_1) = 1$, $\pi_0 = 1$. Carrying this back, we get the face

$$t_1 + 0t_2 + 1t_3 + 0t_4 + 1t_5 \geqslant 1$$

of $P(\mathscr{G}_6, (5))$. Similarly mapping onto $\mathscr{G}_3$ with 0 and $3g_1$ as the kernel, we carry back the face $(\pi_1, \pi_2; \pi_0) = (1, 2; 2)$ into the face

$$t_1 + 2t_2 + 0t_3 + 1t_4 + 2t_5 \geqslant 2$$

of $P(\mathscr{G}_6, (5))$. Many examples of faces of this kind can be seen in the Appendix; in fact, as we shall see shortly, any nontrivial face containing a zero component can be obtained this way.

The groups $\mathscr{G}_{2,2}$ and $\mathscr{G}_{2,2,2}$ in Appendix 5 are also particularly good examples of this construction as *all* their faces are obtained by mapping onto $\mathscr{G}_2$ and by using its one nontrivial face $(\pi; \pi_0) = (1, 1)$ over and over again in different mappings to produce different faces of $P(\mathscr{G}_{2,2}, g_0)$, etc.

It should be borne in mind, however, that the faces corresponding to a fixed $\mathscr{H}$ and π' and different mappings ψ are not always distinct.

Finally, when dealing with groups $\mathscr{G}$ that are direct sums of more than one cyclic group, this theorem allows us to use our knowledge about the cyclic components. For, if $\mathscr{G} = \mathscr{H}_1 \oplus \mathscr{H}_2$, g_0 cannot be both in $\mathscr{H}_1$ and in $\mathscr{H}_2$; so, let us say $g_0 \notin \mathscr{H}_1$. Then $\mathscr{H}_2$ can serve as the kernel $\mathscr{K}$ and the factor group is $\mathscr{H}_1$. Thus every face of $\mathscr{H}_1$ extends to a face of $\mathscr{G}$, etc.

Theorem 19 has a converse which shows that every (nontrivial) face of $P(\mathscr{G}, g_0)$ having a zero component is a face of this type.

THEOREM 20. *Let (π_1, π_0) be a nontrivial face of $P(\mathscr{G}, g_0)$. If $\pi(g) = 0$ for some $g \neq \bar{0}$, then there is a group $\mathscr{H}$, homomorphism ψ, and face (π', π_0) of $P(\mathscr{H}, h_0)$ such that $\mathscr{H} = \psi\mathscr{G}$, and $\pi(g) = \pi'(\psi g)$.*

Proof. Let us suppose that $\pi(g) = 0$. If $\pi(g)$ is zero for two elements g and g' (not necessarily distinct), then

$$0 = \pi(g) + \pi(g') \geqslant \pi(g + g');$$

therefore $\pi(g + g') = 0$; thus the elements for which $\pi(g) = 0$ form a group. If we call this group $\mathscr{K}$ and use it to split $\mathscr{G}$ into cosets, we find, for any two elements g and g' belonging to the same coset, that, since $g' = g + k$, $k \in \mathscr{K}$, then $\pi(g') \leqslant \pi(g) + \pi(k) = \pi(g)$. Since we can also have $\pi(g) \geqslant \pi(g')$, we conclude that elements in the same coset have the same $\pi(g)$. If we now take the homomorphism ψ mapping g onto $\mathscr{G}/\mathscr{K}$, we can assign unambiguously the values $\pi'(h) = \pi(g)$, where $\psi g = h$. If we let $\mathscr{G}/\mathscr{K}$ be the group $\mathscr{H}$, we have provided the $\mathscr{H}$, ψ, and π' of the theorem. What remains is to show that (π', π_0) is a face of $P(\mathscr{H}, \psi g_0)$.

It is easily seen that the minimal path length from $\bar{0}$ to h_0 in $H(\mathscr{H}, h_0)$ is $\pi_1 \geqslant \pi_0$. For, if there were a path in $H(\mathscr{H}, h_0)$ of cost $\pi_1 < \pi_0$, this path would give rise, by the same method used in the proof of Theorem 19, to paths in $\mathscr{H}(\mathscr{G}, g_0)$ of the same length; so π_1 is $\geqslant \pi_0$. However, there are $|H| - 1$ independent paths actually attaining the value π_0. For, if we take the mapping of $(D-1)$-dimensional t-space to $(|H|-1)$-dimensional τ-space induced by ψ (i.e., the vector $t(g)$ goes into $\tau(h) = \sum_{g \in \psi^{-1}(h)} t(g)$, $h \neq 0$), this mapping is clearly linear, and onto, and hence sends the $(D-1)$ independent minimal paths which form a basis for t-space onto a basis for τ-space from which $|H| - 1$ independent paths can be selected. These paths go from $\bar{0}$ to h_0 and have cost π_0; so they are minimal and (π', π_0) is a face of $P(\mathscr{H}, h_0)$.

We turn next to a method of face construction that is applicable when $g_0 \in \mathscr{K}$. Again it is particularly useful in allowing us to carry faces of cyclic groups into faces of more complex ones.

In stating the next theorem we refer to special faces of a polyhedron $P(\mathscr{H}, (0))$ where $\mathscr{H}$ is cyclic. By a special face we mean either the face

$$\pi(ph) = \frac{p}{|H|}, \qquad \pi_0 = 1,$$

or one of the faces obtained from it by an automorphism.

We can now assert the following theorem:

THEOREM 21. *Let ψ be a homomorphism sending $\mathscr{G}$ onto $\mathscr{H}$, a cyclic group of order > 2 with $g_0 \in \mathscr{K}$, the kernel. Then the inequality $(\pi, 1)$ defined by*

$$\pi(g) = \pi_1(g), \qquad g \in K,$$

$$\pi(g) = \pi_2(\psi g), \qquad g \notin K$$

is a face of $P(\mathscr{G}, g_0)$. Here $(\pi_1, 1)$ is any face of $P(\mathscr{K}, g_0)$ and $(\pi_2, 1)$ is a special face of $P(\mathscr{H}, (0))$.

For example, if we take the group $\mathscr{G}_{3,3}$ as consisting of pairs of integers (n_1, n_2) with addition modulo 3, and let $g_0 = (0, 2)$, then the mapping ψ: $(n_1, n_2) \to (n_1, 0)$ is an appropriate ψ, and the kernel $\mathscr{K}$ is the subgroup of elements of the form $(0, n_2)$. Since $(\frac{1}{2}, 1; 1)$ is a face of $P(\mathscr{G}_3, (2))$ and $(\frac{1}{3}, \frac{2}{3}; 1)$ is a special face of $P(\mathscr{G}_3, (0))$, the theorem asserts that the values $\pi((n_1, n_2))$ given by

$n_2 = 2$	1	$\frac{1}{3}$	$\frac{2}{3}$
$n_2 = 1$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$
$n_2 = 0$	0	$\frac{1}{3}$	$\frac{2}{3}$
	$n_1 = 0$	$n_1 = 1$	$n_1 = 2$

form a face of $P(\mathscr{G}_{3,3}, (0, 2))$.

Proof. We prove the theorem by showing that π is a basic feasible solution to the system of equations and inequalities given in Theorem 18.

We start by showing that π is a feasible solution. Clearly $\pi(g) \geqslant 0$. To show $\pi(g) + \pi(g_0 - g) = \pi(g_0)$, we have two cases:

1. $g \in \mathscr{K}$. Then $(g_0 - g) \in \mathscr{K}$; so $\pi = \pi_1$ and therefore the condition is satisfied.

2. $g \notin \mathscr{K}$. Then $(g_0 - g) \notin \mathscr{K}$; so $\pi = \pi_2$ and therefore the condition is satisfied.

Next we must show $\pi(g) + \pi(g') \geqslant \pi(g + g')$. Again, if g and g' are in $\mathscr{K}$, it follows that $g + g'$ is too, so this case is taken care of by $\pi = \pi_1$.

If g, g', and $g + g' \notin \mathscr{K}$, the argument is the same.

If g, $g' \notin \mathscr{K}$ but $(g + g') \in \mathscr{K}$, we have

$$\pi(g) + \pi(g') = \pi_2(\psi g) + \pi_2(\psi g').$$

Since $\psi(g + g') = h_0$,

$$\pi_2(\psi g) + \pi_2(\psi g') = 1 \geqslant \pi(g + g').$$

If $g \in \mathscr{K}$ but $g' \notin \mathscr{K}$, we have

$$\pi(g) + \pi(g') = \pi_1(g) + \pi_2(\psi g').$$

$g + g'$ must $\notin \mathscr{K}$, so

$$\pi(g + g') = \pi_2(\psi(g + g')) = \pi_2(\psi g');$$

therefore

$$\pi(g) + \pi(g') \geqslant \pi(g + g').$$

This covers all cases.

Now, to show that π is basic, we must produce a submatrix of rank $|\mathscr{G}| - 1$, including all the equations. $\pi(g) + \pi(g_0 - g) = 1$ and with all inequalities satisfied as equalities. We split the matrix M into two parts: the columns corresponding to elements in $\mathscr{H}$ and the remaining ones:

$$\left(\begin{array}{c|c} M_1 & 0 \\ \hline 0 & M_2 \end{array}\right).$$

M_1 consists of rows for the relations $\pi(g) + \pi(g_0 - g) = 1$ for the $g \in \mathscr{H}$ and for the set of relations satisfied by π_1 to form a basis for the system (13) applied to $P(\mathscr{H}, g_0)$:

$$M_1 = \begin{pmatrix} 1 & & & & & & & 1 \\ 1 & & & & & 1 & & \\ & & & 2 & & & & \\ 1 & & 1 & & \cdots & 1 & & \\ & 1 & 1 & & & & \cdots & 1 \end{pmatrix}.$$

M_1 has $|\mathscr{H}| - 1$ rows and is of rank $|\mathscr{H}| - 1$.

What remains is to create an M_2 of the proper form and rank. We do this separately for the cases $|\mathscr{H}|$ odd and $|\mathscr{H}|$ even. In what follows we refer to a generator of $\mathscr{H}$ as h_1 and to the zero element in $\mathscr{H}$ as h_0. We recall that the special faces of $P(\mathscr{H}, (0))$ are produced by automorphism from the face $(\pi', 1)$ with $\pi'(sh_1) = s/|\mathscr{H}|$. Thus we can assume that $\pi_2(sh_1) = \pi'(\varphi^{-1}(sh_1))$ when φ is some automorphism of $\mathscr{H}$.

Case 1: $|\mathscr{H}|$ odd. First we partition M_2 into two parts. The first one consists of the columns belonging to elements g for which $\phi^{-1}\psi g = h_s$ with $0 < s < |\mathscr{H}|/2$. The second part contains the columns whose corresponding g is such that $\phi^{-1}\psi g = h_s$, $s > |\mathscr{H}|/2$.

Since $g_0 \in \mathscr{H}$, $\psi(g_0 - g) + \psi(g) = h_0$; so $\phi^{-1}\psi(g_0 - g) + \phi^{-1}\psi(g) = h_0$. Hence, if g belongs to one part of the partition, $\bar{g} = g_0 - g$ belongs to the other.

We proceed to fill out M_2 by putting in the rows corresponding to the $\frac{1}{2}(|\mathscr{G}| - |\mathscr{H}|)$ equations:

$$\pi(g) + \pi(\bar{g}) = 1.$$

The corresponding row in M_2 has a 1 in the g column, a 1 in the $\bar{g}$ column, and is zero elsewhere. (We are not describing the constant terms as part of M_2.) We have already shown that these equations are satisfied by π.

Next we put in rows corresponding to the $\frac{1}{2}(|\mathscr{G}| - |\mathscr{H}|)$ equations:

$$\pi(g) + \pi(\bar{g} - g) - \pi(\bar{g}) = 0,$$

for all g with $\phi^{-1}\psi g = h_s$, $s < |\mathscr{H}|/2$.

We do not, as yet, know that our π satisfies anything except the condition $\pi(g) + \pi(\bar{g} - g) - \pi(\bar{g}) \geqslant 0$. It will be necessary to show equality if our M is to be accepted as a basis.

The appearance of M_2 at this point is roughly:

$$\begin{array}{c}
\text{Part 1} \qquad\qquad\qquad\qquad\qquad\qquad \text{Part 2}\\[1ex]
\left(\begin{array}{cccccccc|}
1 & & & & & & & 1\\
 & 1 & & & & & 1 & \\
 & & 1 & & & 1 & & \\
 & & & \ddots & \cdot^{\cdot^{\cdot}} & & & \\
 & & & & \vdots & & & \\
1 & & 1 & & & -1 & & \\
 & -1 & 1 & & & 1 & & \\
 & 2 & & & & & -1 &
\end{array}\right.
\begin{array}{l}
\\ \frac{1}{2}(|\mathscr{G}| - |\mathscr{H}|) \\ \\ \\ \\ \frac{1}{2}(|\mathscr{G}| - |\mathscr{H}|). \\ \\
\end{array}
\end{array}$$

In the bottom half, most rows have three entries: two 1's and a -1, with one 1 and one -1 under a pair of 1's belonging to a row in the upper half. (The row representing $\pi(g) + \pi(\bar{g}) = 1$ for that g.) Occasionally g and $\bar{g} - g$ can coincide to produce a row with a 2 and a -1. Note that we are implicitly using the odd parity of $|\mathscr{H}|$ here, for, since $g + (\bar{g} - g) - \bar{g} = 0$,

$$\psi g + \psi(\bar{g} - g) - \psi\bar{g} = \bar{0}.$$

If $\psi(\bar{g} - g) = 0$, $\psi g = \psi\bar{g}$; but $\psi\bar{g} = -\psi g$, so $2(\psi g) = \bar{0}$. This is impossible, for 2 does not divide $|\mathscr{H}|$. Therefore $\psi(\bar{g} - g) \neq \bar{0}$; and thus the element

$(\bar{g} - g)$ is not in $\mathscr{H}$ and does appear to be associated with a column of M_2, not of M_1.

Next we show that π satisfies the relations represented by the lower half of M_2, as equalities. For a particular g, with $\phi^{-1}\psi g = h_s$, $s < |\mathscr{H}|/2$, we have

$$\begin{aligned}\pi(g) + \pi(\bar{g} - g) &= \pi_2(\psi g) + \pi_2(\psi(\bar{g} - g)) \\ &= \pi'(\phi^{-1}\psi g) + \pi'(\phi^{-1}\psi(\bar{g} - g) \\ &= \pi'(h_s) + \pi'(h_r) \\ &= s + r,\end{aligned}$$

with $s < |\mathscr{H}|/2$. Also

$$\pi(\bar{g}) = \pi'(\phi^{-1}\psi g) = \pi'(h_{s'}) = (s'/|\mathscr{H}|) > \tfrac{1}{2}.$$

Since

$$h_s + h_r = h_{s'},$$

$$r + s \equiv s', \qquad \bmod(|H|).$$

But, since $s < |\mathscr{H}|/2$ and $s' > |\mathscr{H}|/2$ and $0 < r < |\mathscr{H}|$, the only possibility is $s + r = s'$, which establishes the desired equality.

What remains now is to show that M_2 is nonsingular. We proceed to modify M_2 by row operations. We first modify the row with entries at g, $\bar{g} - g$, and $\bar{g}$ by adding to it the row (from the top half) with entries at g and $\bar{g}$. This, of course, does not affect the rank of the matrix, but M now has the form

$$\left[\begin{array}{cccc:cccc}
1 & & & & & & & 1 \\
 & 1 & & & & & 1 & \\
 & & \ddots & 1 & 1 & \ddots & & \\
\hdashline
a & & 1 & & 0 & & & \\
 & a & & & & 0 & & \\
 & & & & & & 0 & 1 \\
 & & & a & 1 & & & 0
\end{array}\right],$$

where the a's are 2's except when g and $\bar{g} - g$ have coincided; so the 2 is replaced by 3. Now, in the bottom half, if there is a 1 in the second partition, we add -1 times the row having a $+1$ in that column in the upper half, and we obtain all zeros in the lower right partition:

$$\left(\begin{array}{ccc|ccc}
1 & & & & & 1\\
 & 1 & & & 1 & \\
 & & 1 & 1 & & \\ \hline
b & 1 & & & & \\
 & b & -1 & & 0 & \\
 & & b & & &
\end{array}\right).$$

Now b is at least 2 if the (one) other row entry is 1 or -1 and b is at least 1 (actually 3) if it is the only entry in its row. This lower left matrix is nonsingular, by the diagonal dominance criterion,* so the entire matrix is. Therefore π is a basic solution, for the rank of M_2 is $|\mathscr{G}| - |\mathscr{H}|$ and that of M_1 is $|\mathscr{H}| - 1$. So M has rank $|\mathscr{G}| - 1$.

Thus π is feasible, and it satisfies enough of the inequalities of Theorem 18 to form a basic solution. Therefore π is a basic feasible solution and, hence, a face.

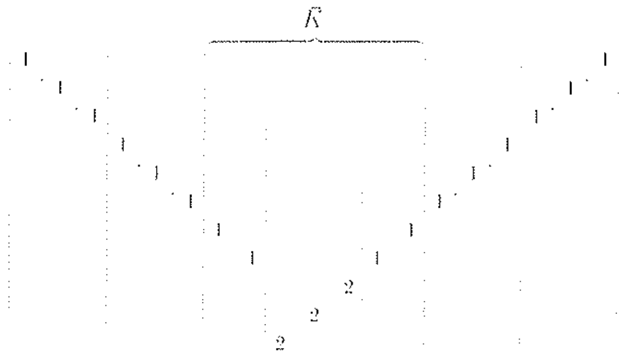
This proves the theorem for $|\mathscr{H}|$ odd. We now take up the case $|\mathscr{H}|$ even and > 2.

Case 2: $|\mathscr{H}|$ even. The matrix M_1 is constructed as before; the construction of M_2 is slightly more complicated. We partition the M_2 matrix into sets of columns corresponding to the cosets of $\mathscr{H}$. M_1 is the part corresponding to $\mathscr{H}$ itself. In the upper portion of M_2 we write the relations

$$\pi(g) + \pi(g_0 - g) = \pi(g_0).$$

Since H is cyclic of order divisible by 2, $2g = g_0$ is now possible for some of the $g \in \psi^{-1}(|\mathscr{H}|/2)h_1$; so the upper portion of M_2 has the form

* A matrix is nonsingular if the absolute value of the diagonal element exceeds the sum of the absolute values of the other row elements.



where the columns belonging to the coset $\bar{K}$ corresponding to $|\mathscr{H}/2|h_1$ are divided into sections $\bar{K}_1$ and $\bar{K}_3$ containing g's with distinct complements $\bar{g}$, and a middle section containing the β elements g for which $2g = g_0$. As before, columns to the left of the center coset $\bar{\mathscr{H}}$ contain elements with $\phi^{-1}\psi g = h_s$, $s < (|\mathscr{H}|/2)$. $(\phi^{-1}h_1|\mathscr{H}|/2 = h_1|\mathscr{H}|/2)$. We obtain in this way $\frac{1}{2}(|\mathscr{H}|(|\mathscr{H}| - 2) + |\mathscr{H}| + \beta)$ rows. In the cosets, other than the middle one, $\bar{K}$, we write the same relations as before, i.e., for each g corresponding to a column to the left of the middle coset, we write the row corresponding to

$$\pi(g) + \pi(\bar{g} - g) - \pi(\bar{g}),$$

and, by the same reasoning as before, the relation

$$\pi(g) + \pi(\bar{g} - g) - \pi(\bar{g}) = 0$$

is fulfilled by our π. However, the reasoning fails and the relation is also false for g in the middle coset; for, whenever $g \in \bar{K}$, so does $\bar{g}$. Therefore $\psi g = \psi\bar{g}$ and $\pi(g) = \pi_2(\psi g) = \pi_2(\psi\bar{g}) = \pi(\bar{g})$. But, since $(g - \bar{g})$ is in $\mathscr{H}$, $\pi(g) = \pi_1(\bar{g} - g)$, which certainly can be $\neq 0$.

At this point we have constructed, then, an additional $\frac{1}{2}|\mathscr{H}|(|\mathscr{H}| - 2)$ rows. $\frac{1}{2}(|\mathscr{H}| - |\beta|)$ rows are needed to make M_2 square. At this point M_2 has the appearance shown in Fig. 8.

By row operations (adding multiples of rows of the upper block U to rows of the lower block L) we can change the matrix, as before, into one having entries $\geqslant 2$ on the lower left diagonal and only zeros in the lower right block. We can also eliminate entries in the lower part of the columns lying in $\bar{K}_2$ and $\bar{K}_3$.

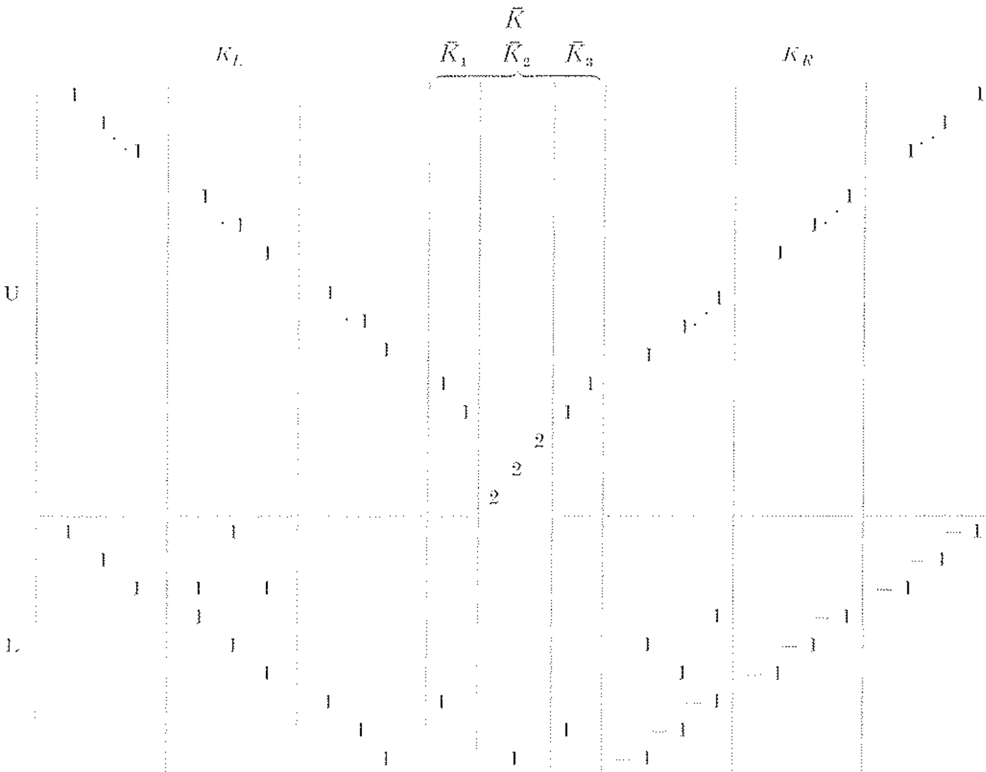


FIG. 8.

We now adjoin the needed additional $\frac{1}{2}(|K| - \beta)$ relations to make M_2 square. We add to M_2: $\frac{1}{2}(|K| - \beta)$ rows; their intersection with $\bar{K}_1$ is a square matrix; and we center -1's on the main diagonal of this square. Then in the row of each -1 we enter two $+1$'s in the left section of the matrix.

The choice of the two $+1$'s, that is, the choice of the actual relations to be added, requires some discussion. The appearance of the matrix at this point is as given in Fig. 9.

Returning to the choice of relations, we note that it is possible to choose the columns originally entered in $\bar{K}_1$ and $\bar{K}_3$ with a certain degree of freedom, a column $c(g)$ can go into either $\bar{K}_1$ or $\bar{K}_3$; it is only necessary that its complementary column $c(\bar{g})$ go into the other partition. We can exploit this freedom to make sure that (a) each $c(g) \in \bar{K}_1$ has at least as many $+1$'s in its L-rows (the only possible entries are 0 and $+1$) as

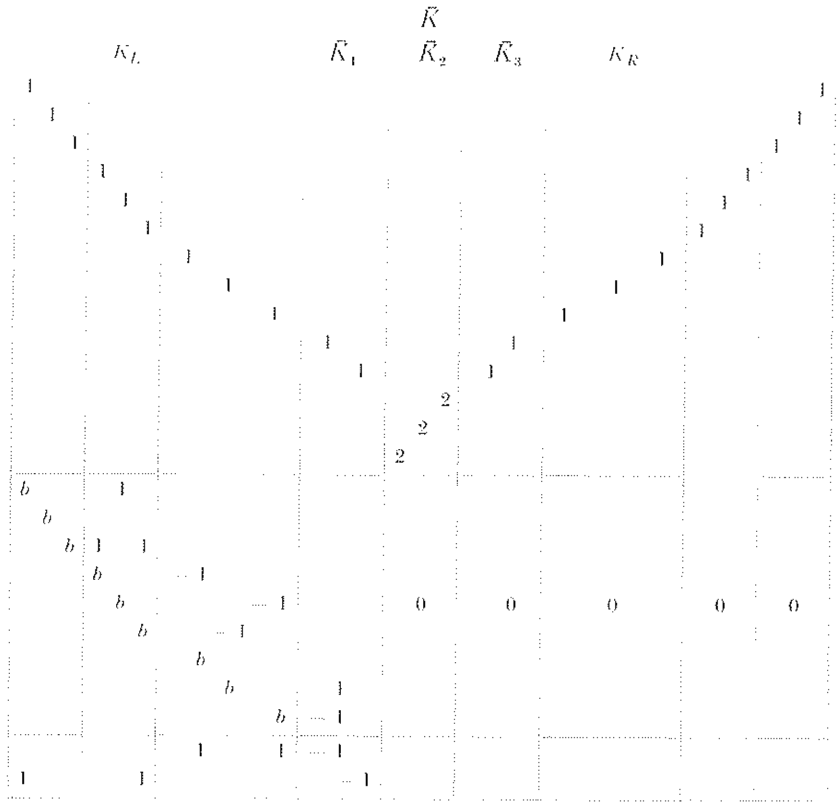


FIG. 9.

does the corresponding $c(\bar{g})$; (b) the $c(g)$ having no $+1$'s in their L-rows are the right-most columns in $\bar{K}_1$. Thus $\bar{K}_1$ splits into two parts, $\bar{K}_1'$ and $\bar{K}_1''$, with the 0-columns of L in $\bar{K}_1''$.

After the matrix has been put into the form shown in Fig. 9, this choice has the following consequences for the part of $\bar{K}_1$ lying in the L-rows: (a) implies that, if there are nonzero entries, there are at least as many $+1$'s as -1's in each column; (b) means there is a block of 0's above the last -1's added below L. The situation is illustrated in Fig. 10, which also explains the obvious notation A' and A''.

We now choose the new relations as follows: If $c(g) \in \bar{K}_1'$ and so has nonzero entries in L, then it contains a $+1$ in L. Say this is in row $s(g)$. The only other nonzero entry in $s(g)$ is b. In the new row $r(g)$, the one containing a -1 in $c(g)$, enter a $+1$ under the diagonal b entry of

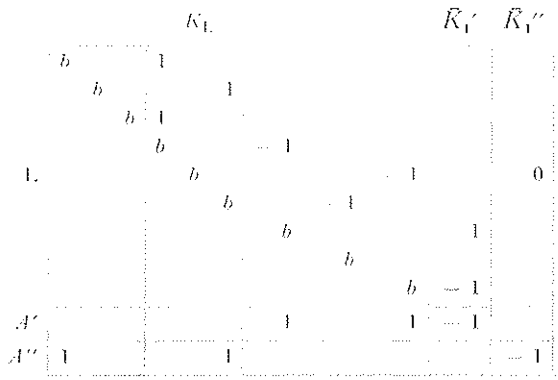


FIG. 10.

row s. Say this $+1$ is in column $c(g_1)$. Enter a second $+1$ in row $r(g)$, column $c(g_2)$, with $g_1 + g_2 = g$. If $c(g_2)$ coincides with $c(g_1)$, enter a $+2$ in row $r(g)$. Since $g_2 = g - g_1$, g_2 certainly lies in the left side of the matrix and, since g_1 and g are in different cosets, $g_2 \notin \mathscr{H}$.

$r(g)$ now represents a new relation.

In the above, we have used the fact that $|\mathscr{H}| > 2$ to give us at least one coset $\neq \mathscr{H}$ to the left of $\bar{K}$.

If the column $c(g) \in \bar{K}_1''$ and so has no nonzero entries, then we choose any g_1 and g_2 from the left-hand cosets and such that $g_1 + g_2 = g$. This always can be done.

We must next show that those new rows are satisfied as equalities, i.e., that $\pi(g_1) + \pi(g_2) - \pi(g) = 0$. But $\pi(g_1) = \pi_2(\psi g_1) = \pi'(\phi^{-1}\psi g_1) = s_1/|\mathscr{H}| < \frac{1}{2}$. Similarly $\pi(g_2) = \pi_2(\psi g_1) = \pi'(\phi^{-1}\psi g_2) = s_2/|\mathscr{H}| < \frac{1}{2}$ and $\pi(g) = \pi_2(\psi g) = \pi'(\phi^{-1}\psi g) = s/|\mathscr{H}| = \frac{1}{2}$. Since $h_{s_1} + h_{s_2} = hs$, $s_1 + s_2 \equiv s \bmod(|\mathscr{H}|)$, which with the inequalities on s_1 and s_2 implies $s_1 + s_2 = s$. This establishes the desired equality.

It remains now to show that the square matrix consisting of $(K_L \cup \bar{K}_1') \cap (L \cup A')$, Fig. 10, is nonsingular. From this the nonsingularity of $(K_L \cup \bar{K}_1) \cap (L \cup A' \cup A'')$ follows and gives the nonsingularity of M_2.

If $g \in \bar{K}_1'$, add a (negative) multiple of the row $s(g)$ to $r(g)$ to make the $+1$ in column $c(g_1)$ vanish. The new $r(g)$ now contains only the other $+1$ and the term in column $c(g)$ itself, which is now strictly < -1 because a negative multiple of $+1$ has been added to it. Doing this for all $g \in \bar{K}_1'$ produces diagonal dominance in these rows and hence in all rows of $(K_L \cup \bar{K}_1') \cap (L \cup A)$ which is, therefore, nonsingular.

This proves the nonsingularity of M_2 and so ends the proof.

E. Characters and Inequalities

The methods described so far have enabled us to produce some faces of $P(\mathscr{G}, g_0)$ rapidly and without making use of the general method of Theorem 18. If we used these methods to provide inequalities for a cutting plane method, we could:

(i) for a given problem calculate by standard methods the factor group $M(I)/M(B) = \mathscr{G}$ for our particular current basis B and find out which group elements correspond to the various nonbasic columns and to the right-hand side;

(ii) knowing what group $\mathscr{G}$ is involved, what g_0 is, and what group elements are in the set $\mathscr{N}$, produce inequalities by (a) creating some faces for $P(\mathscr{G}, g_0)$, (b) deleting the variables corresponding to group elements not in $\mathscr{N}$.

We note that this procedure involves computing $M(I)/M(B)$ to find out what $\mathscr{G}$ is involved. This is in contrast to [4], where the inequalities of the cutting plane method described were obtained without ever examining the group and the fractional parts of certain matrix rows were used directly as inequalities. Remembering this, it is reasonable to ask if it is now possible to produce other inequalities without computing the group.

In fact, it is possible to produce whole new families of inequalities without computing $\mathscr{G}$. To see this we first discuss certain properties of the fractional parts of the matrix. We connect this with group characters, and then with inequalities.

By a group character is often meant a mapping ψ from the group $\mathscr{G}$ to the unit circle in the complex plane such that, if $\psi(g_1) = \zeta_1 = \exp(i\theta_1)$, $\psi(g_2) = \zeta_2 = \exp(i\theta_2)$, then $\psi(g_1 + g_2) = \zeta_3 = \exp(i(\theta_1 + \theta_2)) = \zeta_1 \cdot \zeta_2$. It is easily verified that $\psi(\bar{0})$ must be 1 and, since for any group element $|\mathscr{G}| \cdot g = \bar{0}$, we have, if $\zeta = \psi(g)$, that $\zeta^{|\mathscr{G}|} = \psi(|\mathscr{G}|g) = \psi(\bar{0}) = 1$; so the only ζ ever used are the $|\mathscr{G}|$th roots of unity.

Now the multiplicative group of the $|\mathscr{G}|$th roots of unity, the additive group of the fractions n/D modulo 1 (n integer), and the integers n modulo $|\mathscr{G}|$ are all three isomorphic groups (cyclic of order $|\mathscr{G}|$), so it is possible to define characters as mappings $\psi(g_1 + g_2) = \psi(g_1) + \psi(g_2)$, where the values $\psi(g)$, instead of being on the unit circle, are in one of the last two groups. The usual theorems still hold, i.e., the characters on $\mathscr{G}$ form a group with $(\psi_1 + \psi_2)(g)$ defined as $\psi_1(g) + \psi_2(g)$, and this group of all characters on $\mathscr{G}$ is isomorphic to $\mathscr{G}$.

We shall see that actual numerical characters are readily available to us from the transformed matrix A, or even from a knowledge of B^{-1}. It is convenient in discussing this to use a diagram in which various relevant mappings occur:

$$A = (B, N) \xrightarrow{B^{-1}} (I, B^{-1}N)$$

$$\begin{array}{ccccc} & M(B, N) & \xrightarrow{B^{-1}} & M(I, B^{-1}N) & \\ & \downarrow k_1 & & \downarrow k_2 & \\ \mathscr{G} = & \dfrac{M(B,N)}{M(B)} & \xrightarrow{b} & \dfrac{M(I, B^{-1}N)}{M(I)} & . \end{array}$$

B^{-1} is the matrix sending A into the transformed matrix $(I, B^{-1}N)$ which is used in linear programming. We use B^{-1} again to indicate the isomorphic mapping of the module $M(B, N)$ onto the module $M(I, B^{-1}N)$ which is induced by B^{-1}. k_1 sends $M(B, N) = M(I)$ onto the factor group $M(I)/M(B) = \mathscr{G}$. k_2 is the mapping of $M(I, B^{-1}N)$ onto $M(I, B^{-1}N)/M(I)$.

B^{-1} sends $M(B)$ onto $M(I)$, so the factor groups $M(B, M)/M(B)$ and $M(I, B^{-1}N)/M(I)$ are isomorphic. This isomorphism, induced by B^{-1}, we denote by b. The mappings k_1 and k_2 send the modules into their factor groups. We have assumed throughout that $M(B, N)$ contains a unit matrix; so $M(B, N) = M(I)$ and the factor group is $\mathscr{G}$.

The diagram above shows that there is an isomorphic correspondence b between $\mathscr{G}$ and the group generated by the columns of $B^{-1}N$ modulo $M(I)$, i.e., all components are being treated as elements modulo 1. The group element $g \in \mathscr{G}$ corresponding to a column in $M(I, B^{-1}N)$ is determined by the fractional parts of its components alone. In particular, this is true of the columns of $B^{-1}N$ itself.

We turn next to the group characters.

We define the mapping $\mathscr{F}(m/D)$, where m is any integer and $D = |\det B| = |\mathscr{G}|$, to be that group element in the cyclic group $\mathscr{G}_D$, $0, 1/D, \ldots, p/D, \ldots, (D-1)/D$, for which the numerical value $p/D \equiv m/D$ modulo 1. Thus, for example, if r_i is the ith row of B^{-1}, and c is any integer column, $\mathscr{F}(r_i \cdot c)$ is a mapping of integer columns c into $\mathscr{G}_D$, since the elements of B^{-1} and hence of r_i and $r_i \cdot c$ are all of the form m/D.

Now, for any row r_i, we define the function $\psi_i(g)$, $g \in \mathscr{G}$, by

$$\psi_i(g) = \mathscr{F}(r_i \cdot c),$$

where $k_1 c = g$. To show that this is in fact a function of g and independent of the choice of c, we note that, if $k_1 c = g = k_1 c'$, then $k_1(c - c') = \bar{0} \in \mathscr{G}$, so that $k_2 B^{-1}(c - c') = \bar{0} \in M(I, B^{-1}N)/M(I)$; therefore $B^{-1}(c - c')$ must be an all-integer vector and so, in particular, $r_i(c - c') \equiv 0 \bmod 1$. Hence $\mathscr{F}(r_i \cdot c) = \mathscr{F}(r_i \cdot c')$ for any c' with $k_1 c' = g$. It is easily verified that $\psi_i(g_1 + g_2) = \psi_i(g_1) + \psi_i(g_2)$; thus ψ_i is in fact a group character.

We have shown that ψ_i is a character. It is a routine matter to verify that the mappings $\psi = \sum_{i=1}^{i=m} n_i \psi_i$, n_i integer, are also characters and that the entire character group is obtained this way.

In fact, although we do not need this at the moment, it is easy to show that the columns of B^{-1}, taken modulo 1, generate $\mathscr{G}$, while the rows of B^{-1}, taken modulo 1 and used to define the mappings ψ, generate the (isomorphic) character group.

In dealing with a linear programming problem we need only pick the ith row of $B^{-1}(B, N) = (I, B^{-1}N)$ and take the fractional parts of the entries to obtain the ψ_i character values for those group elements that correspond to the various columns.

Now we connect inequalities and characters.

Let $\mathscr{G}$ be a group of order D, and $\mathscr{H}$ be cyclic of the same order, D. Let (π, π_0) be a face of $P(\mathscr{H}, h_0)$. Define $\pi_\psi(g)$ for a fixed character ψ of $\mathscr{G}$ by

$$\pi_\psi(g) = \pi(\psi(g)).$$

Note that here our character is interpreted as a mapping into $\mathscr{H}$, a cyclic group isomorphic to the group of fractions m/D.

If $t(g)$ gives a path in $H(\mathscr{G}, \pi_\psi)$ from $\bar{0}$ to g_0, i.e.,

$$\sum_{g \in \mathscr{G}^+} g t(g) = g_0,$$

then $\sum_{g \in \mathscr{G}^+} \psi(g) t(g) = \psi(g_0)$ since ψ is a character; so, as usual, the mapping from the path in $\mathscr{G}$ produces a path in $\mathscr{H}$. The cost of the path, either in $H(\mathscr{G}, \pi_\psi)$ or in $H(\mathscr{H}, \pi)$, is the same and is given by

$$\sum_{g \in \mathscr{G}^+} \pi(\psi(g)) t(g).$$

Since π is a face of $P(\mathscr{H}, h_0)$, the π components satisfy the inequality $\pi(h_1) + \pi(h_2) \geqslant \pi(h_1 + h_2)$; so

$$\sum_{g \in \mathscr{G}^-} \pi(\psi(g))t(g) \geqslant \pi(\psi(g_0)),$$

$$\sum_{g \in \mathscr{G}^-} \pi_\psi(g)t(g) \geqslant \pi_\psi(g_0).^*$$

This, then, is an inequality that must be satisfied by any $t(g)$ in $P(\mathscr{G}, g_0)$. Hence it can be used as a cutting plane. Note that we do *not* assume $\psi(g_0) = h_0$.

By varying the character ψ, $D - 1$ inequalities are produced from each face of $P(\mathscr{H}, h_0)$. These inequalities are generally not faces, although that can happen.

In particular, let us consider the face of $P(\mathscr{H}, (D-1))$ obtained by using a special face with the components $\pi(sh_1) = s/D$ and with $\pi_0 = (D-1)/D$. Then the family of inequalities obtained is exactly the family of "fractional cutting planes" of [4].

We illustrate the use of characters to produce inequalities by a numerical linear programming example in Appendix 4.

One relationship among the various inequalities produced is worth noting:

THEOREM 22. *Let T' be any $(D-1)$-vector with nonnegative (but not necessarily integer) components given by $t'(g)$ for all $g \in \mathscr{G}^-$. Then, if T' satisfies for a fixed ψ and g_0,*

$$\sum_{g \in \mathscr{G}^-} \pi(\psi g)t'(g) \geqslant \pi_0,$$

for all faces (π, π_0) of $P(\mathscr{H}, \psi g_0)$, it also satisfies the inequalities

$$\sum_{g \in \mathscr{G}^-} \pi(\psi g)t'(g) \geqslant \pi(\psi g_0)$$

obtained using the π from all faces (π, π_0) of the polyhedra $P(\mathscr{H}, h_0)$ for all $h_0 \in \mathscr{H}$.

One meaning of this theorem is that, as far as cutting planes are concerned, we can limit ourselves to using character inequalities derived

* If $\psi(g_0) = \bar{0}$, $\pi\psi(g_0) = 0$, and the inequality is trivial.

from faces of $P(\mathscr{H}, \psi g_0)$, if these faces are *all* available. If they are not all available, there is usually something to be gained by using the inequalities derived from other $P(\mathscr{H}, h_0)$.

Proof. The proof of the theorem is quite direct. Since T' satisfies the inequalities listed in the theorem, the vector with components $\tau(h)$,

$$\tau(h) = \sum_{g \in \psi^{-1}h} t'(g),$$

satisfies $\sum \tau(h)\pi(h) \geqslant \pi_0$ for all *faces* (π, π_0) of $P(\mathscr{H}, \psi g_0)$. Consequently $\tau(h)$ is a convex combination of vertices τ^i of $P(\mathscr{H}, \psi g_0)$ and they are, of course, integer vectors representing paths to ψg_0 in the graph $H(\mathscr{H}, \pi)$. Now, using for π any face of $P(\mathscr{H}, h_0)$, we have for any path to ψg_0, and hence for the τ^i,

$$\sum_{h \in \mathscr{H}^*} \tau^i(h) \cdot \pi(h) \geqslant \pi(\psi g_0),$$

and, since τ is a convex combination of the τ^i,

$$\sum_{h \in \mathscr{H}^*} \tau(h) \cdot \pi(h) \geqslant \pi(\psi g_0),$$

which implies

$$\sum_{g \in \mathscr{G}^*} t'(g)\pi(\psi g) \geqslant \pi(\psi g_0).$$

This was the desired result.

F. Some Special Groups

In this section we discuss groups of whose elements are all of order 2 or all of order 3. The only such finite Abelian groups are $\mathscr{G}_2$, $\mathscr{G}_{2,2}$, $\mathscr{G}_{2,2,2}$, etc., and $\mathscr{G}_3$, $\mathscr{G}_{3,3}$, $\mathscr{G}_{3,3,3}$, etc. We shall see that in these groups the notions of irreducible solution and of vertex coincide and that these groups have special properties that enable us to count the vertices of the polyhedra $P(\mathscr{G}, g_0)$ for all these $\mathscr{G}$.

In the theorem below we mention sets of independent group elements. A set of group elements $g_1, \ldots, g_r$ is said to be independent if $\sum_{i=1}^{i=r} s_i g_i = \bar{0}$ implies $s_i g_i = \bar{0}$ all i.

THEOREM 23. *If all elements of $\mathscr{G}$ are of order 2 or all of order 3, then $t(g)$, a solution to the group equation, is irreducible if, and only if, it is a vertex. Furthermore, if $g_0 \neq \bar{0}$, the elements g for which $t(g) > 0$ in such a vertex solution are always a set of independent group elements and, hence, part of a group basis.*

The proof consists mainly of the following lemma:

LEMMA. *For $p = 2$ or $p = 3$, if t and s are integers satisfying*

$$p > t > 0,$$

$$p > s \geqslant 0,$$

then there are integers t' and t'' satisfying

$$t \geqslant t' \geqslant 0,$$

$$t \geqslant t'' \geqslant 0,$$

and

$$t' + s \equiv t'' \pmod{p}.$$

This can be verified by simply looking at all cases. Generally, for a given p, t, and s, there is more than one pair (t', t'') satisfying the conditions. One solution pair is $(t', t'') = (0, s)$ for all cases except $p = 3$, $t = 1$, $s = 2$, in which case $(t', t'') = (1, 0)$.

We turn now to the proof of the theorem. Of course, a vertex is irreducible; so it is necessary only to prove that for these circumstances an irreducible point is a vertex.

Let $t(g)$ be an irreducible solution to the group equation and let T be the set of group elements for which $t(g) > 0$. Suppose there is a nontrivial relation $s(g)$ among the $g \in T$, i.e.,

$$\sum_{g \in T} s(g) \cdot g = \bar{0}.$$

Then, by the lemma, there exist a $t'(g) \leqslant t(g)$ and $t''(g) \leqslant t(g)$ with

$$t'(g) + s(g) \equiv t''(g) \bmod p.$$

Summing gives

$$\sum_{g \in T} t'(g) \cdot g + \sum_{g \in T} s(g) \cdot g = \sum_{g \in T} t''(g) \cdot g$$

so $\sum_{g \in T} t'(g) \cdot g = \sum_{g \in T} t''(g) \cdot g$, which contradicts the irreducibility of $t(g)$. So the $g \in T$ must be an independent set.

Now suppose that $u(g)$ is some other distinct solution to the group equation with $u(g) = 0$, $g \notin T$. Choose $s(g)$ to be

$$s(g) \equiv t(g) - u(g)$$

and

$$0 \leqslant s(g) < p.$$

Then $s(g)$ satisfies

$$\sum_{g \in T} s(g) \cdot g = \bar{0}$$

and so the T are not independent, a contradiction. It follows that $t(g)$ is the only group equation solution with components zero on all $g \notin T$.

But $t(g)$ is then the unique solution minimizing the $\pi(g)$ given by $\pi(g) = 0$, $g \in T$, and $\pi(g) = 1$, $g \notin T$. So $t(g)$ is a vertex.

We next proceed to count the vertices of these special groups.

A vector $t(g)$ will be a vertex for some $P(\mathscr{G}, g_0)$, $g_0 \neq \bar{0}$, if and only if the g for which $t(g) > 0$ are independent, and if, also, $t(g) < s$. Here $s = 2$ for the groups $\mathscr{G}_{2^n}$, etc., and $s = 3$ for the groups $\mathscr{G}_{3^n}$.

Now the number of distinct independent p-element subsets is, for the group with s^n elements:

$$\frac{1}{p!}(s^n - 1) \cdot (s^n - s)(s^n - s^2) \cdots (s^n - s^{p-1}).$$

For each such set there are $(s - 1)^p$ different vertices (of $P(\mathscr{G}_{s^n}, g_0)$ with various g_0) that have $t(g) > 0$ on the set and zero elsewhere. Since every vertex is associated uniquely with some p-element set, $p > 0$, we have, for the number of vertices, $v(s^n)$, of all $P(\mathscr{G}_{s^n}, g_0)$:

$$v(s^n) = \sum_{p=1}^{p=n} \frac{(s-1)^p}{p!} \prod_{q=0}^{q=p-1} (s^n - s^q) = \sum_{p=1}^{p=n} \frac{(s-1)^p s^{p \cdot n}}{p!} \prod_{q=0}^{q=p-1} \left(1 - \frac{1}{s^{n-q}}\right).$$

Separating the term $p = n$ in the sum gives

$$= \frac{(s-1)^n s^{n^2}}{n!} \prod_{q=0}^{q=n-1} \left(1 - \frac{1}{s^{n-q}}\right) + \sum_{p=1}^{p=n-1} \frac{(s-1)^p s^{p \cdot n}}{p!} \prod_{q=0}^{q=p-1} \left(1 - \frac{1}{s^{n-q}}\right);$$

so

$$
\begin{aligned}
&\frac{v(s^n)}{(s-1)^n s^{n^2}/n!} - \prod_{q=0}^{q=n-1}\left(1-\frac{1}{s^{n-q}}\right) \\
&= \sum_{p=1}^{p=n-1} \frac{n!(s-1)^{p-n} s^{pn-n^2}}{p!} \prod_{q=0}^{q=p-1}\left(1-\frac{1}{s^{n-q}}\right) \\
&\leqslant \sum_{p=1}^{p=n-1} \frac{n^{n-p}}{(s-1)^{n-p}\cdot s^{n^2-pn}} \leqslant \sum_{p=1}^{p=n-1}\left(\frac{n}{(s-1)\cdot s^n}\right)^{n-p} \\
&= \left\{\frac{(n/((s-1)s^n) - (n/((s-1)s^n))^n}{1-(n/(s-1)s^n}\right\} = O\left(\frac{n}{s^n}\right) \to 0.
\end{aligned}
$$

Since $\prod_{q=0}^{q=n-1}(1-1/s^{n-q})$ is readily shown to decrease from $(1-1/s)$ toward a limit $K_s > 0$ as $n \to \infty$,

$$v(s^n) \sim K_s \frac{(s-1)^n s^{n^2}}{n!},$$

where $\sim$ means that the ratio of the two sides approaches 1. This formula gives the total number of vertices of all polyhedra $P(\mathscr{G}, g_0)$ for a fixed $|\mathscr{G}| = s^n$ and summed over all $g_0 \neq \bar{0}$. However, for the groups we are discussing, there is always an automorphism from g_0 to $g_0' \neq \bar{0}$. Hence, all $D-1$ of these polyhedra have the same number of vertices; so the number of vertices of $P(\mathscr{G}, g_0)$, when $\mathscr{G} = \mathscr{G}_{s^n}$, is given by

$$v(s^n) \sim K_s \frac{s^{n^2}}{n!}.$$

Since $s^{n^2}/n! = s^{n^2(1+O(\log s^n)/n)}$, this grows very rapidly. In terms of D,

$$v(D) = D^{\log_s D(1+\phi(D))},$$

where $\phi(D) \to 0$ as $D \to \infty$.

The number of vertices far exceeds the number of known faces in the case $s = 2$. To see this, choose any independent basis for the group $\mathscr{G}_{2^n}$, including g_0 as one of the basis elements. Any mapping ϕ of these elements onto the elements of $\mathscr{G}_2$ in which $\phi(g_0) = (1)$ defines a homomorphism of all $\mathscr{G}_{2^n}$ onto $\mathscr{G}_2$ for which Theorem 19 applies and hence results in a face of $P(\mathscr{G}_{2^n}, g_0)$. The faces obtained from the various possible mappings ψ involve only 1 and 0 as coefficients and, as different mappings must differ on some basis element, these faces are distinct. There are 2^{n-1} such mappings and hence 2^{n-1} such faces. In the case $s = 2$, we can infer from the work of Edmonds [1] that these are all the faces for many of the subpolyhedra $P(\mathscr{G}_{2^n}, n, g_0)$. This list of known faces grows at the rate of $D/2$ in contrast with the much more rapid $D^{\log_2 D}$ of the vertices in this one case.

G. Conclusion, Algorithms, etc.

One of the most interesting areas for further work is the investigation of properties of the polyhedra $P(\mathscr{G}, g_0)$, especially such points as the rates of growth of the number of vertices, faces, and degree of degeneracy. The relation between $P(\mathscr{G}, g_0)$ and $P(\mathscr{H}, g_0)$, when $\mathscr{H}$ is a subgroup of $\mathscr{G}$ containing g_0, is certainly not completely covered by the theorems given here. For example, in all cases looked at so far every face of $P(\mathscr{G}, g_0)$ is, suitably restricted, a face of $P(\mathscr{H}, g_0)$.

Certainly the simple faces of Section 3C are not the only ones that can be produced by formulas. A glance at Appendix 5 suggests that there are many more. Better methods for obtaining vertices are important and are needed.

In the area of algorithms there is a great variety of possibilities. One approach would be the cutting plane methods. First the group is calculated as a direct sum of cyclic groups, then, using the simple faces of the cyclic components and Theorems 19 and 20, faces are produced. Or Theorems 19 and 20 can be used to produce more faces even within the cyclic components. Or, if the group is such that some components are tabulated, the tabulated faces can be used. Or, if the entire $P(\mathscr{G}, g_0)$ is tabulated, it should be possible to select the vertices lying in $P(\mathscr{G}, \mathscr{N}^+, g_0)$ for the particular $\mathscr{N}$ we are dealing with and, then using incidence matrices, select only faces of $P(\mathscr{G}, g_0)$ incident to at least $|\mathscr{N}|$ of these vertices.

Aside from methods which require determination of $\mathscr{G}$ as a direct sum of cyclic components, there are methods which make use of the group characters. Hence it is only necessary to know the value of $D = |\mathscr{G}| = |\det B|$. Then faces of the $P(\mathscr{G}_D, g_0)$, $\mathscr{G}_D$ cyclic of order D, can be obtained through the combining of simple faces, tabulation, and Theorems 19 and 20. These are then used with the characters which are already available from the simplex calculations, as explained above, to produce inequalities. There are other possibilities in which the prime decomposition of D, $D = P_1^{a_1}, \ldots, P_m^{a_m}$, plays a role.

Along different lines, quite different methods based on the use of vertices rather than faces would seem extremely desirable. There are also primal methods like that in [14] and all-integer methods like that in [4] to be considered as well as methods for the mixed-integer problem.

In most of these algorithms the numerical problems to be encountered can be expected to require separate study.

The various possibilities are numerous, and too little is known about their relative merit to make an extensive catalog of possibilities worthwhile. What seems clear at this point is that further tabulation of the $P(\mathscr{G}, g_0)$ would be helpful both for algorithms and for a further understanding of the properties of the polyhedra.

APPENDIX 1

Asymptotic Integer Programming: A Numerical Example

We illustrate the asymptotic calculation by a numerical example.

If we solve the linear programming problem

$$\begin{aligned} \max z = 2x_1 + x_2 + x_3 + 3x_4 + x_5, \\ 2x_2 + x_3 + 4x_4 + 2x_5 \leqslant 41, \\ 3x_1 - 4x_2 + 4x_3 + x_4 - x_5 \leqslant 47, \\ x_i \geqslant 0, \qquad i = 1, \ldots, 5, \end{aligned}$$

we find that the basic variables are x_1 and x_2, so that the basis matrix

$$B \text{ is } \begin{pmatrix} 0 & 2 \\ 3 & -4 \end{pmatrix}, \qquad N \text{ is } \begin{pmatrix} 1 & 4 & 2 & 1 & 0 \\ 4 & 1 & -1 & 0 & 1 \end{pmatrix}, \qquad \text{and } b \text{ is } \begin{pmatrix} 41 \\ 47 \end{pmatrix}.$$

We can obtain the group $M(I)/M(B)$ by reducing B to the standard elementary divisor form by row and column operations. These operations are confined to permuting rows and columns and to adding integer multiples of rows to other rows and integer multiples of columns to columns. (See, for example, Van der Waerden [11].) Carrying this reduction out on our B, we obtain successively

$$\begin{pmatrix} 0 & 2 \\ 3 & -4 \end{pmatrix}, \ \begin{pmatrix} 0 & 2 \\ 3 & -1 \end{pmatrix}, \ \begin{pmatrix} 6 & 2 \\ 0 & -1 \end{pmatrix}, \ \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}, \ \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}.$$

The row and column operations can be summarized by unimodular matrices P and Q which will have the property

$$P\begin{pmatrix}0 & 2\\ 3 & -4\end{pmatrix}Q = \begin{pmatrix}1 & 0\\ 0 & 6\end{pmatrix}.$$

Here

$$P \text{ is } \begin{pmatrix}0 & -1\\ 1 & 2\end{pmatrix} \quad\text{and}\quad Q \text{ is } \begin{pmatrix}1 & 4\\ 1 & 3\end{pmatrix}.$$

The row operations correspond to unimodular changes of basis in the space of all column vectors, the column operations to changes in the basis of the lattice generated by the columns of B. This lattice, with respect to the new space basis, now consists of all multiples of $\begin{pmatrix}1\\ 0\end{pmatrix}$ and $\begin{pmatrix}0\\ 6\end{pmatrix}$.

Multiplying N and b by P, we bring all these columns over to the new space basis:

$$PN = \begin{pmatrix}0 & -1\\ 1 & 2\end{pmatrix}\begin{pmatrix}1 & 4 & 2 & 1 & 0\\ 4 & 1 & -1 & 0 & 1\end{pmatrix} = \begin{pmatrix}-4 & -1 & 1 & 0 & -1\\ 9 & 6 & 0 & 1 & 2\end{pmatrix},$$

$$Pb = \begin{pmatrix}0 & -1\\ 1 & 2\end{pmatrix}\begin{pmatrix}41\\ 47\end{pmatrix} = \begin{pmatrix}-47\\ 135\end{pmatrix}.$$

To obtain the corresponding elements of $\mathscr{G}$ we regard all the elements of the B-lattice as zero. Thus for each vector the corresponding group element is obtained by replacing the first component by an integer equivalent mod 1 and the second by an integer equivalent mod 6. Therefore

$$\begin{pmatrix}-4\\ 9\end{pmatrix} \to \begin{pmatrix}0\\ 3\end{pmatrix};\quad \begin{pmatrix}-1\\ 6\end{pmatrix} \to \begin{pmatrix}0\\ 0\end{pmatrix};\quad \begin{pmatrix}1\\ 0\end{pmatrix} \to \begin{pmatrix}0\\ 0\end{pmatrix};\quad \begin{pmatrix}0\\ 1\end{pmatrix} \to \begin{pmatrix}0\\ 1\end{pmatrix};\quad \begin{pmatrix}-1\\ 2\end{pmatrix} \to \begin{pmatrix}0\\ 2\end{pmatrix}$$

and

$$\begin{pmatrix}-47\\ 135\end{pmatrix} \to \begin{pmatrix}0\\ 3\end{pmatrix}.$$

The group is, of course, cyclic of order 6. We can refer to the element $\begin{pmatrix}0\\ a\end{pmatrix}$ as (a).

Reviewing, we see that the x_3 column corresponds to group element (3), the x_4 to (0), the x_5 to (0), the x_6, or first slack, to (1), and the x_7, or second slack, to (2). The right-hand side corresponds to (3).

The linear programming solution, which actually appears in Appendix 4, provides us with costs for each column and hence for the group elements. These costs are:

Group Element	Cost
(0)	0
(1)	$1\frac{5}{6}$
(2)	$\frac{4}{6}$
(3)	$3\frac{3}{6}$
(4)	X
(5)	X

We now solve the shortest-path problem in $H(\mathscr{G}, \mathscr{N}, g_0) = H(\mathscr{G}_6, \{(1), (2), (3)\}, (3))$. Using any method (that of Appendix 2 is an example), we find as solution

Shortest Path from (0) to	Cost	Path	
(0)	0	0	
(1)	$1\frac{5}{6}$	$t_1 = 1$	
(2)	$\frac{4}{6}$	$t_2 = 1$	
(3)	$2\frac{3}{6}$	$t_1 = 1$,	$t_2 = 1$
(4)	$1\frac{2}{6}$	$t_2 = 2$	
(5)	$3\frac{1}{6}$	$t_1 = 1$,	$t_2 = 2$

where t_i is the number of times group element (i) is used in the path.

The asymptotic integer programming solution is $x = (x_B, x_N)$, where $x_B = B^{-1}(b - Nx_N)$ and $x_N = (x_3, x_4, x_5, x_6, x_7)$. Using the correspondence $x_6 \to t_1$ and $x_7 \to t_2$, we find for x_N, using the table above and the fact that $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ determines g_0:

$$x_N = (0, 0, 0, 0, 0) \quad \text{if} \quad (1, 2) \cdot (b_1, b_2) \equiv (0) \bmod 6,$$

$$x_N = (0, 0, 0, 1, 0) \quad \text{if} \quad (1, 2) \cdot (b_1, b_2) \equiv (1) \bmod 6,$$

$$x_N = (0, 0, 0, 0, 1) \quad \text{if} \quad (1, 2) \cdot (b_1, b_2) \equiv (2) \bmod 6,$$

$$x_N = (0, 0, 0, 1, 1) \quad \text{if} \quad (1, 2) \cdot (b_1, b_2) \equiv (3) \bmod 6,$$

$$x_N = (0, 0, 0, 0, 2) \quad \text{if} \quad (1, 2) \cdot (b_1, b_2) \equiv (4) \bmod 6,$$

$$x_N = (0, 0, 0, 1, 2) \quad \text{if} \quad (1, 2) \cdot (b_1, b_2) \equiv (5) \bmod 6.$$

For example, for our right-hand side $\begin{pmatrix} 41 \\ 47 \end{pmatrix}$, $(1, 2) \cdot (41, 47) = (135) \equiv (3) \bmod 6$. Therefore $x_N = (0, 0, 0, 1, 1)$, and $Nx_N = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. B^{-1} is $\begin{pmatrix} \frac{4}{6} & \frac{2}{6} \\ \frac{3}{6} & 0 \end{pmatrix}$; so x_B is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = B^{-1} \left\{ \begin{pmatrix} 41 \\ 47 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \begin{pmatrix} 42 \\ 20 \end{pmatrix}.$$

Since x_1 and x_2 are both $\geqslant 0$, the solution is applicable. In fact, as Fig. 4 shows, the solution is applicable for almost all (b_1, b_2) for which B is the optimal basis.

It should be noted that the calculations given here are almost the most laborious possible. Generally it is much easier to obtain the group by working with the fractional parts of B^{-1} if B^{-1} is already available. Many other economies are possible. The method of computation described here was chosen mainly for ease of exposition.

APPENDIX 2

A Group Minimization Calculation

To solve for nonnegative $\pi(g)$, the problem

$$\min \sum_{g \in \mathcal{G}} \pi(g) t(g) = \phi,$$

subject to $\sum g \cdot t(g) = g_0$, we can proceed as follows:

First, as a preliminary, we reduce $\mathscr{G}$ to a direct sum of r cyclic groups $\mathscr{G}_i$ of orders ε_i. Each element of the group can now be represented as an r vector,

$$g = (x_1, \ldots, x_r),$$

with components x_i added modulo ε_i.

With these preliminaries completed, the steps described, which involve the addition of group elements or the ability to check through a list of group elements, can easily be carried out.

The calculation is dynamic programming with some additional modifications to take care of the group structure.

We define ϕ_S for any set of group elements $S \subset \mathscr{N}^+$ by

$$\phi_S(h) = \min \sum_{g \in S} \pi(g) \cdot t(g),$$

$$\sum_{g \in S} g \cdot t(g) = h,$$

and define ϕ for the null set S_0 by $\phi_{S_0}(g) = M$, $g \neq 0$, $\phi(\bar{0}) = 0$, where

$$M > |\mathscr{G}| \max_{g \in S} \pi(g).$$

Then, assuming $\phi_S(g)$ already computed, we describe a computation for $\phi_{S'}$, where $S' = S \cup \bar{g}$, $\bar{g} \in \mathscr{N}$, $g \notin S$.

First we compute $\phi_{S'}$ for the group elements $\bar{0}$, $\bar{g}$, $2\bar{g}, \ldots, s\bar{g}, \ldots, r\bar{g} = \bar{0}$ by $\phi_{S'}(\bar{0}) = 0$ and

$$\phi_{S'}(s\bar{g}) = \min\{\phi_{S'}((s-1)\bar{g}) + \pi(\bar{g}), \phi_S(s\bar{g})\}, \qquad r > s > 0.$$

Next we choose an h for which $\phi_{S'}(h)$ has not been computed and proceed to get $\phi_{S'}$ for h, $h + \bar{g}$, $h + 2\bar{g}, \ldots$, etc., by the following steps:

First we introduce ψ_s by

$$\psi_0 = \phi_S(h),$$

$$\psi_s = \min\{\pi(g) + \psi_{s-1}, \phi_S(h + s\bar{g})\}.$$

Clearly ψ_s could be computed for all integer $s > 0$. These ψ_s are closely related to the sought-after $\phi_{S'}$. Since $\psi_r = \min\{\pi(\bar{g}) + \psi_{r-1}, \phi_S(h)\}$, since $r \cdot g = \bar{0}$, we have $\psi_r \leqslant \psi_0$. It follows from the recursion that

$\psi_{r+p} \leqslant \psi_p$ and that, if $\psi_{r+\bar{p}} = \psi_r$ for some $\bar{p}$, then $\psi_{r+p} = \psi_r$ for all $p \geqslant \bar{p}$.

On the other hand, if $\psi_{r+p} < \psi_p$, then $\psi_{r+p} < \psi_p < \phi_S(h + p\bar{g})$; so $\psi_{r+p} = \psi_{r+p-1} + \pi(\bar{g})$. Now we cannot have

$$\psi_{r+p} < \psi_p$$

for $p = 0, 1, 2, \ldots, r-1$, because this implies $\psi_{2r-1} = \psi_{r-1} + r \cdot \pi(\bar{g})$, which contradicts $\psi_{r+(r-1)} < \psi_{r-1}$. Thus there is always a p, $0 \leqslant p \leqslant r-1$, for which $\psi_{r+p} = \psi_p$.

Let $\bar{p}$ be the first such p; then $\psi_{r+p} = \psi_p$, all $p > \bar{p}$. If we set

$$\phi_{S'}(h + p\bar{g}) = \psi_p, \qquad p \geqslant \bar{p},$$

we obtain values for $\phi_{S'}$ for all elements $h + s\bar{g}$, and the $\phi_{S'}$ are readily seen to satisfy the relation

$$\phi_{S'}(h + s\bar{g}) = \min\{\phi_{S'}(h + (s-1)\bar{g}) + \pi(\bar{g}), \phi_S(h + s\bar{g})\}.$$

Since it is easily shown that $\psi_{\bar{p}} = \phi(h + \bar{p}\bar{g})$, it follows that $\phi_{S'}(h + \bar{p}\bar{g}) = \phi_S(h + \bar{p}g)$. These facts are enough to establish $\phi_{S'}$ as the sought-after minimizing function.

The calculation is repeated until there are no more h for which $\phi_{S'}$ has not been computed. This gives $\phi_{S'}$. The whole process is then repeated until $S' = \mathscr{N}$.

The computation yields the solutions $t(g)$ by backtracking in the usual manner of dynamic programming. It is only necessary to record, when $\phi_{S'}(g)$ is obtained, whether or not $\bar{g}$ was used in the solution. Even for backtracking, it is unnecessary to keep the values of $\phi_{S'}$ once ϕ_S is computed.

APPENDIX 3

A Face Calculation

To calculate a face of $P(\mathscr{G}, \mathscr{N}, g_0)$ we start by setting $\phi_{\mathscr{N}_0}(g) = M > 1$, $g \neq \bar{0}$, and $\phi_{\mathscr{N}_0}(\bar{0}) = 0$. Then, if the coefficients $\pi(g)$ have already been computed for $g \in S$, we compute $\pi(\bar{g})$, $\bar{g} \notin S$, as follows:

(i) Find the values of m, if any, for which $\phi_S(g_0 - m\bar{g}) < 1$. If there are none, set $\pi(\bar{g}) = 0$. If there are values m_i, $i = 1, \ldots, q$, set $\pi(\bar{g})$ by

$$\pi(\bar{g}) = \max_{i=1,\ldots,q} \frac{1 - \phi_S(g_0 - m_i\bar{g})}{m_i}.$$

(ii) Use this value of $\pi(\bar{g})$ to compute $\phi_{S \cup \bar{g}}(g)$, $g \in \mathscr{G}$, as in Appendix 2 (or by any other method).

Repeat this process until $\pi(\bar{g})$ has been obtained for all $g \in \mathscr{N}$.

The inequality

$$\sum_{g \in \mathscr{N}} \pi(g) \cdot t(g) \geqslant 1$$

is a face of $P(\mathscr{G}, \mathscr{N}, g_0)$.

To see this, let us suppose that the first r elements result in $\pi(g) = 0$ and the next, say element $\bar{g}$, results in $\pi(\bar{g}) = (1 - \phi_S(g_0 - m\bar{g}))/m > 0$. We can easily verify that $\phi_S(g_0 - m\bar{g}) = 0$; therefore the zero length path leading to $g_0 - m\bar{g}$ followed by $m\bar{g}$'s is a path of length 1 to g_0, and that any other paths are as long. $r + 1$ independent paths are obtained by adding to this one the loops $s(g) \cdot g = \bar{0}$, where g is any one of the earlier elements, all having $\pi(g) = 0$. Moving on to the $r + 2$ element h', we see two cases:

(i) $\qquad \pi(h') = 0.$

In this case we can use h' in a loop and form a new path.

(ii) $\qquad \pi(h') > 0.$

In this case the maximizing m_i provides a path $m_i h_i$ from $g_0 - m_i h'$ to g_0. This path is of length 1; all others involving h' are of length $\geqslant 1$ because of the choice of $\pi(h')$ which implies $m_i \pi(h') \geqslant 1 - \phi_s(g_0 - mh')$, and so $\phi_s(g_0 - mh') + m_i h' \geqslant 1$, all i. All paths not using h' at all are already known to be of length $\geqslant 1$; so this is a shortest path and, since it used h', it is independent of previous paths.

We can go on in this way until $|\mathscr{N}|$ elements are used and $|\mathscr{N}|$ shortest paths formed. Thus the $\pi(g)$ are a face.

APPENDIX 4

Characters and Inequalities: A Numerical Example

We illustrate the use of characters to produce inequalities by an example.

The linear programming problem of Appendix 1, written in a matrix form that includes the objective function becomes: maximize z subject to

$$\begin{pmatrix} 1 & -2 & -1 & -1 & -3 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 4 & 2 & 1 & 0 \\ 0 & 3 & -4 & 4 & 1 & -1 & 0 & 1 \end{pmatrix} \begin{bmatrix} z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{pmatrix} 0 \\ 41 \\ 47 \end{pmatrix}.$$

The optimal basis consists of the first columns, and the optimal transformed matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 3\frac{3}{6} & 5 & 2 & 1\frac{5}{6} & \frac{4}{6} \\ 0 & 0 & 1 & \frac{3}{6} & 2 & 1 & \frac{3}{6} & 0 \\ 0 & 1 & 0 & 2 & 3 & 1 & \frac{4}{6} & \frac{2}{6} \end{pmatrix} \begin{bmatrix} z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{pmatrix} 106\frac{3}{4} \\ 20\frac{3}{4} \\ 43 \end{pmatrix}.$$

The optimal basis has determinant 6.

Each row of this matrix gives us a mapping into the cyclic group of order 6. These mappings, ψ_1, ψ_2, ψ_3, are characters and are specifically, using the fractional parts as described in Section 3E:

Column Name	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	R.H.S.
$\mathscr{G}_6$ element ψ_1	(0)	(0)	(0)	(3)	(0)	(0)	(5)	(4)	(3)
$\mathscr{G}_6$ element ψ_2	(0)	(0)	(0)	(3)	(0)	(0)	(3)	(0)	(3)
$\mathscr{G}_6$ element ψ_3	(0)	(0)	(0)	(3)	(0)	(0)	(4)	(2)	(0)

Since ψ_1 sends the right-hand side column into (3), we use the faces of $P(\mathscr{G}_6, (3))$ from Appendix 5. According to Section 3E, each face π gives $\sum \pi(\psi_g) \geqslant \pi_0$. Note that, although we have not determined what g of $\mathscr{G} = M(I)/M(B)$ corresponds to each column, this is not necessary, as we do know the values ψ_g. From the faces of $P(\mathscr{G}_6, (3))$, using ψ_1, we obtain:

From face 1: $1x_3 + 1x_6 + 0x_7 \geqslant 1$;

From face 2: $3x_3 + 1x_6 + 2x_7 \geqslant 3$;

From face 3: $3x_3 + 1x_6 + 2x_7 \geqslant 3$;

From face 4: $3x_3 + 2x_6 + 1x_7 \geqslant 3$.

Using ψ_2 and $P(\mathscr{G}_6, (3))$, we obtain:

From face 1: $1x_3 + 1x_6 \geqslant 1$;

From face 2: $3x_3 + 3x_6 \geqslant 3$;

From face 3: $3x_3 + 3x_6 \geqslant 3$;

From face 4: $3x_3 + 3x_6 \geqslant 3$.

Now ψ_3 provides nothing further, as it sends the right-hand side into (0). So, eliminating duplicates, we have obtained the inequalities:

$$x_3 + x_6 \geqslant 1,$$

$$3x_3 + x_6 + 2x_7 \geqslant 3,$$

$$3x_3 + 2x_6 + x_7 \geqslant 3.$$

In this case, because $\mathscr{G} = M(B)/M(I)$ is actually cyclic and ψ_1 is an isomorphism, we have actually obtained all faces of the corner polyhedron.

The fractional inequalities of [4] in this case would have been a weaker set. We would have obtained from ψ_1

$$3x_3 + 5x_6 + 4x_7 \geqslant 3,$$

and from ψ_2

$$3x_3 + 3x_6 \qquad \geqslant 3.$$

APPENDIX 5

$P(\mathscr{G}_2, (0))$

FACES

	π_1	π_0
Row		
1	1	2

VERTICES

1. $(t_1) = (2)$

INCIDENCE MATRIX

Face	1
Vertex	
1.	1

$P(\mathscr{G}_2, (1))$

FACES

	π_1	π_0
Row		
1	1	1

VERTICES

1. $(t_1) = (1)$

INCIDENCE MATRIX

Face	1
Vertex	
1.	1

$P(\mathscr{G}_3, (0))$

FACES

	π_1	π_2	π_0
Row			
1	2	1	3
2	1	2	3

VERTICES

1. $(t_1) = (3)$
2. $(t_1, t_2) = (1, 1)$
3. $(t_2) = (3)$

INCIDENCE MATRIX

Face	1	2	3	4
Vertex				
1.	0	1	0	1
2.	1	1	0	0
3.	1	0	1	0

$P(\mathscr{G}_3, (2))$

FACES

	π_1	π_2*	π_0
Row			
1	1	2	2

VERTICES

1. $(t_1) = (2)$
2. $(t_2) = (1)$

INCIDENCE MATRIX

Face	1	2	3
Vertex			
1.	1	0	1
2.	1	1	0

$P(\mathscr{G}_4, (0))$

FACES

	π_1	π_2	π_3	π_0
Row				
1	3	2	1	4
2	1	2	3	4

VERTICES

1. $(t_1) = (4)$
2. $(t_2) = (2)$
3. $(t_1, t_3) = (1, 1)$
4. $(t_3) = (4)$

INCIDENCE MATRIX

Face	1	2	3	4	5
Vertex					
1.	0	1	0	1	1
2.	1	1	1	0	1
3.	1	1	0	1	0
4.	1	0	1	1	0

$P(\mathscr{G}_4, (2))$

FACES

	π_1	$\pi_2{}^*$	π_3	π_0
Row				
1	1	2	1	2

VERTICES

1. $(t_1) = (2)$
2. $(t_2) = (1)$
3. $(t_3) = (2)$

INCIDENCE MATRIX

Face	1	2	3	4
Vertex				
1.	1	0	1	1
2.	1	1	0	1
3.	1	1	1	0

$P(\mathscr{G}_4, (3))$

FACES

	π_1	π_2	$\pi_3{}^*$	π_0
Row				
1	1	0	1	1
2	1	2	3	3

VERTICES

1. $(t_1) = (3)$
2. $(t_1, t_2) = (1, 1)$
3. $(t_3) = (1)$

INCIDENCE MATRIX

Face	1	2	3	4	5
Vertex					
1.	0	1	0	1	1
2.	1	1	0	0	1
3.	1	1	1	1	0

$P(\mathscr{G}_{2,2}, (0, 0))$

FACES

	$\pi_{1,0}$	$\pi_{0,1}$	$\pi_{1,1}$	π_0
Row				
1	1	1	1	2

VERTICES

1. $(t_{1,0}) = (2)$
2. $(t_{0,1}) = (2)$
3. $(t_{1,1}) = (2)$

INCIDENCE MATRIX

Face	1	2	3	4
Vertex				
1.	1	0	1	1
2.	1	1	0	1
3.	1	1	1	0

$P(\mathscr{G}_{2,2}, (1, 0))$

FACES

	$\pi_{1,0}$	$\pi_{0,1}$	$\pi_{1,1}$	π_0
Row				
1	1	1	0	1
2	1	0	1	1

VERTICES

1. $(t_{1,0}) = (1)$
2. $(t_{0,1}, t_{1,1}) = (1, 1)$

INCIDENCE MATRIX

Face	1	2	3	4	5
Vertex					
1.	1	1	0	1	1
2.	1	1	1	0	0

$P(\mathscr{G}_5, (0))$

FACES

	π_1	π_2	π_3	π_4	π_0
Row					
1	4	3	2	1	5
2	3	1	4	2	5
3	2	4	1	3	5
4	1	2	3	4	5

VERTICES

1. $(t_1) = (5)$
2. $(t_1, t_2) = (1, 2)$
3. $(t_2) = (5)$
4. $(t_1, t_3) = (2, 1)$
5. $(t_2, t_3) = (1, 1)$
6. $(t_3) = (5)$
7. $(t_1, t_4) = (1, 1)$
8. $(t_3, t_4) = (2, 1)$
9. $(t_2, t_4) = (1, 2)$
10. $(t_4) = (5)$

INCIDENCE MATRIX $P(\mathscr{G}_5, (0))$

Face	1	2	3	4	5	6	7	8
Vertex								
1.	0	0	0	1	0	1	1	1
2.	0	1	0	1	0	0	1	1
3.	0	1	0	0	1	0	1	1
4.	0	0	1	1	0	1	0	1
5.	1	1	1	1	1	0	0	1
6.	0	0	1	0	1	1	0	1
7.	1	1	1	1	0	1	1	0
8.	1	0	1	0	1	1	0	0
9.	1	1	0	0	1	0	1	0
10.	1	0	0	0	1	1	1	0

$P(\mathscr{G}_5, (4))$

FACES

	π_1	π_2	π_3	π_4	π_0
Row					
1	1	2	3	4	4
2	4	3	2	6	6

VERTICES

1. $(t_1) = (4)$
2. $(t_2) = (2)$
3. $(t_1, t_3) = (1, 1)$
4. $(t_3) = (3)$
5. $(t_4) = (1)$

INCIDENCE MATRIX $P(\mathscr{G}_5, (4))$

Face	1	2	3	4	5	6
Vertex						
1.	1	0	0	1	1	1
2.	1	1	1	0	1	1
3.	1	1	0	1	0	1
4.	0	1	1	1	0	1
5.	1	1	1	1	1	0

$P(\mathscr{G}_6, (0))$

FACES

	π_1	π_2	π_3	π_4	π_5	π_0
Row						
1	5	4	3	2	1	6
2	4	2	3	4	2	6
3	2	4	3	2	4	6
4	1	2	3	4	5	6

VERTICES

1. $(t_1) = (6)$
2. $(t_2) = (3)$
3. $(t_3) = (2)$
4. $(t_1, t_4) = (2, 1)$
5. $(t_2, t_4) = (1, 1)$
6. $(t_4) = (3)$
7. $(t_1, t_5) = (1, 1)$
8. $(t_2, t_5) = (1, 2)$
9. $(t_5) = (6)$

INCIDENCE MATRIX $P(\mathscr{G}_6, (0))$

Face	1	2	3	4	5	6	7	8	9
Vertex									
1.	0	0	0	1	0	1	1	1	1
2.	0	1	0	1	1	0	1	1	1
3.	1	1	1	1	1	1	0	1	1
4.	0	0	1	1	0	1	1	0	1
5.	1	1	1	1	1	0	1	0	1
6.	1	0	1	0	1	1	1	0	1
7.	1	1	1	1	0	1	1	1	0
8.	1	1	0	0	1	0	1	1	0
9.	1	0	0	0	1	1	1	1	0

$P(\mathscr{G}_6, (3))$

FACES

	π_1	π_2	$\pi_3{}^*$	π_4	π_5	π_0
Row						
1	1	0	1	0	1	1
2	2	1	3	2	1	3
3	1	2	3	2	1	3
4	1	2	3	1	2	3

VERTICES

1. $(t_1) = (3)$
2. $(t_1, t_2) = (1, 1)$
3. $(t_3) = (1)$
4. $(t_1, t_4) = (1, 2)$
5. $(t_2, t_5) = (2, 1)$
6. $(t_4, t_5) = (1, 1)$
7. $(t_5) = (3)$

INCIDENCE MATRIX $P(\mathscr{G}_6, (3))$

Face	1	2	3	4	5	6	7	8	9
Vertex									
1.	0	0	1	1	0	1	1	1	1
2.	1	1	1	1	0	0	1	1	1
3.	1	1	1	1	1	1	0	1	1
4.	1	0	0	1	0	1	1	0	1
5.	1	1	0	0	1	0	1	1	0
6.	1	1	1	1	1	1	1	0	0
7.	0	1	1	0	1	1	1	1	0

$P(\mathscr{G}_6, (4))$

FACES

	π_1	π_2	π_3	$\pi_4{}^*$	π_5	π_0
Row						
1	2	1	0	2	1	2
2	1	2	3	4	2	4

VERTICES

1. $(t_1) = (4)$
2. $(t_2) = (2)$
3. $(t_1, t_3) = (1, 1)$
4. $(t_4) = (1)$
5. $(t_5) = (2)$

INCIDENCE MATRIX $P(\mathscr{G}_6, (4))$

Face	1	2	3	4	5	6	7
Vertex							
1.	0	1	0	1	1	1	1
2.	1	1	1	0	1	1	1
3.	1	1	0	1	0	1	1
4.	1	1	1	1	1	0	1
5.	1	1	1	1	1	1	0

$P(\mathscr{G}_6, (5))$

FACES

	π_1	π_2	π_3	π_4	π_5*	π_6
Row						
1	1	0	1	0	1	1
2	1	2	0	1	2	2
3	1	2	3	4	5	5

VERTICES

1. $(t_1) = (5)$
2. $(t_1, t_2) = (1, 2)$
3. $(t_1, t_3) = (2, 1)$
4. $(t_2, t_3) = (1, 1)$
5. $(t_1, t_4) = (1, 1)$
6. $(t_3, t_4) = (1, 2)$
7. $(t_5) = (1)$

INCIDENCE MATRIX $P(\mathscr{G}_6, (5))$

Face	1	2	3	4	5	6	7	8
Vertex								
1.	0	0	1	0	1	1	1	1
2.	1	0	1	0	0	1	1	1
3.	0	1	1	0	1	0	1	1
4.	1	1	1	1	0	0	1	1
5.	1	1	1	0	1	1	0	1
6.	1	1	0	1	1	0	0	1
7.	1	1	1	1	1	1	1	0

$P(\mathscr{G}_7, (0))$

FACES

	π_1	π_2	π_3	π_4	π_5	π_6	π_0
Row							
1	6	5	4	3	2	1	7
2	5	3	1	6	4	2	7
3	4	1	5	2	6	3	7
4	3	6	2	5	1	4	7
5	2	4	6	1	3	5	7
6	1	2	3	4	5	6	7

VERTICES

1. $(t_1) = (7)$
2. $(t_1, t_2) = (1, 3)$
3. $(t_2) = (7)$
4. $(t_2, t_3) = (2, 1)$
5. $(t_1, t_3) = (1, 2)$
6. $(t_3) = (7)$
7. $(t_1, t_4) = (3, 1)$
8. $(t_1, t_2, t_4) = (1, 1, 1)$
9. $(t_3, t_4) = (1, 1)$
10. $(t_2, t_4) = (1, 3)$
11. $(t_4) = (7)$
12. $(t_1, t_5) = (2, 1)$
13. $(t_2, t_5) = (1, 1)$
14. $(t_3, t_5) = (3, 1)$
15. $(t_4, t_5) = (1, 2)$
16. $(t_5) = (7)$
17. $(t_1, t_6) = (1, 1)$
18. $(t_4, t_6) = (2, 1)$
19. $(t_3, t_5, t_6) = (1, 1, 1)$
20. $(t_5, t_6) = (3, 1)$
21. $(t_2, t_6) = (1, 2)$
22. $(t_3, t_6) = (1, 3)$
23. $(t_6) = (7)$

INCIDENCE MATRIX $P(\mathscr{G}_7, (0))$

Face	1	2	3	4	5	6	7	8	9	10	11	12
Vertex												
1.	0	0	0	0	0	1	0	1	1	1	1	1
2.	0	0	1	0	0	1	0	0	1	1	1	1
3.	0	0	1	0	0	0	1	0	1	1	1	1
4.	0	1	1	0	0	1	1	0	0	1	1	1
5.	0	1	0	1	0	1	0	1	0	1	1	1
6.	0	1	0	0	0	0	1	1	0	1	1	1
7.	0	0	0	0	1	1	0	1	1	0	1	1
8.	0	0	1	0	1	1	0	0	1	0	1	1
9.	1	1	1	1	1	1	1	1	0	0	1	1
10.	0	0	1	0	1	0	1	0	1	0	1	1
11.	0	0	0	0	1	0	1	1	1	0	1	1
12.	0	0	0	1	1	1	0	1	1	1	0	1
13.	1	1	1	1	1	1	1	0	1	1	0	1

INCIDENCE MATRIX $P(\mathscr{G}_7, (0))$ (continued)

Face	1	2	3	4	5	6	7	8	9	10	11	12
14.	0	1	0	1	0	0	1	1	0	1	0	1
15.	1	0	0	1	1	0	1	1	1	0	0	1
16.	0	0	0	1	0	0	1	1	1	1	0	1
17.	1	1	1	1	1	1	0	1	1	1	1	0
18.	1	0	1	0	1	0	1	1	1	0	1	0
19.	1	1	0	1	0	0	1	1	0	1	0	0
20.	1	0	0	1	0	0	1	1	1	1	0	0
21.	1	1	1	0	0	0	1	0	1	1	1	0
22.	1	1	0	0	0	0	1	1	0	1	1	0
23.	1	0	0	0	0	0	1	1	1	1	1	0

$P(\mathscr{G}_7, (6))$

FACES

Row	π_1	π_2	π_3	π_4	π_5	π_6*	π_0
1	1	2	3	4	5	6	6
2	6	5	4	3	2	8	8
3	4	8	5	2	6	10	10
4	9	4	6	8	3	12	12

VERTICES

1.	(t_1)	$= (6)$	6.	(t_4)	$= (5)$
2.	(t_2)	$= (3)$	7.	(t_1, t_5)	$= (1, 1)$
3.	(t_3)	$= (2)$	8.	(t_4, t_5)	$= (2, 1)$
4.	(t_1, t_4)	$= (2, 1)$	9.	(t_5)	$= (4)$
5.	(t_2, t_4)	$= (1, 1)$	10.	(t_6)	$= (1)$

INCIDENCE MATRIX $P(\mathscr{G}_7, (6))$

Face / Vertex	1	2	3	4	5	6	7	8	9	10
1.	1	0	0	0	0	1	1	1	1	1
2.	1	0	0	1	1	0	1	1	1	1
3.	1	1	1	1	1	1	0	1	1	1
4.	1	0	1	0	0	1	1	0	1	1
5.	1	1	1	1	1	0	1	0	1	1
6.	0	0	1	0	1	1	1	0	1	1
7.	1	1	1	1	0	1	1	1	0	1
8.	0	1	1	0	1	1	1	0	0	1
9.	0	1	0	1	1	1	1	1	0	1
10.	1	1	1	1	1	1	1	1	1	0

$P(\mathscr{G}_8, (0))$

FACES

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_0
Row								
1	3	2	1	2	3	2	1	4
2	1	2	3	2	1	2	3	4
3	7	6	5	4	3	2	1	8
4	5	2	3	4	5	6	3	8
5	3	6	5	4	3	2	5	8
6	3	6	1	4	7	2	5	8
7	5	2	7	4	1	6	3	8
8	1	2	3	4	5	6	7	8

VERTICES

1. $(t_1) = (8)$
2. $(t_2) = (4)$
3. $(t_1, t_3) = (2, 2)$
4. $(t_2, t_3) = (1, 2)$
5. $(t_3) = (8)$
6. $(t_4) = (2)$
7. $(t_1, t_5) = (3, 1)$
8. $(t_1, t_2, t_5) = (1, 1, 1)$
9. $(t_3, t_5) = (1, 1)$
10. $(t_1, t_5) = (1, 3)$
11. $(t_5) = (8)$
12. $(t_1, t_6) = (2, 1)$
13. $(t_2, t_6) = (1, 1)$
14. $(t_5, t_6) = (2, 1)$
15. $(t_6) = (4)$
16. $(t_1, t_7) = (1, 1)$
17. $(t_3, t_7) = (3, 1)$
18. $(t_3, t_6, t_7) = (1, 1, 1)$
19. $(t_2, t_7) = (1, 2)$
20. $(t_5, t_7) = (2, 2)$
21. $(t_3, t_7) = (1, 3)$
22. $(t_7) = (8)$

INCIDENCE MATRIX $P(\mathscr{G}_8, (0))$

Face	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Vertex															
1.	0	0	0	0	0	0	0	1	0	1	1	1	1	1	1
2.	0	0	0	1	0	0	1	1	1	0	1	1	1	1	1
3.	0	0	0	0	0	1	0	1	0	1	0	1	1	1	1
4.	1	0	0	1	0	1	0	1	1	0	0	1	1	1	1
5.	0	0	0	0	0	1	0	0	1	1	0	1	1	1	1
6.	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1
7.	0	1	0	0	0	0	0	1	0	1	1	1	0	1	1
8.	0	1	0	0	0	0	1	1	0	0	1	1	0	1	1
9.	1	1	1	1	1	1	1	1	1	1	0	1	0	1	1
10.	0	1	0	0	0	0	1	0	0	1	1	1	0	1	1
11.	0	0	0	0	0	0	1	0	1	1	1	1	0	1	1
12.	0	1	0	0	1	1	0	1	0	1	1	1	1	0	1
13.	1	1	1	1	1	1	1	1	1	0	1	1	1	0	1

INCIDENCE MATRIX $P(\mathscr{G}_8, (0))$ (continued)

Face	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Vertex															
14.	0	1	1	0	1	0	1	0	1	1	1	1	0	0	1
15.	0	0	1	0	1	1	0	0	1	1	1	1	1	0	1
16.	1	1	1	1	1	1	1	1	0	1	1	1	1	1	0
17.	1	0	0	0	0	1	0	0	1	1	0	1	1	1	0
18.	1	0	1	0	0	1	0	0	1	1	0	1	1	0	0
19.	1	0	1	1	0	0	1	0	1	0	1	1	1	1	0
20.	0	0	1	0	0	0	1	0	1	1	1	1	0	1	0
21.	1	0	1	0	0	0	0	0	1	1	0	1	1	1	0
22.	0	0	1	0	0	0	0	0	1	1	1	1	1	1	0

$P(\mathscr{G}_8, (4))$

FACES

	π_1	π_2	π_3	$\pi_4{}^*$	π_5	π_6	π_7	π_0
Row								
1	1	2	3	4	1	2	3	4
2	1	2	3	4	3	2	1	4
3	3	2	1	4	3	2	1	4
4	3	2	1	4	1	2	3	4

VERTICES

1. $(t_1) = (4)$
2. $(t_2) = (2)$
3. $(t_1, t_3) = (1, 1)$
4. $(t_3) = (4)$
5. $(t_4) = (1)$
6. $(t_5) = (4)$
7. $(t_6) = (2)$
8. $(t_5, t_7) = (1, 1)$
9. $(t_7) = (4)$

INCIDENCE MATRIX $P(\mathscr{G}_8, (4))$

Face	1	2	3	4	5	6	7	8	9	10	11
Vertex											
1.	1	1	0	0	0	1	1	1	1	1	1
2.	1	1	1	1	1	0	1	1	1	1	1
3.	1	1	1	1	0	1	0	1	1	1	1
4.	0	0	1	1	1	1	0	1	1	1	1
5.	1	1	1	1	1	1	1	0	1	1	1
6.	1	0	0	1	1	1	1	1	0	1	1
7.	1	1	1	1	1	1	1	1	1	0	1
8.	1	1	1	1	1	1	1	1	0	1	0
9.	0	1	1	0	1	1	1	1	1	1	0

$P(\mathscr{G}_8, (6))$

FACES

	π_1	π_2	π_3	π_4	π_5	$\pi_6{}^*$	π_7	π_0
Row								
1	1	2	1	0	1	2	1	2
2	5	2	3	4	1	6	3	6
3	1	2	3	4	5	6	3	6

VERTICES

1. $(t_1) = (6)$
2. $(t_2) = (3)$
3. $(t_3) = (2)$
4. $(t_1, t_4) = (2, 1)$
5. $(t_2, t_4) = (1, 1)$
6. $(t_1, t_5) = (1, 1)$
7. $(t_4, t_5) = (1, 2)$
8. $(t_5) = (6)$
9. $(t_6) = (1)$
10. $(t_7) = (2)$

INCIDENCE MATRIX $P(\mathscr{G}_8, (6))$

Face	1	2	3	4	5	6	7	8	9	10
Vertex										
1.	0	0	1	0	1	1	1	1	1	1
2.	0	1	1	1	0	1	1	1	1	1
3.	1	1	1	1	1	0	1	1	1	1
4.	1	0	1	0	1	1	0	1	1	1
5.	1	1	1	1	0	1	0	1	1	1
6.	1	1	1	0	1	1	1	0	1	1
7.	1	1	0	1	1	1	0	0	1	1
8.	0	1	0	1	1	1	1	0	1	1
9.	1	1	1	1	1	1	1	1	0	1
10.	1	1	1	1	1	1	1	1	1	0

$P(\mathscr{G}_8, (7))$

FACES

	π_1	π_2	π_3	π_4	π_5	π_6	$\pi_7{}^*$	π_0
Row								
1	1	0	1	0	1	0	1	1
2	1	2	3	0	1	2	3	3
3	1	2	1	2	1	2	3	3
4	3	2	1	4	3	2	5	5
5	1	2	3	4	5	6	7	7
6	7	6	5	4	3	2	9	9
7	9	10	3	12	5	6	15	15

VERTICES

1.	(t_1)	$= (7)$	9.	(t_1, t_5)	$= (2, 1)$
2.	(t_1, t_2)	$= (1, 3)$	10.	(t_2, t_5)	$= (1, 1)$
3.	(t_2, t_3)	$= (2, 1)$	11.	(t_5)	$= (3)$
4.	(t_1, t_3)	$= (1, 2)$	12.	(t_1, t_6)	$= (1, 1)$
5.	(t_3)	$= (5)$	13.	(t_4, t_5, t_6)	$= (1, 1, 1)$
6.	(t_1, t_4)	$= (3, 1)$	14.	(t_3, t_6)	$= (1, 2)$
7.	(t_1, t_2, t_4)	$= (1, 1, 1)$	15.	(t_5, t_6)	$= (1, 3)$
8.	(t_3, t_4)	$= (1, 1)$	16.	(t_7)	$= (1)$

INCIDENCE MATRIX $P(\mathscr{G}_8, (7))$

Face	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Vertex														
1.	0	0	0	0	1	0	0	0	1	1	1	1	1	1
2.	1	0	0	0	1	0	0	0	0	1	1	1	1	1
3.	1	0	0	1	1	0	0	1	0	0	1	1	1	1
4.	0	0	1	1	1	0	1	0	1	0	1	1	1	1
5.	0	0	0	1	0	0	1	1	1	0	1	1	1	1
6.	0	1	0	0	1	0	0	0	1	1	0	1	1	1
7.	1	1	0	0	1	0	0	0	0	1	0	1	1	1
8.	1	1	1	1	1	1	1	1	1	0	0	1	1	1
9.	0	1	1	0	1	0	0	0	1	1	1	0	1	1
10.	1	1	1	1	1	1	1	1	0	1	1	0	1	1
11.	0	1	1	0	0	1	1	1	1	1	1	0	1	1
12.	1	1	1	1	1	1	1	0	1	1	1	1	0	1
13.	1	1	0	0	0	1	0	1	1	1	0	0	0	1
14.	1	0	0	1	0	1	1	1	1	0	1	1	0	1
15.	1	0	0	0	0	1	0	1	1	1	1	0	0	1
16.	1	1	1	1	1	1	1	1	1	1	1	1	1	0

$P(\mathscr{G}_{4,2}, (0, 0))$

FACES

	$\pi_{1,0}$	$\pi_{2,0}$	$\pi_{3,0}$	$\pi_{0,1}$	$\pi_{1,1}$	$\pi_{2,1}$	$\pi_{3,1}$	π_0
Row								
1	3	2	1	2	3	2	1	4
2	1	2	3	2	3	2	1	4
3	3	2	1	2	1	2	3	4
4	1	2	3	2	1	2	3	4

VERTICES

1.	$(t_{1,0})$	$= (4)$	6.	$(t_{1,1})$	$= (4)$
2.	$(t_{2,0})$	$= (2)$	7.	$(t_{2,1})$	$= (2)$
3.	$(t_{1,0}, t_{3,0})$	$= (1, 1)$	8.	$(t_{1,1}, t_{3,1})$	$= (1, 1)$
4.	$(t_{3,0})$	$= (4)$	9.	$(t_{3,1})$	$= (4)$
5.	$(t_{0,1})$	$= (2)$			

INCIDENCE MATRIX $P(\mathscr{G}_{4,2}, (0, 0))$

Face	1	2	3	4	5	6	7	8	9	10	11
Vertex											
1.	0	1	0	1	0	1	1	1	1	1	1
2.	1	1	1	1	1	0	1	1	1	1	1
3.	1	1	1	1	0	1	0	1	1	1	1
4.	1	0	1	0	1	1	0	1	1	1	1
5.	1	1	1	1	1	1	1	0	1	1	1
6.	0	0	1	1	1	1	1	1	0	1	1
7.	1	1	1	1	1	1	1	1	1	0	1
8.	1	1	1	1	1	1	1	1	0	1	0
9.	1	1	0	0	1	1	1	1	1	1	0

$P(\mathscr{G}_{4,2}, (2, 0))$

FACES

	$\pi_{1,0}$	$\overset{*}{\pi}_{2,0}$	$\pi_{3,0}$	$\pi_{0,1}$	$\pi_{1,1}$	$\pi_{2,1}$	$\pi_{3,1}$	π_0
Row								
1	1	2	1	0	1	2	1	2
2	1	2	1	2	1	0	1	2

VERTICES

1.	$(t_{1,0})$	$= (2)$	4.	$(t_{1,1})$	$= (2)$
2.	$(t_{2,0})$	$= (2)$	5.	$(t_{0,1}, t_{2,1})$	$= (1, 1)$
3.	$(t_{3,0})$	$= (3)$	6.	$(t_{3,1})$	$= (2)$

INCIDENCE MATRIX $P(\mathscr{G}_{4,2}, (2, 0))$

Face	1	2	3	4	5	6	7	8	9
Vertex									
1.	1	1	0	1	1	1	1	1	1
2.	1	1	1	0	1	1	1	1	1
3.	1	1	1	1	0	1	1	1	1
4.	1	1	1	1	1	1	0	1	1
5.	1	1	1	1	1	0	1	0	1
6.	1	1	1	1	1	1	1	1	0

$P(\mathscr{G}_{4,2}, (3, 0))$

FACE

	$\pi_{1,0}$	$\pi_{2,0}$	$\overset{*}{\pi}_{3,0}$	$\pi_{0,1}$	$\pi_{1,1}$	$\pi_{2,1}$	$\pi_{3,1}$	π_0
Row								
1	1	0	1	1	0	1	0	1
2	1	0	1	0	1	0	1	1
3	1	2	3	2	3	0	1	3
4	1	2	3	2	1	2	1	3
5	1	2	3	0	1	2	3	3

VERTICES

1. $(t_{1,0}) = (3)$
2. $(t_{1,0}, t_{2,0}) = (1, 1)$
3. $(t_{3,0}) = (1)$
4. $(t_{2,0}, t_{0,1}, t_{1,1}) = (1, 1, 1)$
5. $(t_{1,0}, t_{1,1}) = (1, 2)$
6. $(t_{0,1}, t_{1,1}) = (1, 3)$
7. $(t_{1,0}, t_{0,1}, t_{2,1}) = (1, 1, 1)$
8. $(t_{1,1}, t_{2,1}) = (1, 1)$
9. $(t_{0,1}, t_{3,1}) = (1, 1)$
10. $(t_{2,0}, t_{2,1}, t_{3,1}) = (1, 1, 1)$
11. $(t_{1,0}, t_{3,1}) = (1, 2)$
12. $(t_{2,1}, t_{3,1}) = (1, 3)$

INCIDENCE MATRIX $P(\mathscr{G}_{4,2}, (3, 0))$

Face	1	2	3	4	5	6	7	8	9	10	11	12
Vertex												
1.	0	0	1	1	1	0	1	1	1	1	1	1
2.	1	1	1	1	1	0	0	1	1	1	1	1
3.	1	1	1	1	1	1	1	0	1	1	1	1
4.	1	1	0	0	1	1	0	1	0	0	1	1
5.	1	0	0	1	1	0	1	1	1	0	1	1
6.	1	0	0	0	1	1	1	1	0	0	1	1
7.	0	1	1	0	1	0	1	1	0	1	0	1
8.	1	1	1	1	1	1	1	1	1	0	0	1
9.	1	1	1	1	1	1	1	1	0	1	1	0
10.	1	1	1	0	0	1	0	1	1	1	0	0
11.	1	0	1	1	0	0	1	1	1	1	1	0
12.	1	0	1	0	0	1	1	1	1	1	0	0

$P(\mathscr{G}_{4,2}, (0, 1))$

FACES

	$\pi_{1,0}$	$\pi_{2,0}$	$\pi_{3,0}$	$\overset{*}{\pi}_{0,1}$	$\pi_{1,1}$	$\pi_{2,1}$	$\pi_{3,1}$	π_0
Row								
1	1	0	1	1	0	1	0	1
2	0	0	0	1	1	1	1	1
3	1	2	1	2	1	0	1	2
4	3	2	1	4	3	2	1	4
5	1	2	3	4	1	2	3	4

VERTICES

1. $(t_{0,1}) = (1)$
2. $(t_{1,0}, t_{1,1}) = (3, 1)$
3. $(t_{1,0}, t_{2,0}, t_{1,1}) = (1, 1, 1)$
4. $(t_{3,0}, t_{1,1}) = (1, 1)$
5. $(t_{1,0}, t_{1,1}) = (1, 3)$
6. $(t_{1,0}, t_{2,1}) = (2, 1)$
7. $(t_{2,0}, t_{2,1}) = (1, 1)$
8. $(t_{3,0}, t_{2,1}) = (2, 1)$
9. $(t_{1,1}, t_{2,1}) = (2, 1)$
10. $(t_{1,0}, t_{3,1}) = (1, 1)$
11. $(t_{2,0}, t_{3,0}, t_{3,1}) = (1, 1, 1)$
12. $(t_{3,0}, t_{3,1}) = (3, 1)$
13. $(t_{2,1}, t_{3,1}) = (1, 2)$
14. $(t_{3,0}, t_{3,1}) = (1, 3)$

INCIDENCE MATRIX $P(\mathscr{G}_{4,2}, (0, 1))$

Face	1	2	3	4	5	6	7	8	9	10	11	12
Vertex												
1.	1	1	1	1	1	1	1	1	0	1	1	1
2.	0	1	0	0	1	0	1	1	1	0	1	1
3.	1	1	0	0	1	0	0	1	1	0	1	1
4.	1	1	1	1	1	1	1	0	1	0	1	1
5.	1	0	0	0	1	0	1	1	1	0	1	1
6.	0	1	1	0	1	0	1	1	1	1	0	1
7.	1	1	1	1	1	1	0	1	1	1	0	1
8.	0	1	1	1	0	1	1	0	1	1	0	1
9.	1	0	1	0	1	1	1	1	1	0	0	1
10.	1	1	1	1	1	0	1	1	1	1	1	0
11.	1	1	0	1	0	1	0	0	1	1	1	0
12.	0	1	0	1	0	1	1	0	1	1	1	0
13.	1	0	1	1	0	1	1	1	1	1	0	0
14.	1	0	0	1	0	1	1	0	1	1	1	0

$P(\mathcal{G}_{2,2,2}, (0, 0, 0))$

FACES

	$\pi_{1,0,0}$	$\pi_{0,1,0}$	$\pi_{1,1,0}$	$\pi_{0,0,1}$	$\pi_{1,0,1}$	$\pi_{0,1,1}$	$\pi_{1,1,1}$	π_0
Row								
1	1	1	1	1	1	1	1	2

VERTICES

1. $(t_{1,0,0}) = (2)$
2. $(t_{0,1,0}) = (2)$
3. $(t_{1,1,0}) = (2)$
4. $(t_{0,0,1}) = (2)$
5. $(t_{1,0,1}) = (2)$
6. $(t_{0,1,1}) = (2)$
7. $(t_{1,1,1}) = (2)$

INCIDENCE MATRIX $P(\mathcal{G}_{2,2,2}, (0, 0, 0))$

Face	1	2	3	4	5	6	7	8
Vertex								
1.	1	0	1	1	1	1	1	1
2.	1	1	0	1	1	1	1	1
3.	1	1	1	0	1	1	1	1
4.	1	1	1	1	0	1	1	1
5.	1	1	1	1	1	0	1	1
6.	1	1	1	1	1	1	0	1
7.	1	1	1	1	1	1	1	0

$P(\mathcal{G}_{2,2,2}, (1, 0, 0))$

FACES

	$\pi_{1,0,0}$	$\pi_{0,1,0}$	$\pi_{1,1,0}$	$\pi_{0,0,1}$	$\pi_{1,0,1}$	$\pi_{0,1,1}$	$\pi_{1,1,1}$	π_0
Row								
1	1	0	1	1	0	1	0	1
2	1	1	0	0	1	1	0	1
3	1	1	0	1	0	0	1	1
4	1	0	1	0	1	0	1	1

VERTICES

1. $(t_{1,0,0}) = (1)$
2. $(t_{0,1,0}, t_{1,1,0}) = (1, 1)$
3. $(t_{0,0,1}, t_{1,0,1}) = (1, 1)$
4. $(t_{1,1,0}, t_{0,0,1}, t_{0,1,1}) = (1, 1, 1)$
5. $(t_{0,1,0}, t_{1,0,1}, t_{0,1,1}) = (1, 1, 1)$
6. $(t_{0,1,0}, t_{0,0,1}, t_{1,1,1}) = (1, 1, 1)$
7. $(t_{1,1,0}, t_{1,0,1}, t_{1,1,1}) = (1, 1, 1)$
8. $(t_{0,1,1}, t_{1,1,1}) = (1, 1)$

INCIDENCE MATRIX $P(\mathscr{G}_{2,2,2}, (1, 0, 0))$

Face	1	2	3	4	5	6	7	8	9	10	11
Vertex											
1.	1	1	1	1	0	1	1	1	1	1	1
2.	1	1	1	1	1	0	0	1	1	1	1
3.	1	1	1	1	1	1	1	0	0	1	1
4.	0	1	1	1	1	1	0	0	1	0	1
5.	1	0	1	1	1	0	1	1	0	0	1
6.	1	1	0	1	1	0	1	0	1	1	0
7.	1	1	1	0	1	1	0	1	0	1	0
8.	1	1	1	1	1	1	1	1	1	0	0

$P(\mathscr{G}_9, (0))$

FACES

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_0
Row									
1	2	1	2	2	1	1	2	1	3
2	2	1	1	2	1	2	2	1	3
3	1	2	2	1	2	1	1	2	3
4	1	2	1	1	2	2	1	2	3
5	8	7	6	5	4	3	2	1	9
6	7	5	3	1	8	6	4	2	9
7	5	1	6	2	7	3	8	4	9
8	4	8	3	7	2	6	1	5	9
9	2	4	6	8	1	3	5	7	9
10	1	2	3	4	5	6	7	8	9
11	14	10	6	11	7	12	8	4	18
12	11	4	6	8	10	12	14	7	18
13	10	11	12	4	14	6	7	8	18
14	8	7	6	14	4	12	11	10	18
15	7	14	12	10	8	6	4	11	18
16	4	8	12	7	11	6	10	14	18

VERTICES $P(\mathscr{G}_9, (0))$

1. $(t_1) = (9)$
2. $(t_1, t_2) = (1, 4)$
3. $(t_2) = (9)$
4. $(t_2, t_3) = (3, 1)$
5. $(t_3) = (3)$
6. $(t_2, t_3, t_4) = (1, 1, 1)$
7. $(t_1, t_4) = (1, 2)$
8. $(t_2, t_4) = (1, 4)$
9. $(t_4) = (9)$
10. $(t_1, t_5) = (4, 1)$
11. $(t_2, t_5) = (2, 1)$
12. $(t_1, t_3, t_5) = (1, 1, 1)$
13. $(t_4, t_5) = (1, 1)$
14. $(t_3, t_5) = (1, 3)$
15. $(t_5) = (9)$
16. $(t_1, t_6) = (3, 1)$
17. $(t_1, t_2, t_6) = (1, 1, 1)$
18. $(t_3, t_6) = (1, 1)$
19. $(t_4, t_6) = (3, 1)$
20. $(t_6) = (3)$
21. $(t_1, t_7) = (2, 1)$
22. $(t_2, t_7) = (1, 1)$
23. $(t_5, t_7) = (4, 1)$
24. $(t_5, t_6, t_7) = (1, 1, 1)$
25. $(t_4, t_7) = (1, 2)$
26. $(t_6, t_7) = (1, 3)$
27. $(t_7) = (9)$

VERTICES $P(\mathscr{G}_9, (0))$ (CONTINUED)

28. $(t_1, t_8) = (1, 1)$
29. $(t_5, t_8) = (2, 1)$
30. $(t_4, t_6, t_8) = (1, 1, 1)$
31. $(t_3, t_7, t_8) = (1, 1, 1)$
32. $(t_7, t_8) = (4, 1)$
33. $(t_2, t_8) = (1, 2)$
34. $(t_3, t_8) = (1, 3)$
35. $(t_4, t_8) = (1, 4)$
36. $(t_8) = (9)$

INCIDENCE MATRIX $P(\mathscr{G}_9, (0))$

Face / Vertex	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1.	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1
2.	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1
3.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1	1	1	1	1	1
4.	0	0	0	0	0	0	1	0	0	1	0	1	0	0	0	0	1	0	0	1	1	1	1	1
5.	0	1	0	1	0	1	0	1	0	1	1	1	0	1	0	0	1	1	0	1	1	1	1	1
6.	0	0	0	0	0	1	1	0	0	1	0	1	0	0	0	0	1	0	0	0	1	1	1	1
7.	0	0	1	1	0	1	1	0	0	1	0	0	1	0	0	1	0	1	1	0	1	1	1	1
8.	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	1	0	1	0	1	1	1	1
9.	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	1	1	0	1	1	1	1
10.	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	1	1	1	0	1	1	1
11.	1	1	0	0	0	0	1	0	1	1	0	1	0	1	0	0	1	0	1	1	0	1	1	1
12.	0	0	0	0	0	0	0	1	1	1	0	0	0	1	0	0	0	1	0	1	0	1	1	1
13.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	1	1	1
14.	0	0	0	0	0	0	0	1	1	0	0	0	0	1	0	0	1	1	0	1	0	1	1	1
15.	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	0	1	1	1
16.	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	1	0	1	1	1	1	0	1	1
17.	0	0	0	0	0	0	1	0	1	1	0	0	0	0	0	1	0	0	1	1	1	0	1	1
18.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	0	1	1
19.	0	0	0	0	0	1	1	0	0	0	0	0	1	0	0	0	1	1	1	0	1	0	1	1
20.	1	0	1	0	1	0	1	0	1	0	0	0	1	0	1	1	1	1	1	1	1	0	1	1
21.	0	0	1	1	0	0	0	1	1	1	0	0	0	0	1	1	0	1	1	1	1	1	0	1
22.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	0	1
23.	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	1	1	1	1	0	1	0	1
24.	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	1	1	1	1	0	0	0	1
25.	0	0	1	1	1	1	0	1	0	0	0	0	1	0	1	0	1	1	1	0	1	1	0	1
26.	0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	1	1	1	1	1	0	0	1
27.	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	1
28.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	0
29.	1	1	0	0	1	0	0	1	1	0	1	0	0	1	0	0	1	1	1	1	0	1	1	0
30.	0	0	0	0	1	1	1	0	0	0	0	0	1	0	0	0	1	1	1	0	1	0	1	0
31.	0	0	0	0	1	1	0	1	0	0	1	0	0	0	0	0	1	1	0	1	1	1	0	0
32.	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0
33.	1	1	0	0	1	1	1	0	0	0	1	1	0	0	0	0	1	0	1	1	1	1	1	0
34.	0	0	0	0	1	1	0	0	0	0	1	0	0	0	0	0	1	1	0	1	1	1	1	0
35.	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1	1	1	0	1	1	1	0
36.	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0

$P(\mathscr{G}_9, (6))$

FACE

	π_1	π_2	π_3	π_4	π_5	$\pi_6{}^*$	π_7	π_8	π_0
Row									
1	4	5	3	1	2	6	4	2	6
2	4	2	3	4	2	6	4	2	6
3	1	2	3	4	5	6	4	2	6
4	2	4	3	2	4	6	2	4	6
5	4	2	3	4	2	6	1	5	6
6	5	10	6	2	7	12	8	4	12
7	2	4	6	8	10	12	5	7	12
8	8	7	6	5	4	12	2	10	12

VERTICES

1. $(t_1) = (6)$
2. $(t_2) = (3)$
3. $(t_3) = (2)$
4. $(t_1, t_4) = (2, 1)$
5. $(t_2, t_3) = (1, 1)$
6. $(t_3, t_4) = (1, 3)$
7. $(t_4) = (6)$
8. $(t_1, t_5) = (1, 1)$
9. $(t_5) = (3)$
10. $(t_6) = (1)$
11. $(t_4, t_7) = (2, 1)$
12. $(t_1, t_7) = (1, 2)$
13. $(t_3, t_7) = (1, 3)$
14. $(t_7) = (6)$
15. $(t_7, t_8) = (1, 1)$
16. $(t_8) = (3)$

INCIDENCE MATRIX $P(\mathscr{G}_9, (6))$

Face	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Vertex																
1.	0	0	1	0	0	0	1	0	0	1	1	1	1	1	1	1
2.	0	1	1	0	1	0	1	0	1	0	1	1	1	1	1	1
3.	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1
4.	0	0	1	1	0	1	1	0	0	1	1	0	1	1	1	1
5.	1	1	1	1	1	1	1	1	1	0	1	0	1	1	1	1
6.	1	0	0	0	0	1	0	0	1	1	0	0	1	1	1	1
7.	1	0	0	0	0	1	0	0	1	1	1	0	1	1	1	1
8.	1	1	1	1	1	1	1	1	0	1	1	1	0	1	1	1
9.	1	1	0	0	1	0	0	1	1	1	1	1	0	1	1	1
10.	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1
11.	1	0	0	1	0	1	0	1	1	1	1	0	1	1	0	1
12.	0	0	0	1	1	0	1	1	0	1	1	1	1	1	0	1
13.	0	0	0	0	1	0	0	1	1	1	0	1	1	1	0	1
14.	0	0	0	0	1	0	0	1	1	1	1	1	1	1	0	1
15.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0
16.	1	1	1	0	0	1	0	0	1	1	1	1	1	1	1	0

$P(\mathscr{G}_9, (8))$

FACES

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	$\pi_8{}^*$	π_0
Row									
1	1	2	0	1	2	0	1	2	2
2	2	1	3	2	1	3	2	4	4
3	1	2	3	4	5	6	7	8	8
4	8	7	6	5	4	3	2	10	10
5	4	8	12	7	2	6	10	14	14
6	11	4	6	8	10	12	5	16	16
7	16	5	12	10	8	15	4	20	20

VERTICES

1. $(t_1) = (8)$
2. $(t_2) = (4)$
3. $(t_1, t_3) = (2, 2)$
4. $(t_2, t_3) = (1, 2)$
5. $(t_4) = (2)$
6. $(t_1, t_5) = (3, 1)$
7. $(t_1, t_2, t_5) = (1, 1, 1)$
8. $(t_3, t_5) = (1, 1)$
9. $(t_2, t_5) = (1, 3)$
10. $(t_5) = (7)$
11. $(t_1, t_6) = (2, 1)$
12. $(t_2, t_6) = (1, 1)$
13. $(t_5, t_6) = (1, 2)$
14. $(t_1, t_7) = (1, 1)$
15. $(t_5, t_7) = (2, 1)$
16. $(t_3, t_7) = (1, 2)$
17. $(t_6, t_7) = (2, 2)$
18. $(t_7) = (5)$
19. $(t_8) = (1)$

INCIDENCE MATRIX $P(\mathscr{G}_9, (8))$

Face	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Vertex															
1.	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1
2.	0	1	1	0	0	1	1	1	0	1	1	1	1	1	1
3.	1	0	1	0	0	0	0	0	1	0	1	1	1	1	1
4.	1	0	1	0	0	1	0	1	0	0	1	1	1	1	1
5.	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1
6.	0	0	1	0	1	0	0	0	1	1	1	0	1	1	1
7.	0	1	1	0	1	0	0	0	0	1	1	0	1	1	1
8.	1	1	1	1	1	1	1	1	1	0	1	0	1	1	1
9.	0	1	0	0	1	0	0	1	0	1	1	0	1	1	1
10.	0	0	0	0	1	0	0	1	1	1	1	0	1	1	1
11.	1	0	1	0	1	0	0	0	1	1	1	1	0	1	1
12.	1	1	1	1	1	1	1	1	0	1	1	1	0	1	1
13.	1	0	0	1	1	0	0	1	1	1	1	0	0	1	1
14.	1	1	1	1	1	1	1	0	1	1	1	1	1	0	1
15.	0	1	0	1	1	0	1	1	1	1	1	0	1	0	1
16.	1	0	0	1	0	1	1	1	1	0	1	1	1	0	1
17.	1	0	0	1	0	0	0	1	1	1	1	1	0	0	1
18.	0	0	0	1	0	0	1	1	1	1	1	1	1	0	1
19.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0

$P(\mathscr{G}_{3,3}, (0, 0))$

FACES

Row	$\pi_{1,0}$	$\pi_{2,0}$	$\pi_{0,1}$	$\pi_{1,1}$	$\pi_{2,1}$	$\pi_{0,2}$	$\pi_{1,2}$	$\pi_{2,2}$	π_0
1	2	1	2	2	2	1	1	1	3
2	1	2	2	2	2	1	1	1	3
3	2	1	1	2	2	2	1	1	3
4	1	2	1	2	2	2	1	1	3
5	2	1	2	2	1	1	2	1	3
6	1	2	2	2	1	1	2	1	3
7	2	1	1	2	1	2	2	1	3
8	1	2	1	2	1	2	2	1	3
9	2	1	2	1	2	1	1	2	3
10	1	2	2	1	2	1	1	2	3
11	2	1	1	1	2	2	1	2	3
12	1	2	1	1	2	2	1	2	3
13	2	1	2	1	1	1	2	2	3
14	1	2	2	1	1	1	2	2	3
15	2	1	1	1	1	2	2	2	3
16	1	2	1	1	1	2	2	2	3

VERTICES $P(\mathscr{G}_{3,3}, (0, 0))$

1.	$(t_{1,0})$	$= (3)$	7.	$(t_{0,1}, t_{0,2})$	$= (1, 1)$
2.	$(t_{1,0}, t_{2,0})$	$= (1, 1)$	8.	$(t_{0,2})$	$= (3)$
3.	$(t_{2,0})$	$= (3)$	9.	$(t_{2,1}, t_{1,2})$	$= (1, 1)$
4.	$(t_{0,1})$	$= (3)$	10.	$(t_{1,2})$	$= (3)$
5.	$(t_{1,1})$	$= (3)$	11.	$(t_{1,1}, t_{2,2})$	$= (1, 1)$
6.	$(t_{2,1})$	$= (3)$	12.	$(t_{2,2})$	$= (3)$

INCIDENCE MATRIX $P(\mathscr{G}_{3,3}, (0, 0))$

Face / Vertex	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1.	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	1	1	1	1	1
2.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1
3.	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	1	1	1	1
4.	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	1	1	0	1	1	1	1	1
5.	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1
6.	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1	0	1	1	1
7.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	0	1	1
8.	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	1	1	1	0	1	1
9.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	0	1
10.	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	1	1	0	1
11.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	0
12.	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0

$P(\mathscr{G}_{3,3}, (1, 0))$

FACES

	$\overset{*}{\pi}_{1,0}$	$\pi_{2,0}$	$\pi_{0,1}$	$\pi_{1,1}$	$\pi_{2,1}$	$\pi_{0,2}$	$\pi_{1,2}$	$\pi_{2,2}$	π_0
Row									
1	2	1	1	0	2	2	1	0	2
2	2	1	0	2	1	0	2	1	2
3	2	1	2	1	0	1	0	2	2
4	6	3	4	4	4	2	2	2	6
5	6	3	2	2	2	4	4	4	6

VERTICES

1. $(t_{1,0}) = (1)$
2. $(t_{2,0}) = (2)$
3. $(t_{0,1}, t_{1,1}) = (2, 1)$
4. $(t_{1,1}, t_{2,1}) = (2, 1)$
5. $(t_{0,1}, t_{2,1}) = (1, 2)$
6. $(t_{1,1}, t_{0,2}) = (1, 1)$
7. $(t_{2,1}, t_{0,2}) = (2, 2)$
8. $(t_{0,1}, t_{1,2}) = (1, 1)$
9. $(t_{0,2}, t_{1,2}) = (2, 1)$
10. $(t_{1,1}, t_{1,2}) = (2, 2)$
11. $(t_{2,1}, t_{2,2}) = (1, 1)$
12. $(t_{1,2}, t_{2,2}) = (2, 1)$
13. $(t_{0,1}, t_{2,2}) = (2, 2)$
14. $(t_{0,2}, t_{2,2}) = (1, 2)$

INCIDENCE MATRIX $P(\mathscr{G}_{3,3}, (1, 0))$

Face	1	2	3	4	5	6	7	8	9	10	11	12	13
Vertex													
1.	1	1	1	1	1	0	1	1	1	1	1	1	1
2.	1	1	1	1	1	1	0	1	1	1	1	1	1
3.	1	1	0	0	1	1	1	0	0	1	1	1	1
4.	1	0	1	0	1	1	1	1	0	0	1	1	1
5.	0	1	1	0	1	1	1	0	1	0	1	1	1
6.	1	1	1	1	1	1	1	1	0	1	0	1	1
7.	0	1	1	0	0	1	1	1	1	0	0	1	1
8.	1	1	1	1	1	1	1	0	1	1	1	0	1
9.	0	1	1	1	0	1	1	1	1	1	0	0	1
10.	1	0	1	0	0	1	1	1	0	1	1	0	1
11.	1	1	1	1	1	1	1	1	1	0	1	1	0
12.	1	0	1	1	0	1	1	1	1	1	1	0	0
13.	1	1	0	0	0	1	1	0	1	1	1	1	0
14.	1	1	0	1	0	1	1	1	1	1	0	1	0

$P(\mathscr{G}_{10}, (0))$

FACES

Row	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_0
1	9	8	7	6	5	4	3	2	1	10
2	8	6	4	2	5	8	6	4	2	10
3	7	4	6	8	5	2	4	6	3	10
4	7	4	1	8	5	2	9	6	3	10
5	6	2	8	4	5	6	2	8	4	10
6	6	2	3	4	5	6	7	8	4	10
7	4	8	7	6	5	4	3	2	6	10
8	4	8	2	6	5	4	8	2	6	10
9	3	6	9	2	5	8	1	4	7	10
10	3	6	4	2	5	8	6	4	7	10
11	2	4	6	8	5	2	4	6	8	10
12	1	2	3	4	5	6	7	8	9	10

VERTICES $P(\mathscr{G}_{10}, (0))$

1. $(t_1) = (10)$
2. $(t_2) = (5)$
3. $(t_2, t_3) = (2, 2)$
4. $(t_1, t_3) = (1, 3)$
5. $(t_3) = (10)$
6. $(t_3, t_4) = (2, 1)$
7. $(t_1, t_4) = (2, 2)$
8. $(t_2, t_4) = (1, 2)$
9. $(t_4) = (5)$
10. $(t_1, t_4, t_5) = (1, 1, 1)$
11. $(t_5) = (2)$
12. $(t_1, t_6) = (4, 1)$
13. $(t_2, t_6) = (2, 1)$
14. $(t_1, t_3, t_6) = (1, 1, 1)$
15. $(t_4, t_6) = (1, 1)$
16. $(t_6) = (5)$
17. $(t_1, t_7) = (3, 1)$
18. $(t_1, t_2, t_7) = (1, 1, 1)$
19. $(t_3, t_7) = (1, 1)$
20. $(t_6, t_7) = (1, 2)$
21. $(t_2, t_7) = (1, 4)$
22. $(t_7) = (10)$
23. $(t_1, t_8) = (2, 1)$
24. $(t_2, t_8) = (1, 1)$
25. $(t_3, t_8) = (4, 1)$
26. $(t_6, t_8) = (2, 1)$
27. $(t_5, t_7, t_8) = (1, 1, 1)$
28. $(t_4, t_8) = (1, 2)$
29. $(t_7, t_8) = (2, 2)$
30. $(t_8) = (5)$
31. $(t_1, t_9) = (1, 1)$
32. $(t_5, t_6, t_9) = (1, 1, 1)$
33. $(t_4, t_7, t_9) = (1, 1, 1)$
34. $(t_7, t_9) = (3, 1)$
35. $(t_3, t_8, t_9) = (1, 1, 1)$
36. $(t_2, t_9) = (1, 2)$
37. $(t_6, t_9) = (2, 2)$
38. $(t_3, t_9) = (1, 3)$
39. $(t_4, t_9) = (1, 4)$
40. $(t_9) = (10)$

INCIDENCE MATRIX $P(\mathscr{G}_{10}, (0))$

Face / Vertex	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1.	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1	1	1	1	1	1
2.	0	0	0	0	1	1	0	0	0	0	0	1	1	0	1	1	1	1	1	1	1
3.	0	0	0	1	0	1	0	0	0	0	0	1	1	0	0	1	1	1	1	1	1
4.	0	0	0	1	0	0	0	1	0	0	0	1	0	1	0	1	1	1	1	1	1
5.	0	0	0	1	0	0	0	0	0	0	0	0	1	1	0	1	1	1	1	1	1
6.	0	1	0	1	0	1	0	1	0	1	0	1	1	1	0	0	1	1	1	1	1
7.	0	0	0	0	0	0	0	0	1	1	0	1	0	1	1	0	1	1	1	1	1
8.	0	1	0	0	1	1	0	0	1	1	0	1	1	0	1	0	1	1	1	1	1
9.	0	1	0	0	0	0	0	0	1	1	0	0	1	1	1	0	1	1	1	1	1
10.	0	0	0	0	0	0	0	0	1	1	0	1	0	1	1	0	0	1	1	1	1
11.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1
12.	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	1	1	0	1	1	1
13.	0	0	1	1	1	1	0	0	0	0	1	1	1	0	1	1	1	0	1	1	1
14.	0	0	0	1	0	0	0	1	0	0	1	1	0	1	0	1	1	0	1	1	1
15.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	0	1	1	1
16.	0	0	1	1	0	0	0	0	0	0	1	0	1	1	1	1	1	0	1	1	1
17.	0	0	0	0	0	0	0	0	1	0	1	1	0	1	1	1	1	1	0	1	1
18.	0	0	0	0	1	0	0	0	1	0	1	1	0	0	1	1	1	1	0	1	1
19.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	0	1	1
20.	1	0	1	0	1	0	1	0	1	0	1	0	1	1	1	1	1	0	0	1	1
21.	0	0	0	0	1	0	0	0	1	0	0	0	1	0	1	1	1	1	0	1	1
22.	0	0	0	0	0	0	0	0	1	0	0	0	1	1	1	1	1	1	0	1	1
23.	0	0	0	0	0	0	1	1	1	1	1	1	0	1	1	1	1	1	1	0	1
24.	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	0	1
25.	0	0	0	1	0	0	0	1	0	0	0	0	1	1	0	1	1	1	1	0	1
26.	1	0	1	1	0	0	1	1	0	0	1	0	1	1	1	1	1	0	1	0	1
27.	1	0	0	0	0	0	1	0	1	0	0	0	1	1	1	1	0	1	0	0	1
28.	1	1	0	0	0	0	1	1	1	1	0	0	1	1	1	0	1	1	1	0	1
29.	1	0	0	0	0	0	1	0	1	0	0	0	1	1	1	1	1	1	0	0	1
30.	1	0	0	0	0	0	1	1	0	0	0	0	1	1	1	1	1	1	1	0	1
31.	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	0
32.	1	0	1	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	1	1	0
33.	1	1	0	0	1	0	0	0	1	0	0	0	1	1	1	0	1	1	0	1	0
34.	1	0	0	0	1	0	0	0	1	0	0	0	1	1	1	1	1	1	0	1	0
35.	1	1	0	1	0	0	0	1	0	0	0	0	1	1	0	1	1	1	1	0	0
36.	1	1	1	1	1	1	0	0	0	0	0	0	1	0	1	1	1	1	1	1	0
37.	1	0	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	0	1	1	0
38.	1	1	0	1	0	0	0	0	0	0	0	0	1	1	0	1	1	1	1	1	0
39.	1	1	0	0	0	0	0	0	0	0	0	0	1	1	1	0	1	1	1	1	0
40.	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0

$P(\mathscr{G}_{10}, (5))$

FACES

Row	π_1	π_2	π_3	π_4	π_5^*	π_6	π_7	π_8	π_9	π_0
1	1	0	1	0	1	0	1	0	1	1
2	1	2	1	2	3	2	1	2	1	3
3	4	3	2	1	5	4	3	2	1	5
4	1	2	3	4	5	4	3	2	1	5
5	3	1	4	2	5	3	1	4	2	5
6	2	4	1	3	5	2	4	1	3	5
7	3	4	1	2	5	2	1	4	3	5
8	1	2	3	4	5	1	2	3	4	5
9	3	6	4	7	10	8	6	4	2	10
10	2	4	6	8	10	7	4	6	3	10
11	6	7	3	4	10	6	2	8	4	10
12	4	8	2	6	10	4	3	7	6	10
13	7	4	11	8	15	12	9	6	3	15
14	3	6	9	12	15	8	11	4	7	15
15	9	8	7	6	15	4	3	12	11	15
16	11	12	3	4	15	6	7	8	9	15

VERTICES $P(\mathscr{G}_{10}, (5))$

1. $(t_1) = (5)$
2. $(t_1, t_2) = (1, 2)$
3. $(t_1, t_3) = (2, 1)$
4. $(t_2, t_3) = (1, 1)$
5. $(t_3) = (5)$
6. $(t_1, t_4) = (1, 1)$
7. $(t_3, t_4) = (1, 3)$
8. $(t_5) = (1)$
9. $(t_3, t_6) = (1, 2)$
10. $(t_1, t_6) = (1, 4)$
11. $(t_2, t_7) = (4, 1)$
12. $(t_4, t_7) = (2, 1)$
13. $(t_2, t_6, t_7) = (1, 1, 1)$
14. $(t_6, t_7) = (3, 1)$
15. $(t_1, t_7) = (1, 2)$
16. $(t_7) = (5)$
17. $(t_3, t_4, t_8) = (1, 1, 1)$
18. $(t_1, t_6, t_8) = (1, 1, 1)$
19. $(t_7, t_8) = (1, 1)$
20. $(t_1, t_8) = (1, 3)$
21. $(t_3, t_8) = (1, 4)$
22. $(t_2, t_9) = (3, 1)$
23. $(t_3, t_9) = (2, 1)$
24. $(t_2, t_4, t_9) = (1, 1, 1)$
25. $(t_4, t_9) = (4, 1)$
26. $(t_6, t_9) = (1, 1)$
27. $(t_8, t_9) = (2, 1)$
28. $(t_7, t_9) = (1, 2)$
29. $(t_9) = (5)$

INCIDENCE MATRIX $P(\mathscr{G}_{10}, (5))$

Face	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
Vertex																									
1.	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	1	1	1	1	1	1	1	1
2.	1	0	0	1	1	0	0	1	0	1	0	0	1	1	0	0	0	0	1	1	1	1	1	1	1
3.	0	1	0	1	0	1	0	1	1	1	0	1	0	1	0	0	0	1	0	1	1	1	1	1	1
4.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1
5.	0	0	0	0	0	1	1	0	0	0	0	1	0	0	0	1	1	1	0	1	1	1	1	1	1
6.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	0	1	1	1	1	1
7.	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	1	1	1	1	1
8.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1
9.	1	0	0	0	0	1	1	1	0	0	0	1	0	0	1	1	1	1	0	1	1	0	1	1	1
10.	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	0	1	1	1
11.	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1	1	0	1	1
12.	1	0	1	0	1	0	1	0	0	0	1	0	0	0	1	1	1	1	1	0	1	1	0	1	1
13.	1	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	1	0	1	1	1	0	0	1	1
14.	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1	1	1	1	1	0	0	1	1
15.	0	1	0	0	1	0	1	1	0	1	1	1	0	0	1	0	0	1	1	1	1	1	0	1	1
16.	0	0	0	0	1	0	1	0	0	0	1	0	0	0	1	0	1	1	1	1	1	1	0	1	1
17.	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	0	0	1	1	1	0	1
18.	1	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0	0	1	1	1	1	0	1	0	1
19.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	1
20.	1	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	1	1	1	1	1	0	1
21.	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	1	1	0	1
22.	1	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0	1	1	1	1	1	1	0
23.	0	1	1	0	0	1	1	0	1	0	1	1	0	0	0	1	1	1	0	1	1	1	1	1	0
24.	1	0	1	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0	1	1	1	1	0
25.	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	1	1	1	1	0
26.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	0
27.	1	0	1	1	0	1	0	0	1	0	0	0	1	1	0	0	1	1	1	1	1	1	1	0	0
28.	0	1	1	1	1	0	0	0	1	1	1	0	1	0	0	0	1	1	1	1	1	1	0	1	0
29.	0	0	1	1	0	0	0	0	1	0	0	0	1	0	0	0	1	1	1	1	1	1	1	1	0

$P(\mathscr{G}_{10}, (8))$

FACES

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8^*	π_9	π_0
Row										
1	3	1	4	2	0	3	1	4	2	4
2	2	4	6	3	0	2	4	6	3	6
3	2	4	1	3	5	2	4	6	3	6
4	6	2	3	4	5	6	2	8	4	8
5	1	2	3	4	5	6	7	8	4	8
6	9	8	7	6	5	4	3	12	6	12
7	9	8	2	6	10	4	3	12	6	12

VERTICES

1. $(t_1) = (8)$
2. $(t_2) = (4)$
3. $(t_1, t_2) = (2, 2)$
4. $(t_2, t_3) = (1, 2)$
5. $(t_3) = (6)$
6. $(t_4) = (2)$
7. $(t_1, t_5) = (3, 1)$
8. $(t_1, t_2, t_5) = (1, 1, 1)$
9. $(t_3, t_5) = (1, 1)$
10. $(t_1, t_6) = (2, 1)$
11. $(t_2, t_6) = (1, 1)$
12. $(t_6) = (3)$
13. $(t_1, t_7) = (1, 1)$
14. $(t_5, t_6, t_7) = (1, 1, 1)$
15. $(t_7) = (4)$
16. $(t_8) = (1)$
17. $(t_9) = (2)$

INCIDENCE MATRIX $P(\mathscr{G}_{10}, (8))$

Face / Vertex	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1.	0	0	0	0	1	0	0	0	1	1	1	1	1	1	1	1
2.	1	0	0	1	1	0	0	1	0	1	1	1	1	1	1	1
3.	0	0	1	0	1	0	0	0	1	0	1	1	1	1	1	1
4.	0	0	1	1	1	0	1	1	0	0	1	1	1	1	1	1
5.	0	0	1	0	0	0	1	1	1	0	1	1	1	1	1	1
6.	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1
7.	0	1	0	0	1	0	0	0	1	1	1	0	1	1	1	1
8.	1	1	0	0	1	0	0	0	0	1	1	0	1	1	1	1
9.	1	1	1	1	1	1	1	1	1	0	1	0	1	1	1	1
10.	0	1	1	0	1	0	0	0	1	1	1	1	0	1	1	1
11.	1	1	1	1	1	1	1	1	0	1	1	1	0	1	1	1
12.	0	1	1	0	0	1	1	1	1	1	1	1	0	1	1	1
13.	1	1	1	1	1	1	1	0	1	1	1	1	1	0	1	1
14.	1	1	0	0	0	1	0	1	1	1	1	0	0	0	1	1
15.	1	0	0	1	0	1	1	1	1	1	1	1	1	0	1	1
16.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1
17.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0

$P(\mathscr{G}_{10}, (9))$

FACES

Row	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_0
1	1	0	1	0	1	0	1	0	1	1
2	1	2	1	2	1	2	1	2	3	3
3	1	2	3	4	0	1	2	3	4	4
4	4	3	2	6	0	4	3	2	6	6
5	4	3	2	1	5	4	3	2	6	6
6	2	4	6	3	5	2	4	6	8	8
7	6	7	3	4	5	6	2	3	9	9
8	6	2	3	4	5	6	7	3	9	9

FACES (continued)

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_0
9	1	2	3	4	5	6	7	8	9	9
10	9	8	7	6	5	4	3	2	11	11
11	6	12	8	4	10	6	2	8	14	14
12	9	18	7	6	15	14	3	12	21	21

VERTICES $P(\mathscr{G}_{10}, (9))$

1. $(t_1) = (9)$
2. $(t_1, t_2) = (1, 4)$
3. $(t_2, t_3) = (3, 1)$
4. $(t_3) = (3)$
5. $(t_2, t_3, t_4) = (1, 1, 1)$
6. $(t_1, t_4) = (1, 2)$
7. $(t_3, t_4) = (1, 4)$
8. $(t_1, t_5) = (4, 1)$
9. $(t_2, t_5) = (2, 1)$
10. $(t_1, t_3, t_5) = (1, 1, 1)$
11. $(t_4, t_5) = (1, 1)$
12. $(t_1, t_6) = (3, 1)$
13. $(t_1, t_2, t_6) = (1, 1, 1)$
14. $(t_3, t_6) = (1, 1)$
15. $(t_4, t_6) = (1, 3)$
16. $(t_5, t_6) = (1, 4)$
17. $(t_1, t_7) = (2, 1)$
18. $(t_2, t_7) = (1, 1)$
19. $(t_4, t_7) = (3, 1)$
20. $(t_6, t_7) = (2, 1)$
21. $(t_5, t_7) = (1, 2)$
22. $(t_1, t_7) = (1, 4)$
23. $(t_7) = (7)$
24. $(t_1, t_8) = (1, 1)$
25. $(t_5, t_6, t_8) = (1, 1, 1)$
26. $(t_4, t_7, t_8) = (1, 1, 1)$
27. $(t_7, t_8) = (3, 1)$
28. $(t_3, t_8) = (1, 2)$
29. $(t_5, t_8) = (1, 3)$
30. $(t_7, t_8) = (1, 4)$
31. $(t_9) = (1)$

INCIDENCE MATRIX $P(\mathscr{G}_{10}, (9))$

Face	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Vertex																					
1.	0	0	0	0	0	0	0	0	1	0	0	0	0	1	1	1	1	1	1	1	1
2.	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1
3.	1	0	0	0	0	0	0	1	1	0	0	0	1	0	0	1	1	1	1	1	1
4.	0	1	0	1	1	0	1	1	1	0	0	1	1	1	0	1	1	1	1	1	1
5.	1	0	0	0	1	0	0	1	1	0	0	0	1	0	0	0	1	1	1	1	1
6.	1	0	0	0	1	1	0	0	1	0	1	1	0	1	1	0	1	1	1	1	1
7.	1	0	0	0	1	0	0	0	0	0	0	0	1	1	0	0	1	1	1	1	1
8.	0	0	1	0	0	0	0	0	1	0	0	0	0	1	1	1	0	1	1	1	1
9.	1	0	1	1	0	0	0	1	1	0	0	0	1	0	1	1	0	1	1	1	1
10.	0	1	1	1	0	0	0	0	1	0	0	0	0	1	0	1	0	1	1	1	1
11.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	1	1	1	1
12.	0	0	1	0	0	1	0	0	1	0	0	0	0	1	1	1	1	0	1	1	1
13.	1	0	1	0	0	1	0	0	1	0	0	0	0	0	1	1	1	0	1	1	1
14.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	0	1	1	1
15.	1	0	1	0	0	1	0	0	0	0	0	0	0	1	1	1	1	0	1	1	1
16.	1	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	1	1	1
17.	0	1	1	0	0	1	0	0	1	0	1	1	0	1	1	1	1	1	0	1	1
18.	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	0	1	1
19.	1	0	0	0	1	0	0	0	0	0	1	1	1	1	1	0	1	1	0	1	1
20.	1	0	1	0	0	1	0	0	0	1	1	0	1	1	1	1	1	0	0	1	1

INCIDENCE MATRIX $P(\mathscr{G}_{13}, (9))$ (continued)

Face	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
21.	0	1	1	1	0	0	1	0	0	1	1	1	1	1	1	1	0	1	0	1	1
22.	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	1	1	1	0	1	1
23.	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	1	1
24.	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	0	1
25.	1	0	1	1	0	0	0	0	0	1	0	0	1	1	1	1	0	0	1	0	1
26.	1	0	0	0	1	0	1	0	0	1	1	1	1	1	1	0	1	1	0	0	1
27.	0	0	0	0	0	0	1	0	0	1	1	1	1	1	1	1	1	1	0	0	1
28.	1	0	0	1	1	0	1	1	0	1	0	0	1	1	0	1	1	1	1	0	1
29.	1	0	0	1	0	0	0	0	0	1	0	0	1	1	1	1	0	1	1	0	1
30.	1	0	0	0	0	0	0	0	0	1	0	0	1	1	1	1	1	1	0	0	1
31.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0

$P(\mathscr{G}_{11}, (0))$

FACES

Row	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_0
1	2	1	2	2	2	1	1	1	2	1	3
2	1	2	1	1	1	2	2	2	1	2	3
3	10	9	8	7	6	5	4	3	2	1	11
4	9	7	5	3	1	10	8	6	4	2	11
5	8	5	2	10	7	4	1	9	6	3	11
6	7	3	10	6	2	9	5	1	8	4	11
7	6	1	7	2	8	3	9	4	10	5	11
8	5	10	4	9	3	8	2	7	1	6	11
9	4	8	1	5	9	2	6	10	3	7	11
10	3	6	9	1	4	7	10	2	5	8	11
11	2	4	6	8	10	1	3	5	7	9	11
12	1	2	3	4	5	6	7	8	9	10	11
13	18	14	10	6	13	9	16	12	8	4	22
14	16	10	4	9	14	8	13	18	12	6	22
15	14	6	9	12	4	18	10	13	16	8	22
16	13	4	6	8	10	12	14	16	18	9	22
17	12	13	14	4	16	6	18	8	9	10	22
18	10	9	8	18	6	16	4	14	13	12	22
19	9	18	16	14	12	10	8	6	4	13	22
20	8	16	13	10	18	4	12	9	6	14	22
21	6	12	18	13	8	14	9	4	10	16	22
22	4	8	12	16	9	13	6	10	14	18	22

VERTICES $P(\mathscr{G}_{11}, (0))$

1. $(t_1) = (11)$
2. $(t_1, t_2) = (1, 5)$
3. $(t_2) = (11)$
4. $(t_2, t_3) = (4, 1)$
5. $(t_1, t_3) = (2, 3)$
6. $(t_2, t_3) = (1, 3)$
7. $(t_3) = (11)$
8. $(t_1, t_4) = (3, 2)$
9. $(t_1, t_2, t_4) = (1, 1, 2)$
10. $(t_3, t_4) = (1, 2)$
11. $(t_2, t_4) = (1, 5)$
12. $(t_4) = (11)$
13. $(t_2, t_5) = (3, 1)$
14. $(t_3, t_5) = (2, 1)$
15. $(t_2, t_4, t_5) = (1, 1, 1)$
16. $(t_1, t_5) = (1, 2)$
17. $(t_4, t_5) = (3, 2)$
18. $(t_2, t_5) = (1, 4)$
19. $(t_5) = (11)$
20. $(t_1, t_6) = (5, 1)$
21. $(t_1, t_2, t_6) = (1, 2, 1)$
22. $(t_1, t_3, t_6) = (2, 1, 1)$
23. $(t_2, t_3, t_6) = (1, 1, 1)$
24. $(t_1, t_4, t_6) = (1, 1, 1)$
25. $(t_4, t_6) = (4, 1)$
26. $(t_5, t_6) = (1, 1)$
27. $(t_2, t_6) = (2, 3)$
28. $(t_4, t_6) = (1, 3)$
29. $(t_3, t_6) = (1, 5)$
30. $(t_6) = (11)$
31. $(t_1, t_7) = (4, 1)$
32. $(t_2, t_7) = (2, 1)$
33. $(t_1, t_3, t_7) = (1, 1, 1)$
34. $(t_3, t_7) = (5, 1)$
35. $(t_4, t_7) = (1, 1)$
36. $(t_5, t_7) = (3, 1)$
37. $(t_3, t_6, t_7) = (1, 2, 1)$
38. $(t_1, t_7) = (1, 3)$
39. $(t_6, t_7) = (2, 3)$
40. $(t_5, t_7) = (1, 4)$
41. $(t_7) = (11)$
42. $(t_1, t_8) = (3, 1)$
43. $(t_1, t_2, t_8) = (1, 1, 1)$
44. $(t_3, t_8) = (1, 1)$
45. $(t_4, t_5, t_8) = (1, 2, 1)$
46. $(t_5, t_8) = (5, 1)$
47. $(t_7, t_8) = (2, 1)$
48. $(t_2, t_8) = (3, 2)$
49. $(t_2, t_4, t_8) = (1, 1, 2)$
50. $(t_6, t_8) = (1, 2)$
51. $(t_1, t_8) = (1, 4)$
52. $(t_4, t_8) = (1, 5)$
53. $(t_8) = (11)$
54. $(t_1, t_9) = (2, 1)$
55. $(t_2, t_9) = (1, 1)$
56. $(t_6, t_9) = (4, 1)$
57. $(t_3, t_7, t_9) = (2, 1, 1)$
58. $(t_6, t_7, t_9) = (1, 1, 1)$
59. $(t_7, t_9) = (5, 1)$
60. $(t_5, t_8, t_9) = (1, 1, 1)$
61. $(t_8, t_9) = (3, 1)$
62. $(t_4, t_9) = (1, 2)$
63. $(t_5, t_9) = (3, 2)$
64. $(t_3, t_9) = (2, 3)$
65. $(t_6, t_9) = (1, 3)$
66. $(t_8, t_9) = (1, 4)$
67. $(t_9) = (11)$
68. $(t_1, t_{10}) = (1, 1)$
69. $(t_3, t_{10}) = (4, 1)$
70. $(t_4, t_{10}) = (3, 1)$
71. $(t_6, t_{10}) = (2, 1)$
72. $(t_5, t_7, t_{10}) = (1, 1, 1)$
73. $(t_4, t_8, t_{10}) = (1, 1, 1)$
74. $(t_3, t_9, t_{10}) = (1, 1, 1)$
75. $(t_7, t_9, t_{10}) = (2, 1, 1)$
76. $(t_5, t_9, t_{10}) = (1, 2, 1)$
77. $(t_9, t_{10}) = (5, 1)$
78. $(t_2, t_{10}) = (1, 2)$
79. $(t_5, t_8, t_{10}) = (1, 1, 2)$
80. $(t_8, t_{10}) = (3, 2)$
81. $(t_3, t_{10}) = (1, 3)$
82. $(t_7, t_{10}) = (2, 3)$
83. $(t_4, t_{10}) = (1, 4)$
84. $(t_5, t_{10}) = (1, 5)$
85. $(t_{10}) = (11)$

INCIDENCE MATRIX $P(\mathscr{G}_{11}, (0))$

Face	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
Vertex																																
1.	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
2.	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
3.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1	1	1	1	1	1
4.	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0	1	1	1	1	1	1	1
5.	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1	1	1	1	1
6.	0	0	0	0	1	0	0	0	1	0	0	1	0	1	0	1	0	0	0	0	0	0	1	0	0	1	1	1	1	1	1	1
7.	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	1	1	1	1	1
8.	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	1	1	1	1
9.	0	0	0	0	0	0	1	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1	1	1	1
10.	0	1	0	1	0	0	1	0	1	1	0	1	1	1	0	1	1	0	0	0	0	0	1	1	0	0	1	1	1	1	1	1
11.	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	1	1	1	1	1
12.	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	1	1	1	1	1	1
13.	0	0	0	0	0	1	1	0	0	0	0	1	0	0	1	1	0	0	0	0	0	0	1	0	1	1	0	1	1	1	1	1
14.	0	1	0	1	1	0	0	1	1	0	0	1	0	1	1	1	0	1	0	0	0	0	1	1	0	1	0	1	1	1	1	1
15.	0	0	0	1	0	1	1	0	0	1	0	1	0	0	1	1	0	0	0	0	0	0	1	0	1	0	0	1	1	1	1	1
16.	0	1	0	1	0	1	0	1	0	1	0	1	0	0	1	0	0	1	0	0	1	1	0	1	1	1	0	1	1	1	1	1
17.	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	1	1	1	1	1
18.	0	0	0	1	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1	1	0	1	1	1	1	1
19.	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	1	1	1	1	1
20.	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	1	1	1	1
21.	0	0	0	0	0	0	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	1	1	1	1
22.	0	0	0	0	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0	1	1	1	1
23.	0	0	0	0	1	0	1	0	1	0	1	1	0	1	0	1	0	0	0	0	0	0	1	0	0	1	1	0	1	1	1	1

continued on following page

INCIDENCE MATRIX $P(\mathscr{G}_{11}, (0))$ CONTINUED

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
24.	0	0	0	0	0	0	1	0	1	1	1	1	0	0	0	0	1	0	0	1	0	0	0	1	1	0	1	0	1	1	1	1
25.	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	1	1	1	0	1	0	1	1	1	1
26.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	1	1	1	1
27.	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1	0	1	1	1	1
28.	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0	1	0	0	1	0	0	1	1	1	0	1	0	1	1	1	1
29.	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	0	1	1	1	1
30.	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	1	1	1	1
31.	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	1	0	1	1	1	1	1	0	1	1	1
32.	1	0	0	0	1	1	1	0	0	0	1	1	0	0	1	1	0	1	0	0	0	1	1	0	1	1	1	1	0	1	1	1
33.	0	0	0	0	1	0	0	1	1	0	1	1	0	0	0	0	0	1	0	0	0	1	0	1	0	1	1	1	0	1	1	1
34.	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	1	0	1	1	1
35.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	0	1	1	1
36.	0	0	0	1	0	1	0	1	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	1	1	1	0	1	0	1	1	1
37.	0	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	0	0	1	1	1
38.	0	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	1	1	1	1	1	0	1	1	1
39.	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	1	1	1
40.	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	1	1	1	0	1	0	1	1	1
41.	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	1	1	1
42.	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	1	1	0	1	1	1	1	1	1	0	1	1
43.	0	0	0	0	0	1	1	0	0	1	1	1	0	0	0	0	0	0	0	0	1	1	0	0	1	1	1	1	1	0	1	1
44.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	0	1	1
45.	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	1	1	0	1	1
46.	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	1	1	0	1	1
47.	1	0	1	0	1	1	0	1	0	0	1	0	0	0	0	0	0	1	1	0	1	1	1	1	1	1	1	1	0	0	1	1

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INCIDENCE MATRIX $P(\mathscr{G}_{11}, (0))$ CONTINUED

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
48.	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1	1	1	0	1	1
49.	0	0	0	0	0	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	1	1	0	1	1
50.	1	0	1	0	0	1	1	0	0	1	1	0	0	0	0	0	1	0	1	1	1	0	1	1	1	1	1	0	1	0	1	1
51.	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	1	1	1	1	0	1	1
52.	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	1	1	1	0	1	1
53.	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	1	1
54.	0	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	1	1	1	1	0	1	1	1	1	1	1	1	0	1
55.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	0	1
56.	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	1	0	0	1	1	1	1	1	0	1	1	0	1
57.	0	0	0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	1	0	1	0	1
58.	0	0	1	0	1	0	0	1	1	0	1	0	0	0	0	0	0	0	1	1	0	0	1	1	1	1	1	0	0	1	0	1
59.	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	1	0	1
60.	0	0	1	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	1	0	1	0	1	1	1	1	0	1	1	0	0	1
61.	0	0	1	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	1	0	1	0	1	1	1	1	1	1	1	0	0	1
62.	0	1	1	1	0	0	0	1	1	1	0	0	1	0	0	0	1	0	1	1	0	0	1	1	1	0	1	1	1	1	0	1
63.	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	1	1	1	0	1
64.	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	1	1	1	0	1
65.	0	0	1	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1	1	1	1	0	1	1	0	1
66.	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1	1	1	1	1	1	0	0	1
67.	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	1
68.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	0
69.	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	1	1	0	1	1	1	1	1	1	0
70.	0	0	0	1	0	0	1	0	0	1	0	0	1	0	0	0	1	0	0	0	0	0	1	1	1	0	1	1	1	1	1	0
71.	1	0	1	0	1	0	1	0	1	0	1	0	1	1	0	0	1	0	0	1	0	0	1	1	1	1	1	0	1	1	1	0

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INCIDENCE MATRIX $P(\mathscr{G}_{11}, (0))$ CONTINUED

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
72.	0	0	1	1	1	1	0	1	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	1	1	1	0	1	0	1	1	0
73.	0	0	1	1	0	1	1	0	0	1	0	0	1	0	0	0	1	0	0	0	0	0	1	1	1	0	1	1	1	0	1	0
74.	0	0	1	1	1	0	0	1	1	0	0	0	1	1	0	0	0	0	0	0	0	0	1	1	0	1	1	1	1	1	0	0
75.	0	0	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	1	0	0
76.	0	0	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	1	1	1	0	0
77.	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0
78.	1	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	1	0	1	1	1	1	1	1	1	0
79.	0	0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	1	1	0	1	0
80.	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	1	0
81.	0	0	1	1	1	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	1	1	0	1	1	1	1	1	1	0
82.	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	1	1	0
83.	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	0	1	1	1	1	1	0
84.	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	1	1	1	1	0
85.	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0

$P(\mathscr{G}_{11}, (10))$

FACES*

Row	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	$\overset{*}{\pi_{10}}$	π_0
1	1	2	3	4	5	6	7	8	9	10	10
2	10	9	8	7	6	5	4	3	2	12	12
3	8	5	2	10	7	4	12	9	6	14	14
4	6	12	7	13	8	3	9	4	10	16	16
5	6	12	7	2	8	14	9	4	10	16	16
6	15	8	12	5	9	13	6	10	3	18	18
7	4	8	12	16	9	2	6	10	14	18	18
8	4	8	12	5	9	13	6	10	14	18	18
9	13	4	6	8	10	12	14	16	7	20	20
10	13	15	6	8	10	12	14	5	7	20	20
11	20	7	16	14	12	10	8	17	4	24	24
12	9	18	16	14	12	10	8	6	15	24	24
13	9	18	16	3	12	21	8	6	15	24	24
14	9	18	5	14	12	10	19	6	15	24	24
15	18	14	10	6	13	20	16	12	8	26	26
16	16	21	4	20	14	8	24	7	12	28	28
17	25	6	20	12	15	18	10	24	5	30	30
18	14	6	20	12	15	18	10	24	16	30	30

VERTICES $P(\mathscr{G}_{11}, (10))$

1. $(t_1) = (10)$
2. $(t_2) = (5)$
3. $(t_2, t_3) = (2, 2)$
4. $(t_1, t_3) = (1, 3)$
5. $(t_3) = (7)$
6. $(t_3, t_4) = (2, 1)$
7. $(t_1, t_4) = (2, 2)$
8. $(t_2, t_4) = (1, 2)$
9. $(t_4) = (8)$
10. $(t_5) = (2)$
11. $(t_1, t_6) = (4, 1)$
12. $(t_2, t_6) = (2, 1)$
13. $(t_1, t_3, t_6) = (1, 1, 1)$
14. $(t_4, t_6) = (1, 1)$
15. $(t_3, t_6) = (1, 3)$
16. $(t_6) = (9)$
17. $(t_1, t_7) = (3, 1)$
18. $(t_1, t_2, t_7) = (1, 1, 1)$
19. $(t_3, t_7) = (1, 1)$
20. $(t_7) = (3)$
21. $(t_1, t_8) = (2, 1)$
22. $(t_2, t_8) = (1, 1)$
23. $(t_6, t_8) = (4, 1)$
24. $(t_6, t_7, t_8) = (1, 1, 1)$
25. $(t_8) = (4)$
26. $(t_1, t_9) = (1, 1)$
27. $(t_1, t_9) = (3, 1)$
28. $(t_6, t_9) = (2, 1)$
29. $(t_1, t_8, t_9) = (1, 1, 1)$
30. $(t_3, t_9) = (1, 2)$
31. $(t_5, t_9) = (1, 3)$
32. $(t_9) = (6)$
33. $(t_{10}) = (1)$

* I would like to thank Fred Glover and Ellis Johnson for supplying three faces of $P(\mathscr{G}_{11}, (10))$ that were missing in earlier versions of this paper.

INCIDENCE MATRIX $P(\mathscr{G}_{11}, (10))$

Face	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Vertex																												
1.	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
2.	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	0	1	1	1	1	1	1	1	1
3.	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	1	1	1	1	1	1	1
4.	1	0	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1	0	1	1	1	1	1	1	1
5.	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	1	1	1	1	1	1	1
6.	1	0	1	0	1	0	0	0	1	1	0	0	0	1	1	1	0	0	1	1	0	0	1	1	1	1	1	1
7.	1	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	0	0	0	1	1	0	1	1	1	1	1	1
8.	1	0	0	0	1	1	0	1	1	0	0	0	1	0	1	0	1	1	1	0	1	0	1	1	1	1	1	1
9.	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	1	0	1	1	1	1	1	1
10.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1
11.	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	1	1	1	1
12.	1	0	1	0	0	0	1	0	1	0	1	0	0	0	0	0	1	1	1	0	1	1	1	0	1	1	1	1
13.	1	0	1	1	0	0	1	0	0	0	0	0	0	1	1	1	0	0	0	1	0	1	1	0	1	1	1	1
14.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	0	1	1	1	1
15.	0	0	1	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1	1	0	1	1	0	1	1	1	1
16.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	1	1	1	1
17.	1	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	1	1	1
18.	1	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	1	0	0	1	1	1	1	0	1	1	1
19.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	0	1	1	1
20.	0	1	0	0	0	1	1	1	0	0	1	1	1	0	0	0	1	1	1	1	1	1	1	1	0	1	1	1
21.	1	0	0	1	1	0	1	1	0	0	0	1	1	1	0	0	0	0	0	1	1	1	1	1	1	0	1	1
22.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	0	1	1
23.	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	1	0	1	1

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INCIDENCE MATRIX $P(\mathscr{G}_{11}, (10))$ CONTINUED

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
24.	0	1	0	1	0	0	1	0	0	0	0	1	0	0	0	0	0	0	1	1	1	1	1	0	0	0	1	1
25.	0	1	0	1	1	0	0	0	0	1	0	1	1	1	0	1	0	0	1	1	1	1	1	1	1	0	1	1
26.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	0	1
27.	0	0	0	0	1	1	0	0	0	0	0	0	1	0	1	0	0	0	1	1	1	0	1	1	1	1	0	1
28.	0	1	1	1	0	0	1	0	0	0	1	0	0	0	0	1	0	0	1	1	1	1	1	0	1	1	0	1
29.	0	1	0	0	1	1	0	0	0	1	0	0	1	0	0	0	0	0	1	1	1	0	1	1	1	0	0	1
30.	0	1	1	0	0	1	0	0	1	1	1	0	0	0	1	1	1	0	1	1	0	1	1	1	1	1	0	1
31.	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	1	1	1	1	0	1	1	1	0	1
32.	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	1	1	1	1	1	1	1	1	0	1
33.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0

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