INTEGER AND NONLINEAR PROGRAMMING

Editor

J. ABADIE
Electricité de France, Paris and
Institut de Statistique de l'Université de Paris

PROPERTIES OF A CLASS OF INTEGER POLYHEDRA

R. E. GOMORY

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R. E. GOMORY

IBM Watson Research Center, Yorktown Heights, New York, USA

1. Introduction

Very little is known at present about the polyhedra underlying integer programming. This contrasts with the situation in linear programming where there is a clear-cut connection between the data of the problem and the structure of the feasible polyhedra. In linear programming, the feasible polyhedron is given by a system of inequalities

$$A'x' \leq b \qquad x' \geq 0 \tag{1}$$

and each of these inequalities corresponds to a face of the feasible polyhedron.* In integer programming, we are interested only in the integer points x' satisfying (1). Of course, these points are a discrete set; however, from this discrete set we can form a new polyhedron P by taking the convex hull. It is the vertices (all of them integer points) of this polyhedron that are solutions to an integer programming maximization problem. However, almost nothing is known about this polyhedron. There are a few exceptions. In transportation and assignment problems, P coincides with the feasible region of (1) and for matrices A' corresponding to graph covering problems P has been characterized by Edmonds [1965]. There are a few other cases, all quite special in nature, but in general, almost nothing is known even about such apparently simple questions as the number of faces of P. In this paper, we discuss a class of polyhedra related to P but whose properties are much more accessible.

In general, in this paper we give theorems without proofs. Proofs and more complete statements can be found in Gomory [1969].

The polyhedra we describe are related to the underlying polyhedron P and to the integer programming problem in three ways:

* A' is assumed to be an integer matrix and b an integer vector.

(1) Under certain circumstances, the polyhedra coincide with part of P;
(2) The polyhedra provide the complete solution of the asymptotic integer programming problem;
(3) The faces of these new polyhedra provide what are, in some sense, best possible cutting planes for integer programming.

2. The polyhedra P^*

Let us consider the integer programming problem associated with the inequalities (1) which can be written in equality form as

$$Ax = b \qquad x \geq 0$$

where the x now includes the slacks. Associated with a vertex V of the linear programming problem is a basis B and we can imagine that the matrix A, which we will take to be $m \times (m+n)$, has been partitioned into its basic and nonbasic parts so that $A = (B, N)$. Similarly, we will partition x into x_B and x_N with $x = (x_B, x_N)$. The linear programming equations then are

$$Bx_B + Nx_N = b \qquad x_B \geq 0, x_N \geq 0 .$$

If we now drop the condition $x_B \geq 0$, we replace the feasible region of the linear programming problem by a cone C whose vertex is at V, and if we ask for integer solutions to the system

$$Bx_B + Nx_N = b\,, \quad x_N \geq 0\,, \tag{2}$$

we get a set of points whose convex hull we will call P^*. P^* is the polyhedron we will investigate. As Fig. 1 indicates, P^* can be closely related

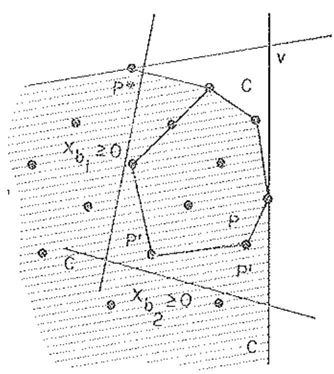

Fig. 1. P^* is shaded region.

to part of P; however, we will see that we can find out a good deal more about P^*.

The necessary and sufficient condition for an integer point x to belong to P^* is that it should be integer and should satisfy (2); however, since there is no non-negativity restriction on the components of x_B, satisfying (2) means merely that

$$N x_N \equiv b \bmod (B).$$

That is to say, that the left- and right-hand sides can differ by any vector that is an integer combination of the columns of B. The columns of N as well as the right-hand side b can be replaced by elements that are equivalent to these mod(B). More precisely, we can replace the equation by one involving only elements of the factor group $\mathscr{G} = M(I)/M(B)$. That is to say, the factor group of all integer vectors modulo the lattice formed from the columns of B. Each column of N corresponds to some element of this group as does b.* Thus, if we number the components of x_N from $1, \ldots, n$ and call the group element, corresponding to the ith column of N, g_i, we can rewrite the equation as

$$\sum_{i=1}^{i=n} g_i x_i = g_0 \tag{3}$$

where g_0 is the group element corresponding to the right-hand side element b.** Any non-negative integer solution to (3) can be extended to an integer

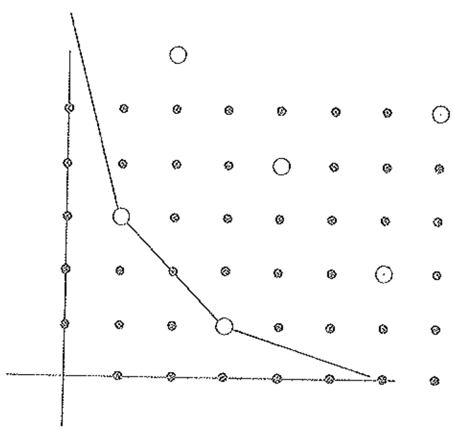

Fig. 2.

* It is important to realize that the corresponding group element can be computed explicitly by standard methods.

** In this paper we assume g_0 is not the zero element $\bar{0}$ of $\mathscr{G}$.

solution to (2) and, hence, gives an integer point of P^*. It is only necessary to take the x_i, $i=1,\ldots,n$, to be the x_N in (2) and choose $x_B=B^{-1}(b-Nx_N)$ which will necessarily turn out to be integer. Since the x_N alone determine the polyhedron P^*, it is convenient to work with these variables alone and to discuss P^* in terms of these coordinates (the nonbasic variables). This is essentially viewing P^* from a coordinate system based on the vertex V. In these coordinates P^* is the convex hull of those first quadrant integer points that satisfy (3). P^* in x_N space is illustrated in Fig. 2.

3. Faces of P^*

We first turn our attention to the faces of P^*. In x_N space, what is a face of P^*? First of all, it is an inequality

$$\sum_{i=1}^{i=n} \pi_i x_i \geq \pi_0 . \tag{4}$$

We will denote such an inequality by (π, π_0). If (π, π_0) is to be a face, we must have all points of P^* lying on one side of it. For this, it is clearly necessary and sufficient that (4) hold for all non-negative integer vectors satisfying (3), the group equation. In addition, there must be a set of n vectors $x^1,\ldots,x^n$ each of which satisfies the group equation and for which $\sum \pi_i x_i^j=\pi_0$. The set of vectors must be of rank n if $\pi_0\neq 0$ and of rank $n-1$ if $\pi_0=0$. Any π satisfying these conditions will be a face.

After these remarks, we can state a very simple theorem which connects the faces of P^* with ordinary linear programming.

THEOREM 1: The inequality (π, π_0) is a face of P^* if and only if it is a basic feasible solution to the system of inequalities

$$\sum \pi_i x_i \geq \pi_0 \tag{5}$$

formed from all x_i that are solutions to the group equation (3).

In the system (5) we can, of course, take $\pi_0=1$ with no loss of generality and, in addition, consider only group equation solutions in which $x_i\leq|\mathscr{G}|$, the order of the group $\mathscr{G}$. Since $|\mathscr{G}|$ is known to be $|\det B|$, this means (5) becomes a finite list of inequalities. Thus, we have connected the faces of the integer polyhedron P^* with the basic feasible solutions (or vertices) of an ordinary linear programming problem. In the above theorem, we have assumed $\pi_0>0$. It is not hard to prove Theorem 2.

THEOREM 2: The only possible faces (π, π_0) of P^* with $\pi_0=0$ are the n hyperplanes $x_i=0$.

We next turn to some of the properties of the π_i making up a face. Each π_i is associated with a group element g_i which is an element of the finite Abelian group $\mathscr{G}$. We have the following properties for the π_i.

THEOREM 3: If $\pi_0 \neq 0$, then

(1) $\pi_i \geq 0$ all i;

(2) if $g_i + g_j = g_k$, then $\pi_i + \pi_j \geq \pi_k$;

(3) if $g_i + g_j = g_0$, then $\pi_i + \pi_j = \pi_0$.

These properties (not proved here), which connect the π_i with the relations between their corresponding group elements, prove to be especially significant when dealing with the larger polyhedra introduced below.

Although Theorem 1 connects faces with basic feasible solution of systems of simultaneous equations, it was also shown in Gomory [1969] that there are always some faces of P^* that can be computed by a recursive or dynamic programming calculation. In this calculation, the π_i are computed one at a time with the value of each π_i depending only on those already computed. The significance of this fact in our present context is that it asserts that some of the basic feasible solutions of (5) always correspond to triangular bases, a fact that is rather surprising. Recent work of Glover [1969] using a dynamic programming calculation has brought out the interesting fact that faces of this type not exist, but are very numerous.

4. A larger polyhedron

In our group equation (3), each variable x_i was associated with a group element g_i. In what follows, we will make the unnecessary, but simplifying, assumption that each x_i is associated with a distinct element g_i and that none of the g_i are $\bar{0}$, the zero of the group $\mathscr{G}$. In the group equation (3), only certain group elements appear; in fact, only those elements appear that correspond to nonbasic columns of the matrix A. We introduce a new higher dimensional polyhedron $P(\mathscr{G}, g_0)$ by means of the equation (6):

$$\sum_{i=1}^{i=|\mathscr{G}|-1} g_i x_i = g_0 \tag{6}$$

where a variable has now been introduced for every non-zero group element. The polyhedron $P(\mathscr{G}, g_0)$ is defined to be the convex hull of the non-negative integer solutions to (6). This is a polyhedron in $|\mathscr{G}|-1$ dimensional space. An equivalent and handier notation is one in which each variable is associated directly with the group element; that is to say, associate with each group element g the integer variable $t(g)$. The x_i of our previous notation is $t(g_i)$. In this notation, equation (6) becomes

$$\sum_{g \in \mathscr{G}^+} g t(g) = g_0 \tag{7}$$

where $\mathscr{G}^+$ stands for the set of all group elements excluding $\bar{0}$ and our old equation (3) becomes

$$\sum_{g \in \mathscr{N}} g t(g) = g_0$$

where $\mathscr{N}$ is the set of group elements corresponding to the non-basic columns.

We note that while P^* depended on $\mathscr{G}$, g_0 and $\mathscr{N}$, the polyhedron $P(\mathscr{G}, g_0)$ is determined by $\mathscr{G}$ and g_0 alone. The relation between the two is given by the following theorem.

THEOREM 4: $P^* = P(\mathscr{G}, g_0) \cap E_{\mathscr{N}}$.

Here, $E_{\mathscr{N}}$ is that subspace of our $|\mathscr{G}|-1$ dimensional space in which $t(g)=0$ for all components $t(g)$ with $g \notin \mathscr{N}$. The meaning of the theorem is that the various P^* possible for fixed group $\mathscr{G}$ and right-hand side g_0 are lower dimensional faces of the one polyhedron $P(\mathscr{G}, g_0)$.

This relationship has consequences which relate the faces and the vertices of $P(\mathscr{G}, g_0)$ with those of any P^*. We will now use the notation (π, π_0) to stand for a face of $P(\mathscr{G}, g_0)$; that is to say, the (π, π_0) are the coefficients in the inequality

$$\sum_{g \in \mathscr{G}^+} \pi(g)\, t(g) \geq \pi_0$$

which yields a face. A consequence of Theorem 4 is that if (π^i, π_0^i) is a complete list of the faces of $P(\mathscr{G}, g_0)$, then the inequalities

$$\sum_{g \in \mathscr{N}} \pi^i(g)\, t(g) \geq \pi_0^i$$

contain among them all the faces of P^* (together with some superfluous inequalities).

Similarly, the vertices of P^* can be found even more simply from the vertices of $P(\mathscr{G}, g_0)$.

One further fact about the $P(\mathscr{G}, g_0)$ is quite important.

THEOREM 5: If (π, π_0) is a face of $P(\mathscr{G}, g_0)$ with components $\pi(g)$, and $\phi: \mathscr{G} \to \mathscr{G}$ is any automorphism of $\mathscr{G}$, then $(\bar{\pi}, \pi_0)$ with components $\bar{\pi}(g) = \pi(\phi^{-1} g)$ is a face of $P(\mathscr{G}, \phi(g_0))$.

This means that two polyhedra $P(\mathscr{G}, g_0)$ and $P(\mathscr{G}, g_0')$ are identical, except for a renumbering of coordinates, as long as g_0 and g_0' are in the same automorphism class; i.e., as long as there is an automorphism ϕ from g_0 to g_0'. It means that there is essentially only one polyhedron for each automorphism class of each Abelian group $\mathscr{G}$; for example, there is only one polyhedron for each cyclic group of prime order.

The $P(\mathscr{I}, g_0)$ clearly have many symmetries lacking in the earlier P^*. This is reflected in the following theorem.

THEOREM 6: (π, π_0), $\pi_0 > 0$, is a face of the polyhedron $P(\mathscr{G}, g_0)$, $g_0 \neq \bar{0}$, if and only if it is a basic feasible solution to the system of equations and inequalities:

$$\begin{aligned} &\pi(g_0) = \pi_0 \\ &\pi(g) + \pi(g_0 - g) = \pi_0 && g \in \mathscr{G}^+, g \neq g_0 \\ &\pi(g) + \pi(g') \geq \pi(g + g') && g, g' \in \mathscr{G}^+ \\ &\pi(g) \geq 0 && g \in \mathscr{G}^+ . \end{aligned} \tag{8}$$

The necessity of these conditions on the π_i appeared earlier in connection with properties of the faces of P^*.

The equations and inequalities of this theorem can easily be written down explicitly. There will be roughly $\frac{1}{2}|\mathscr{G}|$ variables if the equalities are used to eliminate and there are about $\frac{1}{6}|\mathscr{G}|^2$ inequalities aside from the non-negativity conditions. These equations have been used to compute the $P(\mathscr{G}, g_0)$ explicitly for all $|\mathscr{G}| \leq 13$. The faces, vertices, and incidence matrices of the $P(\mathscr{G}, g_0)$ are given in Gomory [1969] for $|\mathscr{G}| \leq 11$ and the two distinct polyhedra associated with the cyclic group of order 9, ($P(\mathscr{G}_9, (8))$ and $P(\mathscr{G}_9, (6))$ are given in Table 1.

5. The graph $H(\mathscr{G}, \pi)$

We will next see that it is possible to create some (but by no means all) of the faces for any group polyhedron very easily and rapidly. To see this we introduce the graph $H(\mathscr{G}, \pi)$. This graph consists of

(i) a node for each group element;

(ii) a directed arc of length $\pi(h)$ from node g to node $g+h$ for each g and $h \in \mathscr{G}$.

In terms of this graph, we can see that any path from node $\bar{0}$ to node g_0 in the graph yields a solution to (7) and any solution to (7) gives several distinct paths from node $\bar{0}$ to node g_0, the different paths being formed by taking the variables in different orders. Now consider a set of arc lengths $\pi(h)$. If π_0 is the length of the shortest path from $\bar{0}$ to g_0 in h, then the inequality

$$\sum_{g \in \mathscr{G}^+} \pi(g)\, t(g) \geq \pi_0$$

must hold for all $t(g)$ that represent paths from $\bar{0}$ to g_0 and, hence, for all

TABLE 1

$P(\mathscr{G}_9, (6))$

Faces

Row	π_1	π_2	π_3	π_4	π_5	π_6^*	π_7	π_8	π_0
1	4	5	3	1	2	6	4	2	6
2	4	2	3	4	2	6	4	2	6
3	1	2	3	4	5	6	4	2	6
4	2	4	3	2	4	6	2	4	6
5	4	2	3	4	2	6	1	5	6
6	5	10	6	2	7	12	8	4	12
7	2	4	6	8	10	12	5	7	12
8	8	7	6	5	4	12	2	10	12

Vertices

1. $(t_1) = (6)$
2. $(t_2) = (3)$
3. $(t_3) = (2)$
4. $(t_1, t_4) = (2, 1)$
5. $(t_2, t_4) = (1, 1)$
6. $(t_3, t_4) = (1, 3)$
7. $(t_4) = (6)$
8. $(t_1, t_5) = (1, 1)$
9. $(t_5) = (3)$
10. $(t_6) = (1)$
11. $(t_4, t_7) = (2, 1)$
12. $(t_1, t_7) = (1, 2)$
13. $(t_3, t_7) = (1, 3)$
14. $(t_7) = (6)$
15. $(t_7, t_8) = (1, 1)$
16. $(t_8) = (3)$

$P(\mathscr{G}_9, (8))$

Faces

Row	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8^*	π_0
1	1	2	0	1	2	0	1	2	2
2	2	1	3	2	1	3	2	4	4
3	1	2	3	4	5	6	7	8	8
4	8	7	6	5	4	3	2	10	10
5	4	8	12	7	2	6	10	14	14
6	11	4	6	8	10	12	5	16	16
7	16	5	12	10	8	15	4	20	20

Vertices

1. $(t_1) = (8)$
2. $(t_2) = (4)$
3. $(t_1, t_3) = (2, 2)$
4. $(t_2, t_3) = (1, 2)$
5. $(t_4) = (2)$
6. $(t_1, t_5) = (3, 1)$
7. $(t_1, t_2, t_5) = (1, 1, 1)$
8. $(t_3, t_5) = (1, 1)$
9. $(t_2, t_5) = (1, 3)$
10. $(t_5) = (7)$
11. $(t_1, t_6) = (2, 1)$
12. $(t_2, t_6) = (1, 1)$
13. $(t_5, t_6) = (1, 2)$
14. $(t_1, t_7) = (1, 1)$
15. $(t_5, t_7) = (2, 1)$
16. $(t_3, t_7) = (1, 2)$
17. $(t_6, t_7) = (2, 2)$
18. $(t_7) = (5)$
19. $(t_8) = (1)$

solutions to (7). The problem of finding a face for $P(\mathscr{G}, g_0)$ is equivalent to the problem of finding a set of arc lengths π such that there is a multiple tie among shortest paths to node g_0; in fact, the tie should be sufficiently multiple that the shortest paths form a maximal linearly independent set of vectors $t(g)$.

We will give a simple construction that does this and thus produce a face, but this is by no means the only such construction.

Let us consider a cyclic group $\mathscr{G}_D$. Let $g_0 = mg_1$ where g_1 is the generator. Then part of the corresponding graph H is shown in Fig. 3.

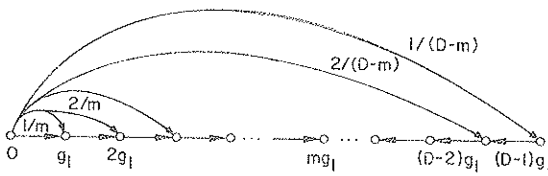

Fig. 3.

Let $D = |\mathscr{G}|$, the order of the group. In H we can form $D-1$ independent paths T_p by using the group element pg_1 once and then completing the path to g_0 by iterating g_1 if $p<m$, or iterating $-g_1 = (D-1)\, g_1$ if $p>m$. This set of paths is clearly independent. Now if we set $\pi_m(pg_1) = p/m$, $p \leq m$, all the T_p, $p \leq m$ have a total length 1. To achieve the same result for the remaining paths, we set $\pi_m(pg) = (D-p)/(D-m)$, for $p>m$. Now it is easily seen that this choice of π provides a face.

Actually, a number of faces can be produced this way for a fixed right-hand side element g_0. It is only necessary to choose a different g_0' from the same automorphism class, carry out the construction, and then do an automorphism carrying g_0' into g_0. It can be shown that distinct faces will result from any distinct g_0' and g_0'' unless $g_0' = -g_0''$.

This construction shows the possibility of making faces for cyclic groups essentially by formula. The next theorem has as a consequence the fact that simple faces are also available for Abelian groups that are not cyclic.

THEOREM 7: Let ψ be a homomorphism of $\mathscr{G}$ onto $\mathscr{H}$ with kernel $\mathscr{K}$ and with $g_0 \notin \mathscr{K}$. Then if (π', π_0) is a face of $P(\mathscr{H}, \psi g_0)$, (π, π_0) is a face of $P(\mathscr{G}, g_0)$ when $\pi(g)$ is given by $\pi(g) = \pi'(\psi g)$. (We take $\pi'(\bar{0}) = 0$; so $\pi(g) = 0$, $g \in \mathscr{K}$.)

For instance, if $\mathscr{G}$ is the direct sum of cyclic groups $\mathscr{G}_i$, then $g_0 = (g_0^1, g_0^2, \ldots, g_0^i, \ldots)$ where the g_0^i are elements in the $\mathscr{G}_i$. Choose any i for which $g_0^i \neq \bar{0}$.

Call this p. We can then take the $\mathscr{H}$ of the theorem to be $\mathscr{G}_p$ and the mapping ψ is simply $\psi(g^1, g^2, \ldots, g^i, \ldots) = g^p$; i.e., map each group element onto its pth component. It is readily seen that all the conditions of the theorem are fulfilled and thus faces of the cyclic group $\mathscr{H}$ carry up into faces of $\mathscr{G}$.

6. Characters and inequalities

Of course, all the inequalities produced by these various methods are valid inequalities for the original integer programming problem, because they are faces for P^*, a polyhedron that includes the original P; however, to obtain any of them, we have to recognize which group element g corresponds to which non-basic column of the linear programming problem. However, in the earlier work (Gomory [1963]) this identification was not necessary. We merely used the fractional parts appearing in the transformed matrix to form "cutting planes". It is reasonable to ask whether some similar procedure can be used here and the answer is that it can.

To see this, construct from the matrix A, whose columns we will designate by C_j, the transformed matrix $A^* = B^{-1}A = (I, B^{-1}N)$. Each C_j has associated with it some group element $g_j \in \mathscr{G} = M(I)/M(B)$; however, we have not worked out what the group $\mathscr{G}$ is or which group element in $\mathscr{G}$ g_j is. We next define a mapping ψ^i from the columns C_j into the cyclic group of order $D = |\det B| = |\mathscr{G}|$ as follows:

$$\psi^i(C_j) = \mathscr{F}(a^*_{i,j}).$$

Here, $\mathscr{F}(x)$ stands for the fractional part of x; i.e., $x - [x]$ and $a^*_{i,j}$ is the entry in the ith row and jth column of A^*. Since $D = |\det B|$, the numbers $a^*_{i,j}$ are all of the form n/D where n is an integer; hence, $\mathscr{F}(a^*_{i,j}) = m/D$ where m is an integer less than D. However, the fractions m/D with addition modulo 1 form the cyclic group of order D. Thus, ψ^i can be considered as mapping C_j onto an element of this cyclic group $\mathscr{G}_D$. ψ^i can also be thought of as mapping the corresponding (unknown) group element g_j onto an element of $\mathscr{G}_D$. Regarded as a mapping from $\mathscr{G}$ into $\mathscr{G}_D$, the mapping ψ^i is readily shown to be a character; i.e., $\psi^i(g_j) + \psi^i(g_k) = \psi^i(g_j + g_k)$. It follows from this property that, if (π, π_0) is a face of $P(\mathscr{G}_D, g_0)$,

$$\sum_{i=1}^{i=m+n} \pi(\psi^i(C_j))\, x_j \geq \pi(\psi^i(b)) \tag{9}$$

is an inequality that must be satisfied by any integer point in P^* and, hence, by any integer in P. In (9) $\psi^i(b)$ is the group element corresponding to the fractional part of the ith component of $B^{-1}b$.

Different rows i give different mappings ψ^i and, hence, different inequalities. Different inequalities also result from using different faces (π, π_0) of the corresponding cyclic group. It follows from what is known about characters that all possible mappings of this sort are obtained by using integer combinations of the row mappings ψ^i and that, in fact, this collection of mappings forms a group isomorphic to $\mathscr{G}$ itself. The "fractional inequalities" of Gomory [1963] can be shown to be a special case of this. They are obtained by using all characters ψ together with one fixed face of $P(\mathscr{G}_D, g_0)$.

7. Asymptotic integer programming

It is reasonable to suppose that if the numbers in the right-hand side vector b become large, the problem of solving an integer programming problem such as max $c \cdot x$

$$Ax = b \qquad x \geq 0 \qquad x \text{ integer} \tag{10}$$

should reduce to the problem of maximizing that same objective function over the polyhedron P^* which comes from the vertex V that maximizes the ordinary linear program. Looking at the problem in its inequality form

$$A'x' \leq b \qquad x' \geq 0$$

we can see that as b gets large, the faces tend to move away from each other and after a while the only inequalities that seem relevant to the vertex V, or to any integer points in its neighborhood are the faces meeting at V. The other faces become quite distant.

There is a theorem which substantiates this intuitive conjecture, but in order to understand its statement, some preliminaries are needed. Let us consider the right-hand sides b as points in m-dimensional space. The set of such points for which B is a feasible basis form the cone K_B made up of the points y for which $B^{-1}y \geq 0$. In fact, for a single objective function C, b space splits up into cones associated with the various optimal bases. If the right-hand side b lies anywhere in the cone K_B, B is the optimal basis. Let us define $K_B(l)$ as the set of points inside K_B at a distance at least l from the frontier of K_B. This is a sort of inner cone.

In what follows, B is the optimal basis for the linear programming problem max $c \cdot x$, $Ax = b$, $x \geq 0$, and $D = |\det B|$ and $l_{\max}$ is the length of the longest vector in N, the non-basic part of A.

THEOREM 8: If $b \in K_B(l_{max}(D-1))$, then there is an optimal integer solution to (10) of the form

$$x = (x_B, x_N) = \left(B^{-1}(b - Nx_N^*(b)), x_N^*(b)\right), \tag{11}$$

where $x_N^*(b)$ is a solution to the group minimization problem

$$\min \sum_{g \in \mathcal{G}} c^*(g)\, t(g)$$

subject to $\sum_{g \in \mathcal{G}} g \cdot t(g) = g_0$. In this formula, the c^* are the relative prices of linear programming; i.e., if $c = (C_B, C_N)$, then the vector of the $c^*(t)$ is the vector $(C_N - C_B B^{-1} N)$.

This theorem has several consequences. First, we note that $x_N^*(b)$ is a periodic function of the right-hand side b; that is to say, that $x_N^*(b) = x_N^*(b + B_i)$ where B_i is any column of the matrix B. This is because x_N^* solves the group equation and the group equation is unaffected by changes in B that are equivalent to 0 mod (B). Thus, (11) consists of two parts, the linear programming solution $(B^{-1}b, 0)$ and a correction $(-B^{-1}Nx_N^*(b), x_N^*(b))$ which is periodic. Inside the cone $K_B(l_{max})$, the relation between the integer and non-integer solutions is a periodic one with $x_N^*(b)$ obtained by minimizing over the group equation.* Since that part of K_B not in $K_B(l_{max})$ is a strip of fixed width, almost any right-hand side b (except for a set of right-hand sides b of lower dimension) multiplied by a large enough scalar, will eventually enter the inner cone $K_B(l_{max})$, and hence, enter the domain of applicability of the theorem.

For any b, we can find the optimal B. b is, of course, in K_B. If it is also in $K_B(l_{max})$, then we know from the theorem that the optimization of any c subject to (10) gives the same result as its optimization over the corresponding P^*. Thus, those vertices of the integer polyhedra associated with (10) which maximize an objective function c maximized at V are the same as the vertices of P^*. Thus, P^* coincides with a portion of the integer polyhedron belonging to (10).

8. Conclusions

This paper has indicated that there is a highly structured class of polyhedra closely related to integer programming problems. Most of the properties of these polyhedra remain to be discovered and exploited. This looks like a promising area for further research.

* This minimization is the problem of finding the shortest path from $\bar{0}$ to g_0 in $\mathscr{H}(\mathscr{G}, c^*)$.

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