SOME CONTINUOUS FUNCTIONS RELATED TO CORNER POLYHEDRA

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INTRODUCTION

A. Inequalities based on the integer nature of some or all of the variables are useful in almost any algorithm for integer programming. They can furnish cut offs for branch and bound or truncated enumeration methods, or cutting planes for cutting plane methods. In this paper we describe methods for producing such inequalities and develop some underlying theory.

We will attempt to outline our general approach, taking the pure integer case first.

Consider a pure integer problem

$$(1) \qquad Ax = b, \quad x \geq 0$$

in which A is an $m \times (m+n)$ matrix, x is an integer $m+n$ vector, and b an m-vector. If we consider a basis B (in most applications this will be an optimal basis) we can write (1) as

$$Bx_B + Nx_N = bx_B \geq 0, \quad x_N \geq 0$$

where x_B is the m-vector of basic variables and x_N the non-basic n-vector. The usual transformed matrix [1, pages 75-80] corresponds to the equations

$$x_B + B^{-1}Nx_N = B^{-1}b, \; x_B \geq 0, \; x_N \geq 0, \text{ or}$$

$$(2) \qquad y + N'z = b', \quad y \geq 0, \quad z \geq 0.$$

Taking the i^{th} row we have

$$y_i + \sum_{j=1}^{j=n} n'_{i,j} z_j = b'_i.$$

We can form a new but related equation by reducing all coefficients modulo 1 and replacing the equality by equivalence modulo 1. This yields

$$(3) \qquad \sum_{j=1}^{j=n} \mathcal{F}(n'_{ij})z_j \equiv \mathcal{F}(b'_i) \qquad (\text{mod } 1).$$

Now any integer vector (y,z) satisfying (2) automatically satisfies (3), so that any inequality

$$\sum_{j=1}^{j=n} \pi_j z_j \geq \pi_0, \qquad \text{or } \pi \cdot z \geq \pi_0$$

which is satisfied by all solutions z to (3) is also satisfied by all solutions to (2), i.e.

$$(0,\pi)\cdot(y,z) \geq \pi_0$$

holds for any integer vector (y,z) satisfying (2).

The approach of this paper is to develop inequalities valid for all solutions to (2) by obtaining those valid for all solutions to the simpler equations like (3).

More generally, we can proceed as follows, let ψ be a linear mapping sending the points of m-space into some other topological group S with addition. If we have an equation (like (2))

$$(4) \qquad \sum_j C_j x_j = C_o$$

in which the C_j and C_o are m vectors, we can obtain a new equation by using the mapping ψ to obtain, by linearity,

$$(5) \qquad \sum_j \psi(C_j x_j) = \psi(C_o)$$

which is an equation involving a set of group elements in S, the elements $\psi(C_j x_j)$. For integer x_j, $\psi(C_j x_j) = \psi(C_j)x_j$ so equivalent group equations are

$$(6) \qquad \sum_j \psi(C_j)x_j = \psi(C_o).$$

In the discussion leading up to equation (3) the C_j were the columns of the matrix (I,N') and ψ was the mapping that sends an m vector into the fractional part of its i^{th} coordinate. The group S was the unit interval with addition modulo 1. Equation (3) was the equation (6).

Again, if $\pi \cdot x \geq \pi_o$ holds for all integer x satisfying (5) or (6) it holds for integer x satisfying (4).

In this paper we study equations such as (6) and develop inequalities for their solutions which are then satisfied by the solutions to (4). Specifically we study the case where S is I, the unit interval mod 1, and develop inequalities for the equations:

$$(7) \qquad \sum_{u \in U} ut(u) = u_o$$

where U represents the set $\psi(C_j) \in I$ and $t(u)$ is a non-negative integer. Equation (7), which we refer to as the problem (or equation), $P(U,u_o)$, is merely (6) rewritten in a different notation.

Returning to equation (4) when some of the x_j are not restricted to be integer, a linear mapping ψ still gives another equation (5) satisfied by all solutions to (4). Thus, any solution to (4) satisfies the equation

$$\sum_j \psi(C_j x_j) = \psi(c_o).$$

Just as before, if any x_j is required to be integer, then $\psi(C_j x_j) = (C_j)x_j$. Let J_1 denote the subset of j for which x_j is required to be integer and J_2 be the j for which x_j is only required to be non-negative. Then, any solution to (4) with x_j integer for $j \in J_1$ satisfies

$$(8) \qquad \sum_{j\in J_1} \psi(C_j)x_j + \sum_{j\in J_2} \psi(C_j x_j) = \psi(C_o).$$

When ψ is the same (fractional) map used to derive (3), we rewrite (8) as

$$(9) \qquad \sum_{j\in J_1} \mathcal{F}(n'_{ij})z_j + \sum_{j\in J_2} \mathcal{F}(n'_{ij}z_j) \equiv \mathcal{F}(b'_i) \qquad (\text{mod } 1).$$

Consider $n'_{ij}z_j$ for $j \in J_2$. If $n'_{ij} = 0$, then z_j does not really enter into the equation. If $n'_{ij} \neq 0$ we can rescale z_j by letting

$$z'_j = |n'_{ij}|z_j, \; j \in J_2.$$

Let $J_2^+ = \{j \in J_2 : n'_{ij} > 0\}$ and $J_2^- = \{j \in J_2 : n'_{ij} < 0\}$. Then $z'_j = n'_{ij}z_j$ for $j \in J_2^+$ and $-z'_j = n'_{ij}z_j$ for $j \in J_2^-$. The restriction $z_j \geq 0$ is equivalent to $z'_j \geq 0$. Hence, (9) becomes

$$(10) \qquad \sum_{j\in J_1} \mathcal{F}(n'_{ij})z_j + \sum_{j\in J_2^+} \mathcal{F}(z'_j) - \sum_{j\in J_2^-} \mathcal{F}(z'_j) \equiv \mathcal{F}(b'_1) \quad (\text{mod } 1).$$

Since

$$\sum_{j\in J_2^+} \mathcal{F}(z'_j) \equiv \mathcal{F}\Big(\sum_{j\in J_2^+} z'_j\Big) \qquad (\text{mod } 1)$$

(10) can be simplified to

$$(11)\qquad \sum_{j\varepsilon J_1} \mathcal{F}(n'_{ij})z_j + \mathcal{F}(z^+) - \mathcal{F}(z^-) \equiv \mathcal{F}(b'_i) \qquad \text{(mod 1)}$$

where

$$z^+ = \sum_{j\varepsilon J_2^+} z'_j,$$

$$z^- = \sum_{j\varepsilon J_2^-} z'_j.$$

We can rewrite (11) in a form similar to (7) to obtain the problem we call $P_-^+(U,u_o)$:

$$(12)\qquad \sum_{u\varepsilon U} ut(u) + \mathcal{F}(s^+) - \mathcal{F}(s^-) = u_o.$$

In this paper, we concentrate on the development of valid inequalities for equations of the form (7) and (12). These inequalities, satisfied by every solution to (7) or (12), are immediately applicable to the original problem (4). In the case of an inequality

$$(13)\qquad \sum_{j\varepsilon J_1} \pi_j z_j + \pi^+ z^+ + \pi^- z^- \geq 1$$

satisfied by every solution to (11), the inequality

$$(14)\qquad \sum_{j\varepsilon J_1} \pi_j z_j + \sum_{j\varepsilon J_2^+} (\pi^+ n'_{ij}) z_j + \sum_{j\varepsilon J_2^-} (\pi^- n'_{ij}) z_j \geq 1$$

is satisfied by every solution to (10), and hence to (4).

B. The Arrangement of the Paper

In Section I we introduce the problems $P(U,u_o)$ and $P_-^+(U,u_o)$ which are the problems (equations) which result from applying the mapping ψ to an integer or mixed integer programming problem. We next introduce

I. Development of Inequalities

IA. Problem Definition

Let I be the group formed by the real numbers on the interval $[0,1)$ with addition modulo 1. Let U be a subset of I and let t be an integer-valued function on U such that (i) $t(u) \geq 0$ for all $u \in U$, and (ii) t has a finite support; that is, $t(u) > 0$ only for a finite subset U_t of U.

The notation and definitions above will be used throughout so that t will always refer to a non-negative integer valued function with finite support.

We say that the function t is a solution to the problem $P(U,u_o)$, for $u_o \in I - \{0\}$, if

$$(1) \quad \sum_{u \in U} ut(u) = u_o .$$

Here, of course, addition and multiplication are taken modulo 1. Let $T(U,u_o)$ denote the set of all such solutions t to $P(U,u_o)$.

Correspondingly, the problem $P_-^+(U,u_o)$ has solutions $t' = (t,s^+,s^-)$ satisfying

$$(2) \quad \sum_{u \in U} ut(u) + \mathfrak{F}(s^+) - \mathfrak{F}(s^-) = u_o$$

where t is, as before, a non-negative integer valued function on U with a finite support, where s^+, s^- are non-negative real numbers, and where $\mathfrak{F}(x)$ denotes the element of I given by taking the fractional part of a real number x. Let $T_-^+(U,u_o)$ denote the set of solutions $t' = (t,s^+,s^-)$ to $P_-^+(U,u_o)$.

It is also possible to define problems $P^+(U,u_o)$ and $P_-(U,u_o)$ in which only s^+ or s^- appear, and these problems are useful in some situations. Their development parallels that of $P^+_-(U,u_o)$.

The notation $u \in I$ will mean that u is a member of the group I so that arithmetic is always modulo 1. If we want to consider u as a point on the real line with real arithmetic, we will write $|u|$. Thus, $|u|$ and $\mathfrak{F}(x)$ are mappings in opposite directions between I and the reals. and, in fact, $\mathfrak{F}(|u|) = u$ but x and $|\mathfrak{F}(x)|$ may differ by an integer.

IB. Inequalities

1. Valid Inequalities For any problem $P(U,u_o)$, we have so far defined the solution set $T(U,u_o)$. A *valid inequality for the problem $P(U,u_o)$* is a real-valued function π defined for all $u \in I$ such that

$$(3) \quad \pi(u) \geq 0, \text{ all } u \in I, \text{ and } \pi(0) = 0,$$

and

$$(4) \quad \sum_{u \in U} \pi(u)t(u) \geq 1, \text{ all } t \in T(U,u_o).$$

For the problem $P^+_-(U,u_o)$, $\pi' = (\pi,\pi^+,\pi^-)$ is a *valid inequality for $P^+_-(U,u_o)$* when π is a real-valued function on I satisfying (3), and π^+, π^- are non-negative real numbers such that

$$(5) \quad \sum_{u \in U} \pi(u)t(u) + \pi^+s^+ + \pi^-s^- \geq 1, \text{ all } t' \in T^+_-(U,u_o).$$

the notion of valid inequality, an inequality which holds for all solutions to $P(U,u_o)$ or $P_-^+(U,u_o)$. Valid inequalities are shown to be arranged in a hierarchy of extreme, minimal or subadditive inequalities. Some theorems relating these properties are then given.

In Section II the preceding theory is applied to a useful special case, the case in which the set U in the problem $P(U,u_o)$ is a regular grid $G_n = \{0,\frac{1}{n},\frac{2}{n},\ldots,\frac{n-1}{n}\}$. (In this case the problem reduces to the finite cyclic group problem studied in [3], and an extreme inequality becomes a face of the polyhedron, $P(G_n,g_o)$ described there).

The theory is also applied to the regular grid problem $P_-^+(G_n,u_o)$. In both the $P(G_n,u_o)$ and $P_-^+(G_n,u_o)$ cases the theory allows the explicit computation of all extreme inequalities of the problems $P(G_u,u_o)$ and $P_-^+(G_u,u_o)$ for small n. For $n \leq 11$ the extreme inequalities for $P(G_n,u_o)$ are included in appendix 5 of [3]. For $P_-^+(G_n,u_o)$, $n \leq 7$, they are Table 2 of the appendix of this paper.

One vital step remains before the inequalities explicitly constructed and tabulated can be used on an arbitrary integer program. The inequalities explicitly computed so far as valid only for integer programs where the resulting problem $P(U,u_o)$ has a U contained in a regular grid G_n(that is, the fractional parts of the coefficients have a common denominator n). In section IIIA we give a simple way of constructing an inequality valid for any $P(U,u_o)$ (and hence for any coefficients N') from an inequality valid for $P(G_n,u_o)$. In section IIIB we give another simple construction which produces valid inequalities for a general $P_-^+(U,u_o)$ from a valid inequality for $P_-^+(G_n,u_o)$.

IIA and IIB allow us to use the tabulated inequalities to construct, essentially by interpolation, valid inequalities for any integer or mixed integer linear program. The mixed integer cut of [1, page 528] emerges from the simplest case where G_n consists of only 1 point. In IIIC we apply the methods of IIIA and IIIB to actual numerical examples of integer and mixed integer problems.

In IV we return to theory and work toward an understanding of the properties of the problems $P(I,u_o)$ and $P_-^+(I,u_o)$ where I is the entire unit interval [0,1). These problems here play the role taken by the master polyhedra in the finite group theory of [3]. In IVA we relate the solutions of $P(I,u_o)$ to those of $P_-^+(I,u_o)$. In IVB we study conditions under which the inequalities constructed in IIIA and IIIB are minimal or extreme for $P(I,u_o)$ and $P_-^+(I,u_o)$. Conditions are given under which inequalities of special form are extreme*, and under which extreme inequalities for $P(I,u_o)$ are extreme for the finite group problems.

In IVC we introduce a more complex method of interpolation which produces a multitude of variant extreme inequalities of $P_-^+(I,u_o)$ from inequalities of $P_-^+(G_n,u_o)$. These results shed light on the rate of growth of the number of faces of the polyhedra $P(G_n,g_o)$ of [3].

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*Theorem IV-12 of this section is the result of collaboration with Alan Konheim.

For a valid inequality π' for $P_{-}^{+}(U,u_o)$ to be subadditive, we require, in addition to (6),

$$(7) \quad \pi(u) + \pi^{+}|v-u| \geq \pi(v), \text{ whenever } u,v \in U,\ |u| < |v|,$$

$$(8) \quad \pi(u) + \pi^{-}|u-v| \geq \pi(v), \text{ whenever } u,v \in U,\ |u| > |v|.$$

Theorem I.2 The minimal valid inequalities are subadditive valid inequalities.

Proof: The theorem will be proven for $P(U,u_o)$, and the extension to the case $P_{-}^{+}(U,u_o)$ is similar.

Suppose π is a minimal valid inequality for $P(U,u_o)$ but is not subadditive. Then there are v,w with v,w, and v+w in U and

$$\pi(v) + \pi(w) < \pi(v+w).$$

Let ρ be defined on I by

$$\rho(u) = \begin{cases} \pi(u), & u \neq v+w, \\ \pi(v) + \pi(w), & u = v+w. \end{cases}$$

We will show that ρ is a valid inequality, thus reaching a contradiction since $\rho < \pi$ and π was assumed minimal.

We show that ρ is a valid inequality by contradiction. Suppose not. Then for some $t \in T(U,u_o)$,

$$\sum_{u \in U} \rho(u)t(u) < 1.$$

Define t* on U by

$$t^*(u) = \begin{cases} t(v) + t(v+w), & u = v \\ t(w) + t(v+w), & u = w \\ 0, & u = v+w \\ t(u), & \text{otherwise.} \end{cases}$$

Clearly, $t^* \in T(U,u_o)$ and

$$\sum_{u\in U} \rho(u)t(u) = \sum_{u\in U} \pi(u)t^*(u).$$

But $\sum\rho(u)t(u) < 1$, so $\sum\pi(u)t^*(u) < 1$ contradicting π being a valid inequality.

Theorems I.1 and I.2 prove the following sequence of inclusions: the set of valid inequalities include the subadditive valid inequalities which include minimal valid inequalities which include extreme valid inequalities. The subadditive valid inequalities form a convex set contained in the larger convex set of valid inequalities. The next theorem says that the extreme points of the set of subadditive valid inequalities include all the extreme valid inequalities. Further, among the extreme subadditive valid inequalities, those which are extreme valid inequalities are the minimal ones. This fact allows us to actually find the extreme valid inequalities for some problems.

<u>Theorem I.3</u> If π (or π') is extreme among the subadditive valid inequalities for $P(U,u_o)$ (or $P_-^+(U,u_o)$), that is, π (or π') is not the mid-point of any two different subadditive valid inequalities, and if π (or π') is also a minimal valid inequality, then it is an extreme valid inequality.

Proof: Suppose π is not an extreme valid inequality. Then $\pi = \frac{1}{2}\rho + \frac{1}{2}\sigma$ for some $\rho \neq \sigma$ which are valid inequalities. Both ρ and σ must be minimal by lemma I.4 which follows. Thus, ρ and σ are subadditive by theorem I.2. But then, π is a mid-point of two subadditive valid inequalities, and a contradiction is reached.

The proof for $P_{-}^{+}(U,u_o)$ is similar, with π', ρ', ρ_1', π_1' replacing π, ρ, ρ_1, π_1.

Lemma I.4 If any minimal valid inequality is a mid-point of two other valid inequalities (and is therefore not an extreme valid inequality)

$$\pi = \frac{1}{2}\rho + \frac{1}{2}\sigma, \text{ or } \pi' = \frac{1}{2}\rho' + \frac{1}{2}\sigma',$$

then ρ and σ (or ρ' and σ') must also be minimal valid inequalities.

Proof: The lemma will be shown for $P(U,u_o)$ and the proof for $P_{-}^{+}(U,u_o)$ is exactly similar.

Suppose one of ρ,σ is not minimal, say ρ is not. Then there is a valid inequality $\rho_1 < \rho$. Hence,

$$\pi_1 = \frac{1}{2}\rho_1 + \frac{1}{2}\sigma$$

is a valid inequality. But $\pi_1 < \pi$, contradicting π being minimal.

IC. Subadditivity for Subgroups U

The problems for which theorem I.3 can be used to find extreme valid inequalities are $P(U,u_o)$ or $P_{-}^{+}(U,u_o)$ where U is a non-empty subgroup of I. We permit U = I and note that 0 is always in U. We will say that a function π defined on I is subadditive on a subgroup U of I if

$\pi(u) \geq 0,\ u \in I,\ \pi(0) = 0$, and

$\pi(u) + \pi(v) \geq \pi(u+v), u,v \in U.$

The function π is not assumed to be a valid inequality. The following theorem establishes the close connection between subadditive functions on U and subadditive valid inequalities.

Theorem I.5 If π is a subadditive function on a subgroup U of I and if $\pi(u_o) \geq 1$ for some $u_o \in U$, $u_o \neq 0$, then π is a valid inequality for $P(U,u_o)$. In fact, the subadditive valid inequalities for $P(U,u_o)$ are precisely the subadditive functions π satisfying $\pi(u_o) \geq 1$. Furthermore, if π is a subadditive function on U and $\pi(u_o) > 0$ for some $u_o \in U$, then π^* defined by

$$(9)\quad \pi^*(u) = \frac{\pi(u)}{\pi(u_o)},\ u \in I,$$

is a valid inequality for $P(U,u_o)$.

Proof: The last statement follows from the first by $\pi^*(u_o) = 1$ and the fact that multiplying a subadditive function by a positive number preserves subadditivity. The second statement is equivalent to the first.

The proof of the first statement consists of showing

$$(10)\quad \sum_{u\in U} \pi(u)t(u) \geq \pi(\sum_{u\in U} ut(u))$$

by induction on $\Sigma_t = \sum_{u\in U} t(u)$ for all non-negative integer valued functions t on U with finite support and, hence, finite Σ_t Since U is a subgroup of I, $\sum_{u\in U} ut(u) \in U$. For any t, Σ_t is a non-negative integer, and $\Sigma_t = 0$ means all $t(u) = 0$, $u \in U$. For such a t, (10) is satisfied trivially. For $\Sigma_t = 1$, all $t(u) = 0$ except for one $v \in U$

having $t(v) = 1$. For such a t, (10) becomes $\pi(v) \geq \pi(v)$ which is true.

Suppose, as induction hypothesis, that (10) is satisfied for all t for which $\sum_t = k$, $k \geq 1$. Consider now, any t having $\sum_t = k+1$. Let $v \in U$ such that $t(v) \geq 1$ and let $w = \sum_{u\in U} ut(u)-(v)$. Then, $v+w = \sum_{u\in U} ut(u)$ and

$$\begin{aligned}\sum_{u\in U} \pi(u)t(u) &= \pi(v) + \sum_{u\in U-\{v\}} \pi(u)t(u) + \pi(v)(t(v)-1)\\ &\geq \pi(v) + \pi(\sum_{u\in U-\{v\}} ut(u)+v(t(v)-1))\\ &= \pi(v) + \pi(w)\\ &\geq \pi(v+w)\\ &= \pi(\sum_{u\in U} ut(u))\end{aligned}$$

by the induction hypothesis and by subadditivity. Thus, (10) is proven by induction.

To complete the proof of the theorem, we need only use (10) and observe that if $t \in T(U,u_o)$, then $u_o = \sum_{u\in U} ut(u)$, and $\pi(u_o) \geq 1$ by hypothesis.

The analogous theorem for $P_-^+(U,u_o)$ will now be developed. Define $\pi' = (\pi, \pi^+, \pi^-)$ to be an <u>extended subadditive function on a subgroup U of I</u> if π is subadditive on U and if, in addition,

(11) $\pi^+|u| \geq \pi(u)$, $u \in U$,

(12) $\pi^-|u| \geq \pi(-u)$, $-u \in U$.

Since the above two conditions are a weakening of (7) and (8), it is obvious that a subadditive valid inequality for $P^{+}_{-}(U,u_o)$, where U is a subgroup, is an extended subadditive function on U. The following theorem establishes a converse result.

Theorem I.5B If π' is an extended subadditive function on a subgroup U of I, if $u_o \in I$, $u_o \neq 0$, and if both of the following hold:

(13) $\pi(u) + \pi^{+}|u_o - u| \geq 1$ whenever $u \in U$ and $|u| \leq |u_o|$,

(14) $\pi(u) + \pi^{-}|u - u_o| \geq 1$ whenever $u \in U$ and $|u| \leq |u_o|$,

then π' is a valid inequality for $P^{+}_{-}(U,u_o)$. In fact, the subadditive valid inequalities are precisely the extended subadditive functions which satisfy (13) and (14).

Proof: The proof closely parallels the proof of theorem I.5 and is by induction on $|t|$. However, three preliminaries are needed.

First, (7) and (8) will be shown to hold for an extended subadditive function π on a subgroup U. We will show (7), and the proof of (8) is similar. By (11) and (6),

$$\pi(u) + \pi^{+}|v-u| \geq \pi(u) + \pi(v-u) \geq \pi(v)$$

whenever $|u| \leq |v|$. Hence, (7) is true. Here, we use the fact that U is a subgroup so v-u is in U.

Secondly, (13) and (14) imply $\pi^{+} \geq 0$ and $\pi^{-} \geq 0$ by taking $u = 0$ and using $|u_o| > 0$.

Thirdly, if $t' = (t,s^{+},s^{-})$ is in $T^{+}_{-}(U,u_o)$ and if $s^{+} > 0$ and $s^{-} > 0$, then one of s^{+},s^{-} is larger, say $s^{+} \geq s^{-}$. Now let $s_1^{+} = s^{+}-s^{-}$ and $s_1^{-} = 0$. Then

$$t_1' = (t, s_1^+, s_1^-)$$

is in $T_-^+(U,u_o)$ since $s_1^+ - s_1^- = s^+ - s^-$. Furthermore, since $\pi^+ \geq 0$ and $\pi^- \geq 0$, if

$$\sum_{u\in U} \pi(u)t(u) + \pi^+ s_1^+ + \pi^- s_1^- \geq 1$$

then the same inequality holds with s^+ replacing s_1^+ and s^- replacing s_1^-.

The second and third observations show that in order to prove that π' is a valid inequality, it suffices to consider $t' = (t, s^+, s^-)$ in $T_-^+(U,u_o)$ for which only one of s^+, s^- is positive and the other is zero. For such t', we wish to show that the hypothesis of the theorem imply that (5) holds, that is,

$$\sum_{u\in U} \pi(u)t(u) + \pi^+ s^+ + \pi^- s^- \geq 1.$$

We already know that subadditivity of π implies (10) and that one of s^+, s^- is zero, say $s^- = 0$. Then

$$\sum_{u\in U} \pi(u)t(u) + \pi^+ s^+ + \pi^- s^-$$

$$= \sum_{u\in U} \pi(u)t(u) + \pi^+ s^+$$

$$\geq \pi\left(\sum_{u\in U} ut(u)\right) + \pi^+ s^+$$

by (10). If $s^+ \geq |u_o|$, then $\pi^+ s^+ \geq \pi^+ |u_o| \geq 1$ by taking $u = 0$ in (13). If $s^+ < |u_o|$, then $s^+ = |u_o - \sum_{u\in U} ut(u)|$, and (13) suffice to show (5) since $\sum_{u \in U} ut(u)$ is in U. Similarly, if $s^- > 0$, then (14) suffices to show (5).

The theorem is, thus, an easy consequence of (10), (13) and (14).

We can now use theorems I.3 and I.5 to characterize the extreme valid inequalities for $P(U,u_o)$ as the extreme subadditive functions subject to $\pi(u_o) \geq 1$ which happen to also be minimal valid inequalities. The next subsection gives a simple condition for minimality when U is a subgroup.

I.D Minimality for Subgroups U

Theorem I.6 If U is a subgroup of I with $u_o \in U$ and if π is a valid inequality for $P(U,u_o)$, then π is a minimal valid inequality if and only if

$$(15) \quad \pi(u) + \pi(u_o - u) = 1, \text{ all } u \in U.$$

Proof: That (15) is sufficient for minimality is obvious since lowering any $\pi(u)$, $u \in U$, would result in

$$\pi(u) + \pi(u_o - u) < 1,$$

by (15), while $t(u) = 1$, $t(u_o - u) = 1$ is a solution to the problem.

The harder part of the proof is to show that (15) holds whenever π is a minimal valid inequality. Clearly, $\pi(u) + \pi(u_o - u) \geq 1$ because $u + (u_o - u) = u_o$ and π is a valid inequality. Hence, if (15) does not hold, then

$$\pi(v) + \pi(u_o - v) = 1 + \delta$$

for some $v \in U$ and some $\delta > 0$. Clearly, at least one of $\pi(v), \pi(u_o - v)$ is positive. Suppose $\pi(v) > 0$.

Define ρ on I by

$$\rho(u) = \begin{cases} \frac{1}{1+\delta}\pi(v), & u = u, \\ \pi(u) & , u \neq v,\ u \in I. \end{cases}$$

Since $\delta > 0$ and $\pi(v) > 0$, it follows that $\rho < \pi$. If ρ can be shown to be a valid inequality, then a contradiction will be reached since π was assumed to be minimal.

By the definition of ρ, we have for any solution t

$$(16) \qquad \sum_{u\in U} \rho(u)t(u) = \sum_{\substack{u\in U \\ u\neq v}} \pi(u)t(u) + \frac{1}{1+\delta}\pi(v)t(v).$$

If $t(v) \geq (1+\delta)/\pi(v)$, then clearly $\sum_{u\in U} \rho(u)t(u) \geq 1$. On the other hand, if $t(v) = 0$, then

$$\sum_{u\in U} \rho(u)t(u) = \sum_{u\in U} \pi(u)t(u) \geq 1.$$

Suppose now that $1 \leq t(v) < (1+\delta)/\pi(v)$. Regrouping the term $(1/(1+\delta))\pi(v)t(v)$ in (16) gives the equation

$$\sum_{u\in U} \rho(u)t(u) = \{\sum_{\substack{u\in U \\ u\neq v}} \pi(u)t(u) + \pi(v)(t(v)-1)\}$$

$$+ \pi(v) - \frac{\delta}{1+\delta}\pi(v)t(v).$$

By subadditivity, the expression in brackets is greater than or equal to $\pi(u_o - v)$, and by $t(v) < (1+\delta)/\pi(v)$ we can write

$$\sum_{u\in U} \rho(u)t(u) \geq \pi(u_o - v) + \pi(v) - \delta$$

$$\geq 1 + \delta - \delta = 1.$$

The proof is complete.

The analogous theorem for $P^{+}_{-}(U,u_{o})$ is separated into two parts. When U is finite, the theorem is given by theorem II.2B, and when U is infinite, the result is given in property IV.7.

II. The Regular Grid $S = G_n$

Let G_n denote the subset

$$G_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\}$$

of I. The elements of G_n will be denoted $g_i = \mathfrak{F}(i/n)$. Each set G_n for $n \geq 1$ is a subgroup, and, in fact, the sets $G_n, n = 1,2,\ldots$ are the only finite subgroups of I. By virtue of G_n being a subgroup, the results of IC. and ID. apply to this section.

We first discuss the problem $P(G_n,u_o)$ and then the problem $P^+_-(G_n,u_o)$.

II.A. The Problem $P(G_n,u_o)$, $u_o \in G_n$

The results from IC. and ID. are specialized in the next two theorems. Theorem II.1 is a direct restatement of theorem I.5.

Theorem II.1 π is a valid inequality for $P(G_n,u_o)$, $u_o \in G_n$, provided π is subadditive on G_n and $\pi(u_o) \geq 1$. In fact, the subadditive functions π on G_n satisfying $\pi(u_o) \geq 1$ are precisely the subadditive valid inequalities for $P(G_n,u_o)$.

Theorem II.2 The extreme valid inequalities for $P(G_n,u_o)$, $u_o \in G_n$, are the extreme points of the solutions to

(1) $\pi(g_i) \geq 0$, $\pi(0) = 0$,

(2) $\pi(g_i) + \pi(g_j) \geq \pi(g_i + g_j)$,

(3) $\pi(u_o) \geq 1$

which satisfy the additional equations.

(4) $\pi(g_i) + \pi(u_o - g_i) = 1, g_i \in G_n$.

In particular, (4) implies $\pi(u_o) = 1$ since $\pi(0) = 0$.

Proof: Theorem II.2 is à specialization of theorems I.5 and I.6 since conditions (1), (2) and (3) are necessary and sufficient for π to be a subadditive valid inequality for $P(G_n,u_o)$, and (4) is necessary and sufficient for a valid inequality to be a minimal valid inequality.

In this case, (1), (2), (3) and (4) can be combined into one linear system whose extreme points are the extreme valid inequalities of $P(G_n,u_o)$. This is a special case of theorem 18 in [3] and the extreme valid inequalities for G_n, $n = 2,\ldots,11$ are included in appendix 5 there.

II.B. The Problem $P_-^+(G_n,u_o)$, $u_o \in I$

The condition I(2) now becomes

$$g_i t(g_1) + \ldots + g_{n-1} t(g_{n-1}) + \mathfrak{J}(s^+) - \mathfrak{J}(s^-) = u_o,$$

where $g_i = \mathfrak{J}(i/n)$ as before and where the $t(g_i)$ must be non-negative integers and s^+, s^- must be non-negative real values. We no longer confine u_o to be in G_n. Let $L(u_o)$ and $R(u_o)$ denote, respectively, the points of G_n immediately below and above u_o. If u_o happens to be in G_n, then $L(u_o) = R(u_o) = u_o$.

Before proceeding to the analogue of theorem II.1, we note that for our special problem $U = G_n$, the conditions for π' to be an extended subadditive function on G_n are that π satisfy (1) and (2) and, in addition,

$$\pi^+ \frac{i}{n} \geq \pi(g_i) \text{ and } \pi^-(1 - \frac{i}{n}) \geq \pi(g_i)$$

The last two inequalities can be weakened here to:

$$(5) \qquad \pi^+ \frac{1}{n} \geq \pi(g_1),$$

$$(6) \qquad \pi^- \frac{1}{n} \geq \pi(g_{n-1}),$$

where $g_1 = \mathfrak{J}(1/n)$ and $g_{n-1} = \mathfrak{J}((n-1)/n)$, since (5) and (2) imply

$$\pi^+ \frac{i}{n} \geq i\pi(g_1) \geq \pi(ig_1) = \pi(g_i).$$

Similarly, (6) and (2) imply $\pi^-(j/n) \geq \pi(g_{n-j})$. Recall from the proof of theorem I.5B that extended subadditivity implies I(7) and I(8), that is, (5) and (6), together with (2), imply

$$\pi(g_i) + \pi^+ \frac{j}{n} \geq \pi(g_i + g_j) \text{ and } \pi(g_i) + \pi^- \frac{j}{n} \geq \pi(g_i + g_{n-j}).$$

The next theorem extends theorem II.1 to the problem $P^+_-(G_n, u_o)$ and is proven by the above argument and by theorem I.5B.

Theorem II.1B $\pi' = (\pi, \pi^+, \pi^-)$ is a valid inequality for $P^+_-(G_n, u_o)$, $u_o \in I$, provided

$$(7) \qquad \pi(L(u_o)) + \pi^+|u_o - L(u_o)| \geq 1$$

$$(8) \qquad \pi(R(u_o)) + \pi^-|R(u_o) - u_o| \geq 1$$

and provided π' is an extended subadditive function on G_n, that is, π is subadditive on G_n and (5) and (6) hold. In fact, the subadditive valid inequalities for $P^+_-(G_n, u_o)$ are precisely the extended subadditive functions on G_n which satisfy (7) and (8).

Proof: The only remaining part of the proof is to show that (7) and (8) above imply conditions I(13) and I(14) of theorem I.5B. Conditions (7) and (8) are a special case of I(13) and I(14).

Suppose $i/n \le |u_o|$. Then $i/n \le |L(u_o)|$

and

$$\begin{aligned}\pi(g_i) + \pi^+(|u_o|-|g_i|) &= \pi(g_i) + \pi^+|u_o-L(u_o)| + \pi^+(|L(u_o)|-\tfrac{i}{n}) \\ &\ge \pi(g_i) + \pi(L(u_o)-g_i) + \pi^+|u_o-L(u_o)| \\ &\ge \pi(L(u_o)) + \pi^+|u_o-L(u_o)| \\ &\ge 1.\end{aligned}$$

Thus, I(13) is proven. The proof of I(14) is similar.

<u>Theorem II.2B</u> The extreme valid inequalities π' for $P_-^+(G_n,u_o)$, $u_o \in I$, are the extreme points of the solutions to the system of linear equations and inequalities (1),(2),(5),(6),(7),(8) which satisfy the additional restrictions:

(9) $\pi(L(u_o)) + \pi^+|u_o-L(u_o)| = 1$

(10) $\pi(R(u_o)) + \pi^-|R(u_o)-u_o| = 1$

(11) for all $g_i \in G_n$, $\pi(g_i) + \pi(L(u_o)-g_i) = \pi(L(u_o))$

or $\pi(g_i) + \pi(R(u_o)-g_i) = \pi(R(u_o))$.

Proof: Conditions (1),(2),(5),(6),(7) and (8) are necessary and sufficient for π' to be a subadditive valid inequality since they are a restatement of the conditions of theorem II.1B. By theorem I.3, the theorem will be proven if (9),(10) and (11) are shown to be necessary and sufficient for a valid inequality for $P_{-}^{+}(G_n,u_o)$ to be a minimal valid inequality. In the case of $P(G_n,u_o)$, the corresponding conditions (4) were known to be necessary and sufficient for minimality by theorem I.6. However, that theorem only applied to problems $P(U,u_o)$ and not to $P_{-}^{+}(U,u_o)$. We now prove its analogue for $P_{-}^{+}(G_n,u_o)$ and, thereby, complete the proof of theorem II.2B.

<u>Theorem II.3</u> A valid inequality π' for $P_{-}^{+}(G_n,u_o)$ is a minimal valid inequality if and only if (9),(10), and (11) hold.

Proof: The proof parallels that of theorem I.6. First, assume π' is a valid inequality for $P_{-}^{+}(G_n,u_o)$ and (9),(10) and (11) hold. Since $t(L(u_o)) = 1$, $s^+ = |u_o-L(u_o)|$ is a solution, (9) assures that any $\rho' \leq \pi'$ having $\rho^+ < \pi^+$ or $\pi(L(u_o)) < \pi(L(u_o))$ is not a valid inequality. Similarly, (10) implies that $\rho^- = \pi^-$ and $\rho(R(u_o)) = \pi(R(u_o))$ for any valid inequality $\rho' \leq \pi'$.

For any $g_i \in G_n$, (11) together with (9) and (10) imply

$$\pi(g_i) + \pi(L(u_o)-g_i) + \pi^+|u_o-L(u_o)| = 1, \text{ or}$$

$$\pi(g_i) + \pi(R(u_o)-g_i) + \pi^-|R(u_o)-u_o| = 1.$$

Let us suppose that the first of the two equations holds. Clearly, $t(g_i) = 1$, $t(L(u_o)-g_i) = 1$, $s^+ = |u_o-L(u_o)|$ is a solution. If $\rho' \leq \pi'$, and $\rho(g_i) < \pi(g_i)$, then ρ cannot be a valid inequality because

$$\rho(g_i) + \rho(L(u_o)-g_i) + \pi^+|u_o-L(u_o)| < 1.$$

Hence, any valid inequality satisfying (9), (10) and (11) is a minimal valid inequality for $P_{-}^{+}(G_n,u_o)$.

Assume, now, that π' is a minimal valid inequality for $P_{-}^{+}(G_n,u_o)$. We will show that (9), (10), and (11) hold.

By theorem I.2, we know that π' is a subadditive valid inequality. By theorem II.1B, we know that (1),(2),(5),(6),(7), and (8) hold, and therefore

(12) $$\sum_{g \in G_n} \pi(g)t(g) \geq \pi(\sum_{g \in G_n} gt(g))$$

holds by I(10), and

(13) $$\pi^+ \frac{i}{n} \geq \pi(g_i),\ \pi^-(1-\frac{i}{n}) \geq \pi(g_i),\ g_i \in G_n,$$

holds since, in fact, (5) and (6) imply I(7) and I(8).

First, (9) will be shown. Suppose (9) does not hold. Then, by (7),

$$\pi(L(u_o)) + \pi^+|u_o - L(u_o)| = 1 + \delta$$

for some $\delta > 0$. Define ρ' on I by $\rho(u) = \pi(u)$, $u \in I$, $\rho^- = \pi^-$, and

$$\rho^+ = \pi^+ - \frac{\delta}{|u_o|}.$$

Clearly, $\rho^+ < \pi^+$ so $\rho' < \pi'$. We must show ρ' to be a valid inequality for $P_-^+(G_n, u_o)$ in order to reach a contradiction to π' being minimal. First of all, $\rho^+ \geq 0$ will be shown.

By (13), $\pi^+|L(u_o)| \geq \pi(L(u_o))$ so by definition of δ,

$$\begin{aligned} 1 + \delta &= \pi(L(u_o)) + \pi^+|u_o - L(u_o)| \\ &\leq \pi(L(u_o)) + \pi^+|u_o| - \pi(L(u_o)) \\ &\leq \pi^+|u_o|. \end{aligned}$$

Hence, $\pi^+ \geq (1+\delta)/|u_o|$, and $\rho^+ = \pi^+ - \delta/|u_o| \geq 1/|u_o| > 0$.

In showing ρ' to be a valid inequality, it suffices to consider solutions t' for $P^+_-(G_n, u_o)$ for which only one of s^+, s^- is positive by the same argument as used in the proof of theorem I.5B. Clearly, if $s^+ = 0$, then

$$\sum_{g\in G_n} \rho(g)t(g) + \rho^+ s^+ + \rho^- s^- = \sum_{g\in G_n} \pi(g)t(g) + \pi^+ s^+ + \pi^- s^- \geq 1.$$

Suppose $s^+ > 0$ and $s^- = 0$. Then,

$$\begin{aligned}
&\sum_{g\in G_n} \rho(g)t(g) + \rho^+ s^+ + \rho^- s^- \\
&\qquad = \sum_{g\in G_n} \pi(g)t(g) + \pi^+ s^+ - \frac{\delta}{|u_o|} s^+ \\
&\qquad \geq \pi\Big(\sum_{g\in G_n} gt(g)\Big) + \pi^+ s^+ - \frac{\delta}{|u_o|} s^+ \\
&\qquad = \pi(v) + \pi^+ s^+ - \frac{\delta}{|u_o|} s^+,
\end{aligned}$$

where $v = \sum_{g\in G_n} gt(g)$, by (12). In order to complete the proof that ρ' is valid, $\pi(v) + \pi^+ s^+ - (\delta/|u_o|)s^+ \geq 1$ must be shown.

If $s^+ \geq |u_o|$, then by $\pi(v) \geq 0$ and $\pi^+ - \delta|u_o| \geq 0$,

$$\begin{aligned}
\pi(v) + \pi^+ s^+ - \frac{\delta}{|u_o|} s^+ &\geq \pi^+ s^+ - \frac{\delta}{|u_o|} s^+ \\
&\geq \Big(\pi^+ - \frac{\delta}{|u_o|}\Big)|u_o| \\
&= \pi^+|u_o| - \delta \\
&= \pi^+|u_o - L(u_o)| + \pi^+|L(u_o)| - \delta \\
&\geq \pi^+|u_o - L(u_o)| + \pi(L(u_o)) - \delta \\
&= 1 + \delta - \delta = 1,
\end{aligned}$$

by (13) and the definition of δ.

Suppose now $0 < s^+ < |u_o|$. Then by $t' \in P_-^+(G_n,u_o)$, $|v| + s^+ = |u_o|$, so $|v|<|u_o|$. Since $L(u_o)$ is the largest element of G_n below u_o, $|v| \le |L(u_o)|$. Hence,

$$\pi(v) + \pi^+s^+ - \frac{\delta}{|u_o|}s^+$$

$$= \pi(v) + \pi^+(|u_o-v|) - \frac{\delta}{u_o}|u_o-v|$$

$$= \pi(v) + \pi^+|u_o-L(u_o)| + \pi^+|L(u_o)-v| - \frac{\delta}{|u_o|}|u_o-v|$$

$$\ge \pi(v) + \pi(L(u_o)-v) + \pi^+|u_o-L(u_o)| - \delta$$

by (13) and $|u_o-v| \le |u_o|$. Now, by subadditivity of π,

$$\pi(v) + \pi^+s^+ - \frac{\delta}{|u_o|}s^+ \ge \pi(L(u_o)) + \pi^+|u_o-L(u_o)| - \delta$$

$$\ge 1 + \delta - \delta = 1.$$

Therefore, ρ' is a valid inequality for $P_-^+(G_n,u_o)$ and (9) is proven. The proof of (10) is similar. The proof of (11) is close to the proof of theorem I.6 and is as follows.

Suppose that (11) does not hold for $g_i \in G_n$ and that $\pi(g_i) > 0$. Define $\delta = \min\{\delta_1, \delta_2\}$ where δ_1 and δ_2 are given by

$$\pi(g_i) + \pi(L(u_o)-g_i) = \pi(L(u_o)) + \delta_1,$$

$$\pi(g_i) + \pi(R(u_o)-g_i) = \pi(R(u_o)) + \delta_2.$$

Then, $\delta_1 > 0$, $\delta_2 > 0$ and $\delta > 0$. Now, define $\rho' = (\rho, \rho^+, \rho^-)$ with $\rho^+ = \pi^+$, $\rho^- = \pi^-$ and ρ defined as in the proof of theorem I.6. The proof now proceeds exactly as the proof of theorem I.6. The only part remaining is to show that if (11) is violated for some $g_i \in G_n$, then it is violated for a $g_i \in G_n$ having $\pi(g_i) > 0$. However, that part is easy and is shown below.

Suppose (11) is violated for $g_i \in G_n$ having $\pi(g_i) = 0$. Then,

$$\pi(L(u_o)-g_i) = \pi(L(u_o)) + \delta_1, \text{ and}$$

$$\pi(R(u_o)-g_i) = \pi(R(u_o)) + \delta_2,$$

for some $\delta_1 > 0$ and $\delta_2 > 0$. Clearly $\pi(L(u_o)-g_i) > 0$ and $\pi(R(u_o)-g_i) > 0$. If (11) is violated for either $g = L(u_o)-g_i$ or $g = R(u_o)-g_i$, then we are done. Suppose instead (11) is satisfied for both. Then

$$\pi(L(u_o)-g_i) + \pi(g_i+\Im(\tfrac{1}{n})) = \pi(R(u_o)), \text{ and}$$

$$\pi(R(u_o)-g_i) + \pi(g_i+\Im(-\tfrac{1}{n})) = \pi(L(u_o)).$$

Substituting for $\pi(L(u_o)-g_i)$ and $\pi(R(u_o)-g_i)$ and adding gives

$$\delta_1 + \delta_2 + \pi(g_i + \tfrac{1}{n}) + \pi(g_i - \tfrac{1}{n}) = 0,$$

a contradiction. Hence, (11) is violated for either $g = L(u_o)-g_i$ or $g = R(u_o)-g_i$, and the proof is completed.

In the appendix, we discuss the computation of the extreme valid inequalities for $P^+_-(G_n,u_o)$ and give these inequalities for

$n = 1,2,\ldots,7$. A comparison with the extreme valid inequalities for $P(G_n, L(u_o))$ and $P(G_n, R(u_o))$ is interesting. However, it is not the case that the π of an extreme valid inequality $\pi' = (\pi, \pi^+, \pi^-)$ for $P_-^+(G_n, u_o)$ is necessarily an extreme valid inequality for some $P(G_n, g)$, $g \in G_n$. In section IV, we discuss the connection between the problems $P_-^+(U, u_o)$ and $P(U, u_o)$. In the appendix, we give further discussion of the two problems $P(G_n, g)$ and $P_-^+(G_n, u_o)$.

At this point, we can give one further result connecting $P(G_n, g_o)$ and $P_-^+(G_n, u_o)$. This result says that the relationship between the vertices of the convex hull of solutions to the two problems is a simple one, by contrast with the situation for faces.

<u>Theorem II .4</u> If $t(g_1),\ldots,t(g_{n-1})$ is a vertex of the convex hull of solutions to $P(G_n, L(u_o))$, then both

$$t' = (t(g_1),\ldots,t(g_{n-1}), |u_o - L(u_o)|,\ 0), \text{ and}$$

$$t' = (0, t(g_2),\ldots,t(g_{n-1}), |u_o - L(u_o)| + \frac{1}{n}t(g_1), 0)$$

are vertices of the convex hull of solutions to $P_-^+(G_n, u_o)$. A similar statement holds for $P(G_n, R(u_o))$ with $s^- = |R(u_o) - u_o|$ or $s^- = |R(u_o) - u_o| + t(g_{n-1})/n$. Furthermore, all of the vertices of the convex hull of solutions to $P_-^+(G_n, u_o)$ are of this form.

Proof: Clearly every such t' is a solution to $P_-^+(G_n, u_o)$. Suppose one is not extreme. Then,

$$(t(g_1),\ldots,t(g_{n-1}), |u_o - L(u_o)|, 0)$$
$$= \sum_{i=1}^{K} \lambda_i (t_i(g_1),\ldots,t_i(g_n), s_i^+, s_i^-)$$

where $\lambda_i \geq 0$, $\sum\lambda_i = 1$, and each of $(t_i(g_1),\ldots,t_i(g_n),s_i^+,s_i^-)$ is a solution for $P_-^+(G_n,u_o)$. By $s_i^- \geq 0$ and $0 = \sum\lambda_i s_i^-$, every $s_i^- = 0$. But then $s_i^+ \geq |u_o-L(u_o)|$. Also, $\sum\lambda_i s_i^+ = |u_o-L(u_o)|$ so each $s_i^+ = |u_o-L(u_o|$. Then, $(t_i(g_1),\ldots,t_i(g_n))$ is a solution to $P(G_n,L(u_o))$, contradicting $(t(g_1),\ldots,t(g_n))$ being a vertex of the convex hull of solutions.

The proof is similar for $t' = (0,t(g_2),\ldots,t(g_{n-1}), |u_o-L(u_o)| + t(g_1)/n,0)$, but uses $t_i(g_1) = 0$ and exhibits $(t(g_1),\ldots,t(g_n))$, as a convex combination of $(\lfloor ns_i^+\rfloor,\ t_i(g_2),\ldots,t_i(g_n))$, where $\lfloor ns_i^+\rfloor$ denotes the largest integer less than or equal to ns_i^+. The proof for $P(G_n,R(u_o))$ is exactly similar.

The remainder of the proof is to show that every vertex for $P_-^+(G_n,u_o)$ is of the form given here. Clearly, only one of s^+,s^- will be positive for a vertex. Let us assume $s^+ \geq 0$ and $s^- = 0$. Then, $s^+ > 1/n$ implies $t(g_1) = 0$ since if both $s^+ > 1/n$ and $t(g_1) \geq 1$, then

$$(t(g_1),\ldots,t(g_n),s^+,s^-) = \frac{1}{2}(t(g_1)+1,\ldots,t(g_n),s^+ - \frac{1}{n},s^-)$$
$$+ \frac{1}{2}(t(g_1)-1,\ldots,t(g_n),s^+ + \frac{1}{n},s^-).$$

Thus, (t,s^+,s^-) is a convex combination of solutions for $P_-^+(G_n,u_o)$ contradicting it being a vertex of the convex hull of solutions to $P_-^+(G_n,u_o)$.

If $s^- = 0$, then clearly $s^+ = |u_o-L(u_o)| + k/n$ for some integer k. If $k \geq 1$ then $t(g_1) = 0$ by the above argument. Hence, we need only show that

$$(t(g_1) + k, t(g_2), \ldots, t(g_n))$$

is a vertex of the convex hull of solutions to $P(G_n, L(u_o))$. If not, we can express it as a convex combination of other solutions to $P(G_n, L(u_o))$,

$$(t(g_1)+k,\ t(g_2), \ldots, t(g_n))$$

$$= \sum_{i=1}^{K} \lambda_i (t_i(g_1), \ldots, t_i(g_n)).$$

Clearly, we can find c_i with $0 \leq c_i \leq t_i(g_i)$ such that

$$(t(g_1), \ldots, t(g_n), |u_o - L(u_o)| + \frac{k}{n}, 0)$$

$$= \sum_{i=1}^{K} \lambda_i (t_i(g_1) - c_i, t(g_2), \ldots, t(g_n), |u_o - L(u_o)| + \frac{c_i}{n}, 0),$$

but the c_i may not be integers. If they were all integer, $(t(g_1), \ldots, t(g_n), |u_o - L(u_o)| + k/n, 0)$ would be exhibited as a convex combination of solutions to $P^+_-(G_n, u_o)$, giving a contradiction and completing the proof. However, if some c_i is not integer, then

$$(t_i(g_1) - c_i, t_i(g_2), \ldots, t_i(g_n), |u_o - L(u_o)| + \frac{c_i}{n}, 0)$$

$$= \mathfrak{F}(c_i)(t_i(g_1) - \lceil c_i \rceil, t_i(g_2), \ldots, t_i(g_n), |u_o - L(u_o)| + \frac{\lceil c_i \rceil}{n}, 0)$$

$$+ \mathfrak{F}(1 - (c_i))(t_i(g_1) - \lfloor c_i \rfloor, t_i(g_2), \ldots, t_i(g_n), |u_o - L(u_o)| + \frac{\lfloor c_i \rfloor}{n}, 0),$$

where $\lceil c_i \rceil$ denotes the smallest integer above or equal to c_i. Making this substitution into the above convex combination shows that $(t(g_1), \ldots, t(g_n), |u_o - L(u_o)| + k/n, 0)$ is a convex combination of solutions to $P^+_-(G_n, u_o)$, completing the proof.

III. Valid Inequalities for General Problems

III.A. $P(U,u_o)$

We now connect the results about $P(G_n,u_o)$ with the general problem $P(U,u_o)$. Here, U can be any subset of the unit interval including the interval I itself.

We will see that valid inequalities can be obtained for the general problem simply by interpolating from the inequalities we already have. As in section II, the element $\mathfrak{J}(i/n)$ of G_n will be denoted g_i. Also, if u is a point on I, R(u) will denote the first point of G_n on or to the right of u and L(u) the first point on or to the left of u.

Theorem III.1 Let π be a subadditive function on G_n. Define $\pi(u)$ for $u \in I-G_n$ by

$$(1) \quad \pi(u) = n\{|u-L(u)|\pi(R(u)) + |R(u)-u|\pi(L(u))\}.$$

Then, π is a subadditive function on I, and π^* defined on I by

$$\pi^*(u) = \frac{\pi(u)}{\pi(u_o)}, \; u \in I,$$

is a valid inequality for any $P(U,u_o)$, U a subset of I, provided $\pi(u_o) > 0$.

Proof: We remark, first of all, that the proof of theorem I.5 actually proved more than the theorem. The additional result is stated here as a corollary.

Corollary III.2 If π is a subadditive function on a closed subset U of I and if $\pi(u_o) \geq 1$ for some $u_o \in U$, then π is a valid inequality for $P(U',u_o)$ for any subset U' of U. The set U' need not be a subgroup, and u_o need not be in U' but must be in U.

Proof: The proof is based on I(10), which was proven from the same hypothesis as here. To complete the proof, we need only remark that if

$$\sum_{u \in U'} ut(u) = u_o,$$

then

$$\sum_{u \in U'} \pi(u)t(u) \geq \pi(u_o)$$

follows from I(10) and U' contained in U. Since $\pi(u_o) \geq 1$, the result follows.

To return to the proof of the theorem, we need only show that π, defined by (1) on $I-G_n$, is a subadditive function. Then, by corollary III.2 and the scaling of π^* so that $\pi^*(u_o) = 1$, π^* must be a valid inequality for any $P(U,u_o)$.

Clearly, π is continuous and piecewise linear with breaks only at points $u \in G_n$. By (1), if $|u| \to |R(u)|$, $|u| < |R(u)|$, then $|R(u)-u| \to 0$, $|u-L(u)| \to 1/n$, and, hence, $\pi(u) \to \pi(R(u))$.

Define ∇ on $I \times I$ by

$$\nabla(u,v) = \pi(u) + \pi(v) - \pi(u+v), \ u \in I, \ v \in I.$$

Then $\nabla(u,v) \geq 0$, all u,v, if, and only if, π is subadditive on I.

Suppose $\nabla(u_1,v_1) < 0$. Suppose, further, that neither u_1 nor v_1 is in G_n. Then, $\nabla(u_1+\delta, \ v_1-\delta) + \nabla(u_1-\delta, \ v_1+\delta) = 2\nabla(u_1,v_1)$ for some $\delta > 0$ by the linearity of π. Since $\nabla(u_1,v_1) < 0$, at least one of $\nabla(u_1+\delta, v_1-\delta)$, $\nabla(u_1-\delta, v_1+\delta)$ is also negative for some $\delta > 0$, say $\nabla(u_1+\delta, v_1-\delta) < 0$. Then δ can be increased until either $u_1 + \delta = R(u_1)$ or $v_1 - \delta = L(v_1)$. By this argument, we have shown that if $\nabla(u_1,v_1) < 0$, then $\nabla(u_1,v_1) < 0$ for a pair u_1,v_1 having at least one of u_1,v_1 in G_n.

If both u_1 and v_1 are in G_n, then $\nabla(u_1,v_1) \geq 0$ by subadditivity of π on G_n. Therefore, assume $u_1 \in G_n$ and $v_1 \notin G_n$. Then, $u_1 + v_1 \notin G_n$. As before,

$$\nabla(u_1,v_1+\delta) + \nabla(u_1,v_1-\delta) = 2\nabla(u_1,v_1)$$

for some $\delta > 0$, and we can find u_1,v_1 such that $\nabla(u_1,v_1) < 0$ and $u_1 \in G_n$ and one of $v_1,u_1 + v_1$ is also in G_n. But then all three $u_1,v_1,u_1 + v_1$ must be in G_n, contradicting subadditivity of π on G_n.

A contradiction is, thus, reached, and the theorem is proven.

Three remarks are worth making about π^*.

First, all the inequalities for $P(U,u_o)$ obtained by using subadditive π on G_n are convex combinations of those obtained by using only the extreme subadditive functions π on G_n.

Second, intuitively speaking, the power of the resulting inequality will be determined by the size of $\pi(u_o)$ relative to the other $\pi(u)$, $u \in U$. In other words, we want $\pi^*(u) = \pi(u)/\pi(u_o)$ to be small for $u \neq u_o$. To achieve small values $\pi^*(u)$, $\pi(u_o)$ should be large relative to $\pi(u)$. It will usually help to choose π to be a minimal valid inequality for $P(G_n, L(u_o))$ or $P(G_n, R(u_o))$. Even better, π may be chosen to be an extreme valid inequality for one of those two problems.

Third, the valid inequality π^* obtained here can be extended to a valid inequality for $P_-^+(U,u_o)$ by defining

$$\pi^+ = n\pi^*(g_1),$$

$$\pi^- = n\pi^*(g_{n-1}).$$

This result is a special case of the more general property IV.2 given in section IV.

III.B. $P_-^+(U,u_o)$

From valid inequalities for $P_-^+(G_n,u_o)$, a different method for generating valid inequalities for $P_-^+(U,u_o)$ is available.

Theorem III.3 Let $\pi' = (\pi, \pi^+, \pi^-)$ be an extended subadditive function on G_n. Define $\pi(u)$ for $u \in I-G_n$ by

$$(2) \quad \pi(u) = \min\{\pi(L(u)) + \pi^+|u-L(u)|, \pi(R(u)) + \pi^-|R(u)-u|\}.$$

Then, π' is an extended subadditive function on I, and ρ' defined by

$$\rho' = \frac{1}{\pi(u_o)}(\pi,\pi^+,\pi^-)$$

is a valid inequality for $P_-^+(U,u_o)$ provided $\pi(u_o) > 0$.

Proof: As in the proof of theorem III.1, we need only show that π' is an extended subadditive function because of the following corollary to theorem I.5B.

<u>Corollary III.4</u> If π' is an extended subadditive function on a closed set U of I and if both I(11) and I(12) hold, then π' is a valid inequality for $P_{-}^{+}(U',u_o)$ where U' is a subset of U. Furthermore, if $u_o \in U$, then $\pi(u_o) \geq 1$ can replace I(11) and I(12).

Proof: The first statement follows from the proof of theorem I.5B in exactly the same way that corollary III.2 followed from theorem I.5. The second statement is an easy consequence of the definition of extended subadditivity, in particular I(7) and I(8).

We return to the proof of the theorem. We need only prove that π' is an extended subadditive function on I. The function π is piecewise linear on I with only two slopes: π^+ and $-\pi^-$. Furthermore, π is continuous. To prove continuity, we need

$$\pi(g_i) + \pi^+(\frac{i+1}{n} - \frac{i}{n}) \geq \pi(g_{i+1}), \text{ and}$$

$$\pi(g_{i+1}) + \pi^-(\frac{i+1}{n} - \frac{i}{n}) \geq \pi(g_i),$$

which follow from π' being an extended subadditive function on G_n. The fact that π is piecewise linear and continuous on I with only two slopes, π^+ and $-\pi^-$, makes I(7) and I(8) obvious.

The remaining part of the proof is to show that π is subadditive on I. Consider $u \in G_n$ and any $v \in I$. If v also is in G_n, then $\pi(u) + \pi(v) \geq \pi(u+v)$ follows from subadditivity of π on G_n. There are two cases for $\pi(v)$:

Case 1: $\pi(v) = \pi(L(v)) + \pi^+(v-L(v))$;

Case 2: $\pi(v) = \pi(R(v)) + \pi^-(R(v)-v)$.

Case 1 will be considered and the other case is similar. By $u \in G_n$, $L(u+v) = u + L(v)$, $R(u+v) = u + R(v)$, and $(u+v) - L(u+v) = v - L(v)$. By definition of π and by the above,

$$\pi(u+v) \le \pi(L(u+v)) + \pi^+|u+v - L(u+v)|$$
$$\le \pi(u+L(v)) + \pi^+|v-L(v)|.$$

By subadditivity of π on G_n,

$$\pi(u+v) \le \pi(u) + \pi(L(v)) + \pi^+|v-L(v)|,$$

and by case 1,

$$\pi(u+v) \le \pi(u) + \pi(v).$$

We, thus, know $\pi(u) + \pi(v) \ge \pi(u+v)$ provided at least one of u,v is in G_n. Suppose now that neither u nor v is in G_n. Again there are two possible cases for $\pi(u)$, and we will suppose that $\pi(u) = \pi(L(u)) + \pi^+|u-L(u)|$. Then, by the case just considered and by $L(u)$ in G_n,

$$\pi(u) + \pi(v) = \pi(L(u)) + \pi(v) + \pi^+|u-L(u)|$$
$$\ge \pi(L(u) + v) + \pi^+|u-L(u)|.$$

We have already shown that I(7) holds for all $u \in I$ so

$$\pi(u) + \pi(v) \ge \pi(L(u)+v+u-L(u)) = \pi(u+v).$$

The proof is completed.

If π' was minimal for $P^+_-(G_n,u_o)$, then

$\pi(L(u_o)) + \pi^+|u_o-L(u_o)| = 1$, and

$\pi(R(u_o)) + \pi^+|R(u_o)-u_o| = 1$,

and, hence, $\pi(u_o) = 1$ by the construction (2) of π. Therefore π' requires no normalization. However, π' need not be a minimal valid inequality for $P_-^+(U,u_o)$. This question is discussed in the next section.

III.C <u>Numerical Examples</u>

<u>Example 1</u>: Consider the integer linear program

$$x_j \geq 0,\ x_j \text{ integer},\ j = 1,2,3,4,5$$

$$x_1 + 2x_2 + x_3 + x_4 + 5x_5 = 10$$

$$3x_1 - 3x_2 + 2x_3 - 3x_4 + 3x_5 = 5$$

$$x_1 + x_2 + x_3 + 2x_4 + 4x_5 = Z(\min)$$

The optimum linear programming tableau is

$$x_1 \quad + \frac{7}{9}x_3 - \frac{1}{3}x_4 + 2\frac{1}{3}x_5 = 4\frac{4}{9}$$

$$x_2 + \frac{1}{9}x_3 + \frac{2}{3}x_4 + 1\frac{1}{3}x_5 = 2\frac{7}{9}$$

$$1\frac{1}{3}x_3 + 1\frac{2}{3}x_4 + \frac{5}{9}x_5 = Z(\min)$$

The optimum linear programming solution is $x_1 = 4\frac{4}{9}$, $x_2 = 2\frac{7}{9}$, $x_3 = x_4 = x_5 = 0$, $z = 6\frac{1}{3}$. From the first row of the tableau, using as the mapping $\psi : \psi\ (x) = \mathfrak{F}(a)$, we obtain

$$\frac{7}{9}x_3 + \frac{2}{3}x_4 + \frac{1}{3}x_5 = \frac{4}{9} \pmod 1.$$

That is, $U = \{7/9,\ 2/3,\ 1/3\}$, $u_o = 4/9$. The mapping used in the introduction is simply this; that is, we get a problem $P(U,u_o)$ from every row of an optimum linear programming tableau for which the basic variable is integer constrained but at a fractional value.

From appendix 5 of [3], we find the following three extreme valid inequalities among those for $P(G_n,u_o)$, $n = 2,3,6$.

$P(G_2, \frac{1}{2})$: $\pi_1(0) = 0$, $\pi_1(\frac{1}{2}) = 1$;

$P(G_3, \frac{1}{3})$: $\pi_2(0) = 0$, $\pi_2(\frac{1}{3}) = 1$, $\pi_2(\frac{2}{3}) = \frac{1}{2}$;

$P(G_6, \frac{1}{2})$: $\pi_3(0) = 0$, $\pi_3(\frac{1}{6}) = \frac{1}{3}$, $\pi_3(\frac{2}{6}) = \frac{2}{3}$,

$\pi_3(\frac{3}{6}) = 1$, $\pi_3(\frac{4}{6}) = \frac{1}{3}$, $\pi_3(\frac{5}{6}) = \frac{2}{3}$.

We could take any of the faces for cyclic groups from appendix 5 and use them in the following way. The linear interpolation of III.A. extends π_1, π_2, π_3 to the interval I:

$$\pi_1(u) = \begin{cases} 2u, & 0 \le u \le \frac{1}{2} \\ 2-2u, & \frac{1}{2} < u < 1 \end{cases}$$

$$\pi_2(u) = \begin{cases} 3u, & 0 \le u \le \frac{1}{3} \\ \frac{3}{2}-\frac{3}{2}u, & \frac{1}{3} < u < 1 \end{cases}$$

$$\pi_3(u) = \begin{cases} 2u, & 0 \le u \le \frac{1}{2} \\ 3-4u, & \frac{1}{2} < u < \frac{2}{3} \\ -1+2u, & \frac{2}{3} < u \le \frac{5}{6} \\ 4-48, & \frac{5}{6} < u < 1 \end{cases}$$

Our congruence problem has $u_o = 4/9$, and so $\pi_1(u_o) = 8/9$, $\pi_2(u_o) = 5/6$, $\pi_3(u_o) = 8/9$. Since $U = \{7/9, 2/3, 1/3\}$, the valid inequalities from π_1, π_2, and π_3 are of the form

$$\frac{\pi_i(\frac{7}{9})}{\pi_i(u_o)}x_3 + \frac{\pi_i(\frac{2}{3})}{\pi_i(u_o)}x_4 + \frac{\pi_i(\frac{1}{3})}{\pi_i(u_o)}x_5 \geq 1,$$

for $i = 1,2,3$, and are given below:

$$\frac{1}{2}x_3 + \frac{3}{4}x_4 + \frac{3}{4}x_5 \geq 1;$$

$$\frac{2}{5}x_3 + \frac{3}{5}x_4 + \frac{6}{5}x_5 \geq 1;$$

$$\frac{5}{8}x_3 + \frac{3}{8}x_4 + \frac{3}{4}x_5 \geq 1.$$

The 'fractional cutting plane' from [1] is, here,

$$\frac{7}{9}x_3 + \frac{2}{3}x_4 + \frac{1}{3}x_5 \geq \frac{4}{9}, \text{ or } \frac{7}{4}x_3 + \frac{3}{2}x_4 + \frac{3}{4}x_5 \geq 1.$$

That inequality is obtained from the subadditive function π on I given by $\pi(u) = u$. The figure III.1 illustrates the functions π, π_1, π_2, π_3.

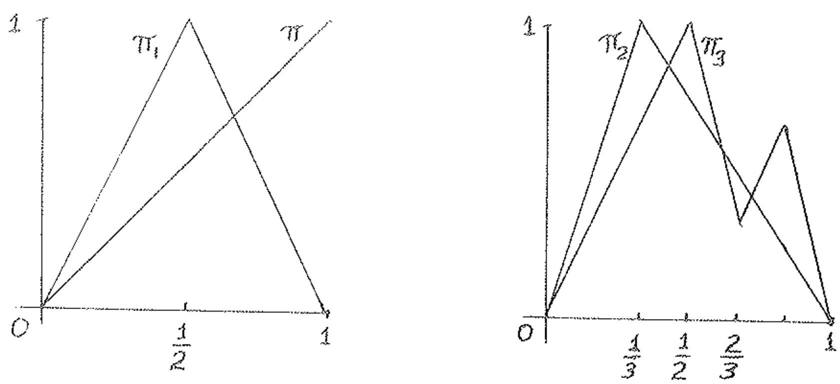

Figure III.1

Example 2: Consider the same integer program but without the integrality restriction on x_5. The first row of the optimal linear programming tableau now gives the congruence:

$$\frac{7}{9}x_3 + \frac{2}{3}x_4 + s^+ \equiv \frac{4}{9} \pmod 1)$$

where $s^+ = \frac{7}{3}x_5$. Thus, $U = \{\frac{7}{9},\frac{2}{3}\}$ and $u_o = \frac{4}{9}$.

From table 2 of the appendix, for n = 1 the only extreme valid inequality has

$$\pi^+ = \frac{1}{|u_o|},\ \pi^- = \frac{1}{|-|u_o||}.$$

Here, $u_o = 4/9$ so $\pi^+ = 9/4$ and $\pi^- = 9/5$. Another extreme valid inequality, this time for n = 3, is

$$\pi(\tfrac{1}{3}) = \frac{1}{3|u_o|},\ \pi(\tfrac{2}{3}) = \frac{1}{6|u_o|},\ \pi^+ = \frac{1}{|u_o|},\ \pi^- = \frac{6|u_o|-1}{4|u_o|-6|u_o|^2}.$$

Since $u_o = 4/9$ here,

$$\pi(\tfrac{1}{3}) = \frac{3}{4},\ \pi(\tfrac{2}{3}) = \frac{3}{8},\ \pi^+ = \frac{9}{4},\ \pi^- = \frac{45}{1.6}$$

The construction III.B extends these two inequalities to functions π_1 and π_2 on the unit interval:

$$\pi_1(u) = \begin{cases} \frac{9}{4}|u| , & 0 \le |u| \le \frac{4}{9} , \\ \frac{9}{5}(1-|u|) , & \frac{4}{9} \le |u| \le 1 , \end{cases}$$

$$\pi_2(u) = \begin{cases} \frac{9}{4}|u|, & 0 \le |u| \le \frac{4}{9}, \\ \frac{3}{8} + \frac{45}{16}\left(\frac{2}{3} - |u|\right), & \frac{4}{9} \le |u| \le \frac{2}{3} \\ \frac{3}{8} + \frac{9}{4}\left(|u| - \frac{2}{3}\right), & \frac{2}{3} \le |u| \le \frac{7}{9} \\ \frac{45}{16}(1 - |u|), & \frac{7}{9} \le |u| \le 1 \end{cases}$$

Since $U = \{\frac{7}{9}, \frac{2}{3}\}$, a valid inequality is

$$\pi_i(\tfrac{7}{9})x_3 + \pi(\tfrac{2}{3})x_4 + \pi_i^+ s^+ \ge 1, \ i = 1,2, \text{ or}$$

$$\pi_i(\tfrac{7}{9})x_3 + \pi_i(\tfrac{2}{3})x_4 + \tfrac{7}{3}\pi_i^+ x_5 \ge 1, \ i = 1,2.$$

Evaluating π_i at $\frac{7}{9}$ and $\frac{2}{3}$ gives the two valid inequalities

$$\tfrac{2}{5}x_3 + \tfrac{3}{5}x_4 + \tfrac{21}{4}x_5 \ge 1, \text{ and}$$

$$\tfrac{5}{8}x_3 + \tfrac{3}{8}x_4 + \tfrac{21}{4}x_5 \ge 1.$$

Other inequalities can be generated in the same way from Table 2.

IV $P(I,u_o)$ and $P^+_-(I,u_o)$

Let the set U now be the entire interval I. The problem $P(I,u_o)$ involves the congruence

$$(1) \qquad \sum_{u \in I} ut(u) = u_o$$

and $P^+_-(I,u_o)$ has the constraint

$$(2) \qquad \sum_{u \in I} ut(u) + \mathfrak{F}(s^+) - \mathfrak{F}(s^-) = u_o,$$

where t is a non-negative integer valued function on I having finite support.

This section intends to reveal something about the extreme valid inequalities for these problems. Such information could be useful in dealing with problems involving subsets of I. The relation to $P(U,u_o)$ is the same as the relation between the master polyhedra and the corner polyhedra corresponding to subsets of a group [3]. Here, every finite cyclic group G_n is a subset of I. In particular, if π is a valid inequality for $P(I,u_o)$, then trivially π is also a valid inequality for $P(U,u_o)$ for every subset U of I, including all cyclic groups $U = G_n$ or subset U of G_n. Furthermore, if π' is a valid inequality for $P^+_-(I,u_o)$, then π is a valid inequality for $P(U,u_o)$, (π,π^+) is a valid inequality for $P^+(U,u_o)$, (π,π^-) is a valid inequality for $P_-(U,u_o)$, and $\pi' = (\pi,\pi^+,\pi^-)$ is a valid inequality for $P^+_-(U,u_o)$ for any subset U of I.

The property of being a valid inequality is hereditary, that is, if π is a valid inequality for $P(S,u_o)$, then it is also valid for any $P(S',u_o)$, $S' \subset S$, and subadditivity for a valid inequality is also hereditary. However, minimality and extremeness are <u>not</u> hereditary properties. That is, π can be a minimal or extreme valid inequality for $P(U,u_o)$ and still, not be for $P(U',u_o)$, $U' \subset U$.

IV.A. Properties and Relation between $P(I,u_o)$ and $P_-^+(I,u_o)$

<u>Property IV.1</u> If $\pi' = (\pi,\pi^+,\pi^-)$ is a valid inequality for $P_-^+(I,u_o)$, then π is a valid inequality for $P(I,u_o)$.

Proof: If π is not a valid inequality for $P(I,u_o)$, then there is a t satisfying (1) with $\sum \pi(u)t(u) < 1$. Clearly, (t,0,0) solves (2) as well contradicting π' being a valid inequality for $P_-^+(I,u_o)$ and completing the proof.

Recall that we define $|u|$, the absolute value of u, as the real number corresponding to $u \in I$. We can then define right and left limits,

$$\lim_{u \downarrow u_o} \text{ and } \lim_{u \uparrow u_o}$$

as the point $|u|$ approaches $|u_o|$ on the real line from the right ($|u| > |u_o|$) or the left ($|u| < |u_o|$).

<u>Property IV.2</u> If π is a valid inequality for $P(I,u_o)$ and if

$$\ell^{+} = \underset{u \downarrow 0}{\text{limit}} \frac{\pi(u)}{|u|} \text{ and } \ell^{-} = \underset{u \uparrow 1}{\text{limit}} \frac{\pi(u)}{1-|u|}$$

both exists (that is, if π has right and left derivatives at 0 and 1), then $\pi' = (\pi, \ell^{+}, \ell^{-})$ is a valid inequality for $P^{+}_{-}(I,u_o)$.

Proof: Suppose $t' = (t,s^{+},s^{-})$ solves (2) but

$$\sum_{u \in I} \pi(u)t(u) + \ell^{+}s^{+} + \ell^{-}s^{-} = 1 - \epsilon, \ \epsilon > 0.$$

We can assume, just as in the proof of theorem I 5B, that only one of s^{+}, s^{-} is positive, say $s^{+} > 0$ and $s^{-} = 0$.

Choose an integer M large enough that

$$\left| \ell^{+} - \frac{\pi(\mathcal{F}(\frac{s^{+}}{M}))}{\frac{s^{+}}{M}} \right| < \frac{\epsilon}{s^{+}},$$

which can be done by existence of ℓ^{+}. Let

$$t_1(u) = \begin{cases} t(u), & u \neq s^{+}/M, \\ t(u) + M, & u = s^{+}/M. \end{cases}$$

Clearly, t_1 satisfies (1) since t' satisfied (2). But,

$$\sum_{u \in I} \pi(u)t_1(u) = \sum_{u \in I} \pi(u)t(u) + M\pi(\tfrac{s^{+}}{M})$$

$$< \sum_{u \in I} \pi(u)t(u) + \ell^{+}s^{+} + \epsilon = 1,$$

contradicting π being a valid inequality for $P(I,u_o)$.

Lemma IV.3 If π is a subadditive function on I and if

$$\limsup_{u\downarrow 0} \frac{\pi(u)}{|u|} = \beta < \infty,$$

then

$$\operatorname{limit}_{u\downarrow 0} \frac{\pi(u)}{|u|} = \beta \ .$$

Proof: If the limit does not exists, then

$$\liminf_{u\downarrow 0} \frac{\pi(u)}{|u|} \neq \beta,$$

that is, there are points v arbitrarily close to 0 with $\pi(v)/|v| \leq \alpha < \beta$. By the lim sup being β, there are also points u arbitrarily close to 0 with $\pi(u)/|u| > \alpha$.

Choose any u with $\pi(u)/|u| > \alpha$ and choose $0 < v < u$ with $\pi(v)/|v| \leq \alpha < \beta$. Then, $|u|$ can be written as an integer multiple of $|v|$ and a remainder:

$$|u| = \left\lfloor \frac{u}{v} \right\rfloor |v| + \gamma(u), \quad 0 \leq \gamma(u) < |v|.$$

Since π is subadditive on I,

$$\pi(u) \leq \pi(\left\lfloor \frac{u}{v} \right\rfloor v) + \pi(\gamma(u))$$

$$\leq \left\lfloor \frac{u}{v} \right\rfloor \pi(v) + \pi(\gamma(u)).$$

Hence, by $\pi(v)/|v| \leq \alpha$,

$$\pi(u) \leq \left[\frac{u}{v}\right] \alpha|v| + \pi(\gamma(u))$$

$$\leq \alpha|u| + \pi(\gamma(u))$$

Since the lim sup exists, $\pi(\gamma(u)) \leq (\beta + \delta)|\gamma(u)| \leq (\beta + \delta)|v|$, for some $\delta > 0$ provided v is small enough. Hence,

$$\pi(u) \leq \alpha|u| + (\beta + \delta)|v|,$$

and as $v \to 0$, we have $\pi(u)/|u| \leq \alpha$, a contradiction to $\pi(u)/|u| > \alpha$. The lemma is, thus, proven.

Clearly, we have the same property for lim sup $\pi(u)/(1 -|u|)$ and limit $\pi(u)/(1 -|u|)$ as $u \uparrow 1$.

Lemma IV.4 If π is a subadditive function on I and if

$$\underset{u \downarrow 0}{\text{limit}} \frac{\pi(u)}{|u|} = \beta,$$

then

$$\limsup_{u \downarrow v} \frac{\pi(u)-\pi(v)}{|u|-|v|} \leq \beta$$

for any $v \in I$.

Proof: By subadditivity, $\pi(u) \leq \pi(v) + \pi(u-v)$. By $\beta = \text{limit}_{u \downarrow 0}\, \pi(u)/|u|$, for any $\epsilon > 0$ there is a $\delta > 0$ such that $\pi(u-v) \leq (\beta + \epsilon)(|u| - |v|)$ for $|u| > |v|$ and $|u| - |v| \leq \delta$. For such u,v in I,

$$\pi(u) \leq \pi(v) + (\beta + \epsilon)(|u| - |v|), \text{ or}$$

$$\frac{\pi(u) - \pi(v)}{|u| - |v|} \leq \beta + \epsilon.$$

The lemma is thus proven.

Clearly, a similar statement holds for limit $_{u\uparrow 1}$ $\pi(u)/(1-|u|)$ and

$$\limsup_{u\uparrow v} \frac{\pi(u)-\pi(v)}{|v|-|u|} .$$

<u>Property IV.5</u> If $\pi' = (\pi, \pi^+, \pi^-)$ is a minimal valid inequality for $P^+_-(I, u_o)$, then

$$\pi^+ = \underset{u\downarrow 0}{\text{limit}} \frac{\pi(u)}{|u|}, \text{ and}$$

$$\pi^- = \underset{u\uparrow 1}{\text{limit}} \frac{\pi(u)}{1-|u|} .$$

Proof: By subadditivity of π', $\pi(u) \le \pi^+|u|$ so

$$\limsup_{u\downarrow 0} \frac{\pi(u)}{|u|} \le \pi^+ .$$

Then, lemma IV.3 implies that limit $_{u\downarrow 0}$ $\pi(u)/|u|$ exists and is less than or equal to π^+. Similarly, limit $_{u\uparrow 1}$ $\pi(u)/1-|u|$ exists and is less than or equal to π^-. If either limit is less than π^+, or π^-, respectively, then property IV.2 implies that π' is not a minimal valid inequality, and the proof is complete.

<u>Property IV.6</u> If π is a subadditive function on I and if $\pi(u) \to 0$ as $u \downarrow 0$ and $\pi(u) \to 0$ as $u \uparrow 1$, then π is continuous at every $u \in I$.

Proof: For any $u \in I$,

$$\pi(u+\delta) - \pi(\delta) \le \pi(u) \le \pi(u+\delta) + \pi(-\delta)$$

$$-\pi(\delta) \le \pi(u) - \pi(u+\delta) \le \pi(-\delta).$$

As $\delta \downarrow 0$, we have $-\delta \uparrow 1$ (since δ is a group element and $-\delta = 1 - \delta$) and $u + \delta \downarrow u$. Therefore, $\pi(u + \delta) \to \pi(u)$ as $u + \delta \downarrow u$. Now, letting $\delta \uparrow 1$ gives $-\delta \downarrow 0$ and $u + \delta \uparrow u$ so that $\pi(u + \delta) \to \pi(u)$ as $u + \delta \uparrow u$.

Theorem I.6 applies here since I is closed under addition and says that a valid inequality π for $P(I,u_o)$ is minimal if and only if $\pi(u) + \pi(u_o - u) = 1$ for all $u \in I$. The analogous result for $P^+_-(I,u_o)$ will now be given.

Property IV.7 A valid inequality $\pi' = (\pi, \pi^+, \pi^-)$ for $P^+_-(I,u_o)$ is minimal if and only if

(3) $\pi(u) + \pi(u_o - u) = 1, \; u \in I,$

(4) $\pi^+ = \lim_{u \downarrow 0} \frac{\pi(u)}{|u|}$, and

(5) $\pi^- = \lim_{u \uparrow 1} \frac{\pi(u)}{1-|u|}$.

Proof: Suppose π' is a minimal valid inequality. Then by property IV.5, (4) and (5) hold. Furthermore, property IV.1 implies π is a valid inequality for $P(I,u_o)$. If π is not a minimal valid inequality for $P(I,u_o)$, then there is a valid inequality $\rho < \pi$ and (ρ, π^+, π^-) is a valid inequality for $P^+_-(I,u_o)$ by

$$\sum_{u\in I} \rho(u)t(u) + \pi^+ s^+ + \pi^- s^-$$

$$\geq \sum_{u\in I} \rho(u)t(u) + \pi(\mathcal{F}(s^+)) + \pi(\mathcal{F}(-s^-))$$

$$\geq \sum_{u\in I} \rho(u)t(u) + \rho(\mathcal{F}(s^+)) + \rho(\mathcal{F}(-s^-))$$

$$\geq 1,$$

since ρ is a valid inequality. We can use $\pi^+ s^+ \geq \pi(\mathcal{F}(s^+))$ and similarly for $\pi^- s^-$ because π' is minimal and, hence, subadditive by theorem I.2. Therefore, π must be minimal, and (3) must hold. We have shown that if π' is a minimal valid inequality for $P_-^+(I,u_o)$, then (3), (4) and (5) must hold. As a corollary, we have seen that π must be a minimal valid inequality for $P(I,u_o)$.

Now suppose (3), (4) and (5) hold for a valid inequality π' for $P_-^+(I,u_o)$. If $\rho' < \pi'$ for $\rho' = (\rho, \rho^+, \rho^-)$ a valid inequality for $P_-^+(I,u_o)$, then at least one of $\rho^+ < \pi^+$, $\rho^- < \pi^-$, or $\rho(u) < \pi(u)$ for some $u \in I$ must hold. The latter possibility is ruled out by (3), just as in proving theorem I. 6. Hence, $\rho(u) = \pi(u)$, all $u \in I$. Hence, at least one of $\rho^+ < \pi^+$, $\rho^- < \pi^-$ must hold. We will reach a contradiction by supposing $\rho^+ < \pi^+$, and the proof is similar if $\rho^- < \pi^-$.

Suppose $\rho^+ < \pi^+$. By (4), there is some $v \in I$ with $\rho^+ < \pi(v)/|v|$, and, hence, $\rho^+|v| < \pi(v)$. But then $t(u_o-v) = 1$, $s^+ = |v|$ is a solution for $P_-^+(I,u_o)$ having

$$\rho(u_o-v)t(u_o-v) + \rho^+ s^+ = \pi(u_o-v) + \rho^+|v|$$

$$< \pi(u_o-v) + \pi(v) = 1$$

by (3). Hence, ρ' is not a valid inequality for $P_-^+(I,u_o)$ completing the proof.

<u>Property IV.8</u> If π is an extreme valid inequality for $P(I,u_o)$ and π^+, π^- are given by (4) and (5), then $\pi' = (\pi, \pi^+, \pi^-)$ is an extreme valid inequality for $P_-^+(I,u_o)$.

Proof: By property IV.2, π' is a valid inequality since we are assuming existence of the limits in (4) and (5). By theorem I.1, π is minimal so (3) holds. Hence, by the previous property, π' is a minimal valid inequality for $P_-^+(I,u_o)$.

Suppose π' is not extreme. Then, there are valid inequalities ρ' and σ' for $P_-^+(I,u_o)$ with

$$\pi' = \frac{1}{2}\rho' + \frac{1}{2}\sigma' = \frac{1}{2}(\rho,\ \rho^+,\ \rho^-) + \frac{1}{2}(\sigma,\ \sigma^+,\ \sigma^-).$$

Now, ρ' and σ' must both be minimal by lemma I.4 since π' is minimal. By property IV.1, ρ and σ are valid inequalities for $P(I,u_o)$. By hypothesis, π is an extreme valid inequality for $P(I,u_o)$ so $\rho = \sigma = \pi$. By ρ' and σ' being minimal valid inequalities for $P_-^+(I,u_o)$ and by property IV.5, $\rho^+ = \sigma^+ = \pi^+$ and $\rho^- = \sigma^- = \pi^-$ because $\rho = \sigma = \pi$. Thus, π' is extreme.

<u>Property IV.9</u> If $\pi' = (\pi,\ \pi^+,\ \pi^-)$ is an extreme valid inequality for $P_-^+(I,u_o)$, then π is an extreme valid inequality for $P(I,u_o)$.

Proof: Since π' is extreme, it is also minimal, and by properties IV.1 and IV.7, π is a minimal valid inequality for $P(I,u_o)$. It is, therefore, a subadditive function on I by theorems I.2 and I. 5. Suppose π is not an extreme valid inequality for $P(I,u_o)$. Then,

$$\pi = \frac{1}{2}\rho + \frac{1}{2}\sigma,\ \rho \neq \sigma,$$

where ρ and σ must be minimal valid inequalities for $P(I,u_o)$ by lemma I.4. Then, $\frac{1}{2}\rho \leq \pi$ and

$$\limsup_{u \downarrow 0} \frac{\rho(u)}{|u|} \le \limsup_{u \downarrow 0} \frac{2\pi(u)}{|u|} = 2\pi^+.$$

Hence, limit $_{u\downarrow 0}\ \rho(u)/|u|$ exists by lemma IV.3. Call it ℓ_1^+. Similarly, limit $_{u\uparrow 1}\rho(u)/(1-|u|)$ exists, and let us call it ℓ_1^-. Obviously, the same limits exists for σ, and let us call them ℓ_2^+ and ℓ_2^-. By property IV.5 and by $\pi = \frac{1}{2}\rho + \frac{1}{2}\rho$, it follows that $\pi^+ = \frac{1}{2}\ell_1^+ + \frac{1}{2}\ell_2^+$ and $\pi^- = \frac{1}{2}\ell_1^- + \frac{1}{2}\ell_2^-$. Hence, $\rho' = (\pi,\ \ell_1^+,\ \ell_1^-)$ and $\sigma' = (\sigma,\ \ell_2^+,\ \ell_2^-)$ are valid inequalities for $P_-^+(I,u_o)$ by property IV.2. But, $\pi' = \frac{1}{2}\rho' + \frac{1}{2}\sigma'$, a contradiction to π' being extreme. Thus, the property is proven.

These nine results give a fairly complete picture of the relation between extreme valid inequalities for the two problems $P_-^+(I,u_o)$ and $P(I,u_o)$. In addition, the results give some idea as to what these extreme valid inequalities are like.

IV.B. Construction of Some Extreme Inequalities

Next, we will see how to construct some extreme valid inequalities for $P(I,u_o)$ and $P_-^+(I,u_o)$ from extreme valid inequalities for $P_-^+(G_n,u_o)$. Let $\pi' = (\pi,\ \pi^+,\ \pi^-)$ be the valid inequality for $P_-^+(I,u_o)$ obtained by the method of theorem III.3 from an extreme valid inequality for $P_-^+(G_n,u_o)$.

Theorem IV.10 π' is an extreme valid inequality for $P_-^+(U,u_o)$ for any subset U of I which contains G_n and for which

$$\pi(u) + \pi(u_o - u) = 1, \text{ all } u \in U.$$

Proof: We know by theorem III.3 that π' is a valid inequality for $P_-^+(U,u_o)$. We first show that it is also a minimal valid inequality for $P_-^+(U,u_o)$. Suppose it is not minimal. Then there is a valid inequality ρ' for $P_-^+(U,u_o)$ with $\rho' < \pi'$.

By construction of π', π' is an extreme valid inequality for $P_-^+(G_n,u_o)$, and, hence, π' is a minimal valid inequality for $P_-^+(G_n,u_o)$. Since ρ' is a valid inequality for $P_-^+(G_n,u_o)$ because $G_n \subseteq S$, ρ' must agree with π' on G_n and, as well, $\rho^+ = \pi^+$, $\rho^- = \pi^-$. Hence, $\rho(v) < \pi(v)$ for some $v \in U - G_n$. By the construction of π', for the complementary point $u_o - v$,

$$\pi(u_o-v) = \min\{\pi(L(u_o-v)) + \pi^+(|u_o-v| - |L(u_o-v)|),$$

$$\pi(R(u_o-v) + \pi^-(|R(u_o-v)| - |u_o-v|)\}.$$

Suppose the first term in brackets gives $\pi(u_o-v)$. Then, $s^+ = |u_o-v| - |L(u_o-v)|$, $t(v) = 1$, $t(L(u_o-v)) = 1$ is a solution to $P_-^+(S,u_o)$, but

$$\sum_{u\in S} \rho(u)t(u) + \rho^+s^+ + \rho^-s^- = \rho(v) + \pi(L(u_o-v))$$

$$+ \pi^+(|u_o-v| - |L(u_o-v)|)$$

$$= \pi(u_o-v) + \rho(v)$$

$$\pi(u_o-v) + \pi(v) = 1,$$

contradicting ρ' being a valid inequality for $P_-^+(U,u_o)$. When $\pi(u_o-v)$ is equal to the second term in the brackets, the proof is similar but uses the solution $s^- = |R(u_o-v)| - |u_o-v|$, $t(R(u_o-v)) = 1$, $t(v) = 1$.

Next we show that π' is extreme among the subadditive valid inequalities for $P_-^+(U,u_o)$. This result, together with minimality, will show that π' is an extreme valid inequality for $P_-^+(U,u_o)$ by theorem I.3.

Suppose π' is not an extreme subadditive valid inequality. Then, $\pi' = \frac{1}{2}\rho' + \frac{1}{2}\sigma'$ for subadditive valid inequalities ρ' and σ'. Just as in the proof of minimality, π' is an extreme valid inequality for $P_-^+(G_n,u_o)$ so

$$\pi^+ = \rho^+ = \sigma^+,$$

$$\pi^- = \rho^- = \sigma^-,$$

$$\pi(g_i) = \rho(g_i) = \sigma(g_i),$$

and, hence, $\rho' \neq \sigma'$ means that $\rho(v) \neq \sigma(v)$ for some $v \in U - G_n$. Since $\pi(v) = \frac{1}{2}\rho(v) + \frac{1}{2}\sigma(v)$, one of $\rho(v)$, $\sigma(v)$ is larger than $\pi(v)$ and one is smaller. Without loss of generality, we can assume $\rho(v) > \pi(v) > \sigma(v)$. Again, by the construction of $\pi(v)$, $\pi(v)$ is either

$$\pi(L(v)) + \pi^+(|v| - |L(v)|), \text{ or}$$

$$\pi(R(v)) + \pi^-(|R(v)| - |v|).$$

Let us assume π (v) is given by the first expression, and the proof in the second case is similar.

By subadditivity of ρ' and by $\rho^+ = \pi^+$,

$$\rho(L(v)) + \pi^+(|v| - |L(v)|) \geq \rho(v).$$

But $\rho(L(v)) = \pi(L(v))$ by $L(v) \in G_n$. Hence,

$$\pi(L(v)) + \pi^+(|v| - |L(v)|) \geq \rho(v).$$

But the left-hand side above is equal to $\pi(v)$ by our assumption of case 1 above. Hence, $\pi(v) \geq \rho(v)$ contradicting $\rho(v) > \pi(v)$. The proof is, thus, completed.

We can apply this theorem to table 2 of the appendix. Corresponding to each extreme valid inequality for $P_-^+(G_n,u_o)$, $n = 1,\ldots,6$, we can easily give the set U_c on which $\pi(u) + \pi(u_o-u) = 1$, $u \in S_c$. Then, for any set U, $G_n \subset U \subset U_c$, the inequality given by theorem III.3 is an extreme valid inequality for $P_-^+(U,u_o)$. For G_o,G_1,G_2,G_3,G_4, and and G_6, $U_c = I$ for all extreme valid inequalities of $P_-^+(G_n,u_o)$; the first exception occurs at G_5. There are four exceptions for G_5 among the 26 faces given by table 2 and the reflections. These exceptions are discussed further following corollary IV. 18.

The unique extreme valid inequality for $P_-^+(G_o,u_o)$, where G_o is the subset consisting of only the point 0, is of particular interest. It is readily seen that this inequality π' when used in conjunction with a mapping φ gives the mixed integer cut of [2]. We see at once that for this π', $\pi(u) + \pi(u_o-u) = 1$ for all u so that π' is an extreme valid inequality for $P_-^+(U,u_o)$ for any set $S \subset I$ provided $0 \in S$, which is actually not a restriction since 0 can always be adjoined to S without changing the problem.

When the inequalities π' given by theorem III.3 satisfy $\pi(u) + \pi(u_o-u) = 1$, then the theorem just proven says that π' is an extreme valid inequality for $P_-^+(I,u_o)$. By property IV.9, π is an extreme valid inequality for $P(I,u_o)$. For subsets U of I, we know that π is a valid inequality for $P(U,u_o)$, but we do not know that π is extreme for $P(U,u_o)$. The following theorem establishes that result for some U and applies, in fact, for any extreme valid inequality for $P(I,u_o)$, not just those given by theorem III.3.

Theorem IV.11 If π is an extreme valid inequality for $P(I,u_o)$ and consists of straight line segments connected at values u belonging to a regular grid G_m with $u_o \in G_m$, then π is an extreme valid inequality for $P(G_m,u_o)$.

Proof: Since π is extreme for $P(I,u_o)$, it cannot be written as $\frac{1}{2}\rho + \frac{1}{2}\sigma$ for different valid inequalities ρ,σ for $P(I,u_o)$. Certainly π is a valid inequality for $P(G_m,u_o)$, and if it is not extreme for $P(G_m,u_o)$, then $\pi = \frac{1}{2}\rho + \frac{1}{2}\sigma$ for different valid inequalities ρ,σ for $P(G_m,u_o)$. If both ρ and σ are valid inequalities for $P(I,u_o)$, a contradiction is reached. However, both can be extended to valid inequalities for $P(I,u_o)$ by defining them on $I - G_m$ as in theorem III.1. Furthermore, such a construction maintains $\pi = \frac{1}{2}\rho + \frac{1}{2}\sigma$ on all of I since π also consists of straight line segments joined at points of G_m. The proof is, thus, completed.

This theorem enables some extreme valid inequalities (faces) of the polyhedra $P(G,g_o)$ of [3] to be constructed. It is of particular interest when one extreme inequality of $P_{-}^{+}(G_n,u_o)$ gives rise to many slight variants, all of which are extreme for $P(I,u_o)$ and all of which in turn give rise to apparently unrelated faces of $P(G,u_o)$. Before showing that possibility, we digress to give a theorem related to the two-slope construction of theorem III.3.

Theorem IV.12 Let π be a continuous function on I consisting of a finite number of straight line segments, each line segment having a slope $\pi^+ > 0$ or else $-\pi^- < 0$. If π is a subadditive function on I with $\pi(u_o) = 1$ for some $u_o \in I$, then π is extreme among the subadditive valid inequalities ρ for $P(I,u_o)$, which have $\rho(u_o) = 1$.

Proof: The theorem asserts that if $\pi = \frac{1}{2}\rho + \frac{1}{2}\sigma$ where ρ and σ are subadditive valid inequalities for $P(I,u_o)$ with $\rho(u_o) = \sigma(u_o) = 1$, then $\rho(u) = \sigma(u)$ for all $u \in I$. We know from theorem I.5 that π is a subadditive valid inequality for $P(I,u_o)$.

Suppose $\pi = \frac{1}{2}\rho + \frac{1}{2}\sigma$ for subadditive valid inequalities ρ,σ for $P(I,u_o)$ with $\rho(u_o) = \sigma(u_o) = 1$. Since π has a right-hand derivative π^+ at 0,

$$\limsup_{u\downarrow 0} \left(\frac{\rho(u)}{|u|}\right) \leq \limsup_{u\downarrow 0} \left(\frac{2\pi(u)}{|u|}\right) = 2\pi^+,$$

and similarly for σ. By lemma IV.3, ρ and σ both have right-hand derivatives ρ^+ and σ^+ at 0. Similarly, the left-hand derivatives ρ^- and σ^- at 1 exist.

We next show that ρ and σ have the same form as π; that is, continuous line segments of slope ρ^+ or ρ^- (σ^+ or σ^-). Choose a point u within an interval where π has slope π^+. Let $\delta > 0$ be small enough that $u + \delta$ is in the same interval and that δ' itself lies in the very first interval. Then, $\pi(u) + \pi(\delta) = \pi(u+\delta)$ by the fact that π has the same slope π^+ on $(0,\delta)$ and $(u,u+\delta)$. Hence,

$$\frac{1}{2}\rho(u) + \frac{1}{2}\sigma(u) + \frac{1}{2}\rho(\delta) + \frac{1}{2}\sigma(\delta) = \frac{1}{2}\rho(u+\delta) + \frac{1}{2}\sigma(u+\delta),$$

or

$$\frac{1}{2}(\rho(u) + \rho(\delta) - \rho(u+\delta)) + \frac{1}{2}(\sigma(u) + \sigma(\delta) - \sigma(u+\delta)) = 0.$$

By subadditivity, each of $\rho(u) + \rho(\delta) - \rho(u+\delta)$ and $\sigma(u) + \sigma(\delta) - \sigma(u+\delta)$ is non-negative. Since they sum to zero, each must be zero. Hence,

$$\underset{\delta \downarrow 0}{\text{limit}} \frac{\rho(u+\ )-\rho(u)}{|\delta|} = \underset{\delta \downarrow 0}{\text{limit}} \frac{\rho(\delta)}{|\delta|} = \rho^{+}, \text{ and}$$

$$\underset{\delta \downarrow 0}{\text{limit}} \frac{\sigma(u+\delta)-\sigma(u)}{|\delta|} = \underset{\delta \downarrow 0}{\text{limit}} \frac{\sigma(\delta)}{|\delta|} = \sigma^{+}.$$

Similarly, we can show that the left-hand derivatives of ρ and σ and u is ρ^{+} and σ^{+}. Therefore, $\rho(\sigma)$ has a constant derivative $\rho^{+}(\sigma^{+})$ on the interval, and so it is a straight line with this slope. A similar result is obtained for any x on an interval where π has slope $-\pi^{-}$. Here, one works with subadditivity through the inequality $\rho(-\delta)+ \rho(u+\delta) \geq \rho(u)$, and concludes that both the left and right derivatives at u are ρ^{-}. Hence, both ρ and σ are of the same form as π with two slope straight line segments over the same intervals.

We now show that $\rho^{+} = \sigma^{+} = \pi^{+}$ and $\rho^{-} = \sigma^{-} = \pi^{-}$.

Let ℓ_L^{+} be the total length of the intervals on which the slope of π is π^{+} and which lie to the left of u_o. Similarly, let ℓ_R^{+} be the length of those intervals to the right of u_o on which π has slope π^{+}, and let ℓ_L^{-} and ℓ_R^{-} be the corresponding lengths for intervals on which π has slope $-\pi^{-}$. Since $\pi(u_o) = 1$,

$$\pi^+ \ell_L^+ - \pi^- \ell_L^- = 1,$$

$$\pi^+ \ell_R^+ - \pi^- \ell_R^- = -1,$$

and the same equations hold for ρ^+, ρ^- and σ^+, σ^-. But these two equations have only the solution π^+, π^- because in order for them to have more than one solution, one equation would have to be a linear multiple of the other. But then, $\ell_L^+ + \ell_R^+ = 0$ and $-\ell_L^- - \ell_R^- = 0$ implying that all of ℓ_L^+, ℓ_R^+, ℓ_L^-, and ℓ_R^- are zero. Hence, $\rho^+ = \pi^+$, $\rho^- = \pi^-$, and $\sigma^+ = \pi^+$, $\sigma^- = \pi^-$.

We have two immediate corollaries.

Corollary IV.13 If π meets the conditions of theorem IV.12 and if $\pi(u) + \pi(u_o-u) = 1$ for all $u \in I$, then π is an extreme valid inequality for $P(I,u_o)$.

Proof: If $\pi(u) + \pi(u_o-u) = 1$ for all $u \in I$, then by theorem I.6, π is a minimal valid inequality for $P(I,u_o)$. The subadditive valid inequalities ρ for which $\rho(u_o) = 1$ includes the minimal valid inequalities by theorems I.2 and I.6. Since, by the theorem IV.12, π is extreme among those inequalities, π cannot be written as a mid-point of two other minimal valid inequalities. By lemma I.4 and minimality of π, π is an extreme valid inequality for $P(I,u_o)$.

Corollary IV.14 If π meets the conditions of theorem IV.12, if $\pi(u) + \pi(u_o-u) = 1$ for all $u \in I$, and if π^+, $-\pi^-$ are the two slopes of π with $\pi^+ > 0$, $\pi^- > 0$, then $\pi' = (\pi, \pi^+, \pi^-)$ is an extreme valid inequality for $P_-^+(I,u_o)$.

Proof: The corollary is immediate from corollary IV.13 and property IV.8.

IV.C. Generating Extreme Inequalities and Exponential Growth for Faces of Some $P(G,u_o)$

We begin by discussing some of the possibilities for creating extreme inequalities for $P(I,u_o)$ from extreme inequalities of $P_-^+(G_n,u_o)$ when the condition $\pi(u) + \pi(u_o-u) = 1$ does not hold for all $u \in I$ for the π constructed in theorem III.3.

By way of background, we observe that the π given by the two-slope fill in of theorem III.3 does satisfy $\pi(u) + \pi(u_o-u) = 1$ when $u \in G_n$. This fact is a consequence of II(9), (10), and (11) because they imply

$$\begin{aligned}\pi(g_i) &= \min\{\pi(L(u_o)) - \pi(L(u_o) - g_i),\\ &\qquad \pi(R(u_o)) - \pi(R(u_o) - g_i)\}\\ &= \min\{1 - \pi^+(|u_o - L(u_o)|) - \pi(L(u_o) - g_i),\\ &\qquad 1 - \pi^-(|R(u_o) - u_o|) - \pi(R(u_o) - g_i)\}\end{aligned}$$

by I(9) and (10). Hence,

$$\begin{aligned}\pi(g_i) + \min\{&\pi(L(u_o) - g_i) + \pi^+(|u_o - L(u_o)|),\\ &\pi(R(u_o) - g_i) + \pi^-(|R(u_o) - u_o|)\} = 1.\end{aligned}$$

By the construction of π on $I-G_n$ and by $L(u_o-g_i) = L(u_o) - g_i$ and $R(u_o-g_i) = R(u_o) - g_i$, the min in the equation above is precisely $\pi(u_o-g_i)$.

Since $\pi(u) + \pi(u_o-u) = 1$ for $u \in G_n$, equality also clearly holds for $u = u_o - g_i$, $g_i \in G_n$. These points are located between consecutive grid points $L(u)$, $R(u)$ in the same relative position as u_o is between $L(u_o)$ and $R(u_o)$.

Figures 1(a) and (b) illustrates the possibilities for π on the intervals g_{i-1}, g_i, g_{i+1} and the complementary intervals u_o-u_{i+2}, u_o-u_{i+1}, u_o-u_i, u_o-u_{i-1}, where we

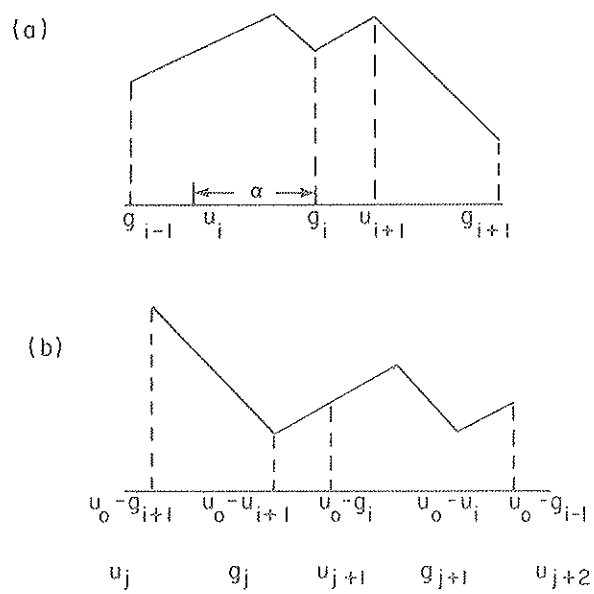

Figure 1.

let $u_{i+1} = g_i+u_o-L(u_o)$ and $u_i = g_i-R(u_o)+u_o$. Then $u_o-u_{i+1} \in G_n$, say $g_j = u_o-u_{i+1}$ and $g_{j+1} = u_o-u_i \in G_n$. If, as in figure 1, the maximum of π in (g_i,g_{i+1}) occurs at $u = u_{i+1}$, then $\pi(u) + \pi(u_o-u) = 1$ for all $u \in (g_i,g_{i+1})$. In order to see this result, consider any interval (u_{i+1},g_{i+1}) where $u_{i+1} = g_i+u_o-L(u_o) = g_{i+1}-R(u_o)+u_o$, and the complementary interval $(u_o-g_{i+1},u_o-u_{i+1})$. The difference $\pi(u_{i+1})-\pi(g_{i+1})$ must be the same as $\pi(u_o-g_{i+1})- \pi(u_o-u_{i+1})$ because $\pi(g_{i+1})+ \pi(u_o-g_{i+1}) = 1$ and $\pi(u_i)+\pi(u_o-u_i) = 1$. Since π can have only two slopes, π must be the same, except for a constant difference in height, in the two intervals (u_{i+1},g_{i+1}) and $(u_o-g_{i+1},u_o-u_{i+1})$.

The second possibility is illustrated in figure 1 by the interval $(u_i\ ,\ g_i)$ and its complementary interval $(u_o-g_i, u_o-u_i\ )$. In both intervals, π has two slopes and a relative maxima occurs within the interval. In this case, we must have $\pi(u) + \pi(u_o-u) > 1$ for all u within either interval. For at $u = u_i$, $\pi(u) + \pi(u_o-u) = 1$, but as u is increased, both $\pi(u)$ and $\pi(u_o-u)$ increase until one of $\pi(u)$, $\pi(u_o-u)$ reaches a maxima. Then, $\pi(u) + \pi(u_o-u)$ remains constant as u increases since one of $\pi(u)$, $\pi(u_o-u)$ is increasing while the other is decreasing at the same rate. When the other $\pi(u)$, $\pi(u_o-u)$ reaches its maxima, then $\pi(u) + \pi(u_o-u)$ decreases until u reaches g_i and u_o-u reaches u_o-g_i at which point $\pi(u) + \pi(u_o-u) = 1$.

An interval (u_i, g_i) or (g_i, u_{i+1}) with only one slope for π will be called an <u>interval of the first type</u>. Here, $u_{i+1} = g_i+u_o-L(u_o)$. The complementary interval will also be an interval of the first type, and for u in an interval of this type, $\pi(u) + \pi(u_o-u) = 1$. An interval (u_{i-1}, g_i) or (g_i, u_{i+1}) with two slopes for π will be called an <u>interval of the second type</u>. Then, its complementary interval is also of the second type, and for u within an interval of the second type, $\pi(u) + \pi(u_o-u) > 1$.

We note that the intervals $(L(u_o), u_o)$ and $(u_o, R(u_o))$ are of the first type, and so are their complementary intervals $(0,\ u_o-L(u_o))$, $(1-R(u_o)+u_o,\ 1)$.

An interval $(u_i\ ,\ g_i)$ will be its own complement if $g_i + g_i = R(u_o)$ since then $u_o - g_i = u_o - R(u_o) + g_i = u_i$. The interval (g_i, u_{i+1}) will be its own complement if $g_i + g_i = L(u_o)$ since then $u_o - g_i = u_o - L(u_o) + g_i = u_{i+1}$. These self-complementary intervals may be of either the first or second type. In what follows, we will exclude the self-complementary intervals in discussion of intervals of the second type.

With this background, we can construct a function π_α from π which will lead to some interesting results. Let $\alpha = (g_i, u_{i+1})$ be an interval of the second type and let β be its complementary interval. We assume α is not its own complement, so $\alpha \neq \beta$. Then, $\pi(u) + \pi(u_o - u) > 1$ for u within either α or β.

Define π_α on I by

$$\pi_\alpha(u) = \begin{cases} \pi(u), & u \in I - \alpha \\ 1 - \pi(u_o - u), & u \in \alpha. \end{cases}$$

Figure 2 illustrates π_α in this case. Let u_α denote the u where $\pi_\alpha(u)$ is smallest in α.

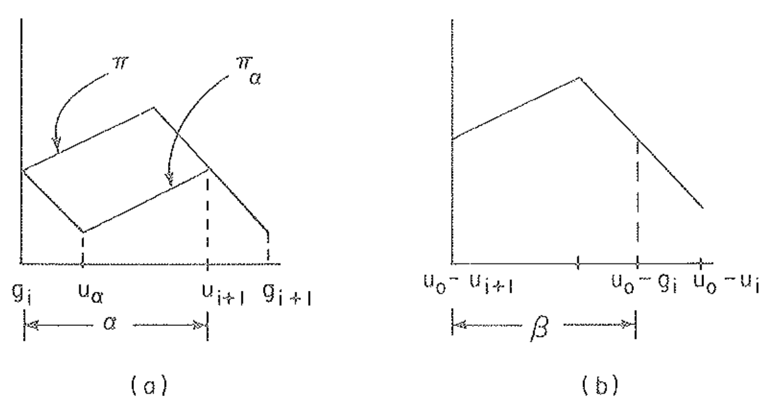

Figure 2.

First, two lemmas are needed. The first applies to any π, and does not depend on the particular construction here.

Lemma IV.15 Let S be a subset of I and let π be a subadditive valid inequality for $P(S,u_o)$. If

$$\pi(u) + \pi(u_o - u) \geq 1 \text{ for all } u \in I - S,$$

$$\pi(u) + \pi(v) \geq \pi(u+v), \text{ and } u \in I-S,\ v \in I-S,$$

then π is a valid inequality for $P(I,u_o)$.

Proof: Consider any t solving $P(I,u_o)$. If $t(u) > 0$ and $t(v) > 0$ for both u and v in I-S, then we can change t by reducing t(u) by 1, reducing t(v) by 1, and increasing t(u+v) by 1. The new t is still a solution, and, since $\pi(u) + \pi(v) \geq \pi(u+v)$, $\sum_{u\epsilon I} \pi(u)t(u)$ has not increased. This process can be continued until $\sum_{u\epsilon I-S} t(u) \leq 1$. At that point, $\sum_{u\epsilon I} \pi(u)t(u) = \pi(v) + \sum_{u\epsilon S}\pi(u)t(u)$, where $v \in I-S$. By subadditivity of π on S,

$$\sum_{u\epsilon I}\pi(u)t(u) \geq \pi(v) + \pi(\textstyle\sum_{u\epsilon S} u t(u))$$

$$= \pi(v) + \pi(u_o - v) \geq 1,$$

by $\pi(u) + \pi(u_o - u) \geq 1$ for $u \in I-S$. The lemma is therefore proven.

The second lemma applies to the particular function π_α constructed here. It actually applies to any two-slope function π in an interval in which the function first decreases and then increases.

Lemma IV.16 If $2\pi_\alpha(u_\alpha) \geq \pi(2u_\alpha)$, then $\pi_\alpha(u) + \pi_\alpha(v) \geq \pi_\alpha(u+v)$ for all $u,v \in \alpha$.

Proof: For any $u \in \alpha$, $u \neq u_\alpha$, either $|u| < |u_\alpha|$ or $|u| > |u_\alpha|$. Let us assume $|u| > |u_\alpha|$. The other case is similar. Then, $\pi_\alpha(u) = \pi_\alpha(u_\alpha) + \pi^+(|u| - |u_\alpha|)$ and for $v \in \alpha$,

$$\begin{aligned}\nabla_\alpha(u,\alpha) &= \pi_\alpha(u) + \pi_\alpha(v) - \pi_\alpha(u+v) \\ &= \pi_\alpha(u_\alpha) + \pi^+(|u| - |u_\alpha|) + \pi_\alpha(v) - \pi_\alpha(u+v) \\ &= \pi_\alpha(u_\alpha) + \pi_\alpha(v) - (\pi_\alpha(u+v) - \pi^+(|u+v| - |u_\alpha+v|)) \\ &\geq \pi_\alpha(u_\alpha) + \pi_\alpha(v) - \pi_\alpha(u_\alpha+v) = \nabla_\alpha(u_\alpha,v),\end{aligned}$$

by $\pi_\alpha(u_\alpha+v) + \pi^+(|u+v| - |u_\alpha+v|) \geq \pi_\alpha(u_\alpha+v)$. Similarly, we can show $\nabla_\alpha(u_\alpha,v) \geq \nabla_\alpha(u_\alpha,u_\alpha)$. Hence if $\nabla_\alpha(u_\alpha, u_\alpha) \geq 0$, then $\nabla_\alpha(u,v) \geq 0$ for all $u,v \in \alpha$.

These two lemmas suffice to prove the following theorem.

<u>Theorem IV.17</u> If $2\pi_\alpha(u_\alpha) \geq \pi_\alpha(2u_\alpha)$, then π_α is a valid inequality for $P(I,u_o)$.

Proof: By lemma IV.15, we need only show that $\pi_\alpha(u) + \pi_\alpha(u_o-u) \geq 1$ for all $u \in \alpha$ and $\pi(u) + \pi(v) \geq \pi(u+v)$ for all $u, v \in \alpha$. The first inequality is true, in fact with equality, by the construction of π_α. The second is true by $2\pi_\alpha(u_\alpha) \geq \pi_\alpha(2u_\alpha)$ and lemma IV.16.

<u>Corollary IV.18</u> If α and its complement β are the only two intervals of the second type, then π_α is an extreme valid inequality for $P(I,u_o)$ if, and only if $2\pi_\alpha(u_\alpha) \geq \pi(2u_\alpha)$.

Proof: If $2\pi_\alpha(u_\alpha) \geq \pi_\alpha(2u_\alpha)$, then, by theorem IV.17, π_α is a valid inequality for $P(I,u_o)$. Furthermore, if α and β are the only two intervals of the second type, then $\pi_\alpha(u) + \pi_\alpha(u_o-u) = 1$ for all $u \in I$, so π_α is minimal. By corollary IV.13, π_α is an extreme valid inequality for $P(I,u_o)$.

We now consider in more detail the case described in corollary IV.18. To begin, two cases will be shown from table 2. When $n = 5$ and $u_o \in (0,\frac{1}{5})$, face 2 from table 2 is illustrated in figure 3. Of course, when $u_o \in (\frac{4}{5},1)$, the reflection is also a face of $P_-^+(G_5,u_o)$. Figure 3 actually shows the construction of theorem III.3 for $u_o = 1/10$.

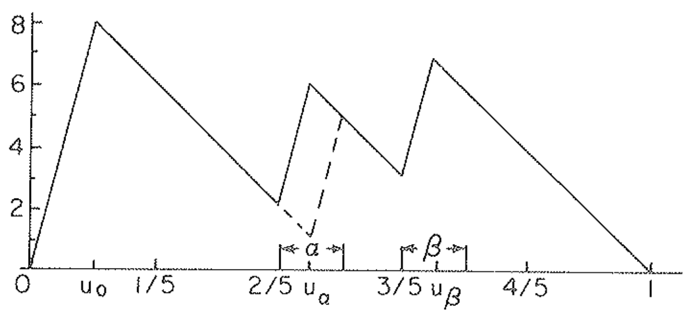

Figure 3.

It is easily verified directly that the two complementary intervals α and β are the only two on which $\pi(u) + \pi(u_o-u) = 1$ does not hold and that $2\pi_\alpha(u_\alpha) \geq \pi(2u_\alpha)$. Here, $u_\alpha = 9/20$.

Figure 4 shows another example for n = 5 and $u_o \varepsilon (\frac{1}{5},\frac{2}{5})$. Its reflection is, again, another example. This figure is face 6.

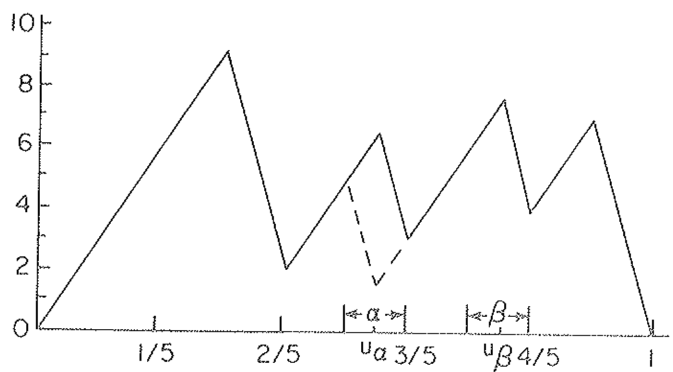

Figure 4

As in Figure 3, α and β are the only two intervals of the second type, and $2\pi_\alpha(u_\alpha) \geq \pi(2u_\alpha)$.

In both figures 3 and 4, the role of α and β can be reversed, and we still have $2\pi_\alpha(u_\alpha) \geq \pi(2u_\alpha)$. In other words, if π_β is defined analogously to π_α with $u_\beta = 13/20$ in Figure 3 and $u_\beta = 15/20$ in Figure 4, then $2\pi_\beta(u_\beta) \geq \pi(2u_\beta)$. The next theorem shows that in this case, a great many extreme valid inequalities can be generated which differ from π only in the intervals α and β.

Theorem IV.19 If α and β are the only two complementary intervals of the second type and if π_α and π_β are each valid inequalities for $P(I,u_o)$, then any continuous, piecewise linear function ρ on I having only the two slopes π^+ and π^- satisfying

$$\rho(u) = \pi(u), \quad u \varepsilon I - (\alpha \cup \beta),$$

$$\rho(u) = 1 - \rho(u_o - u), \quad u \varepsilon \alpha,$$

is an extreme valid inequality for $P(I,u_o)$.

Figure 5 illustrates such a function ρ in the example shown in Figure 3.

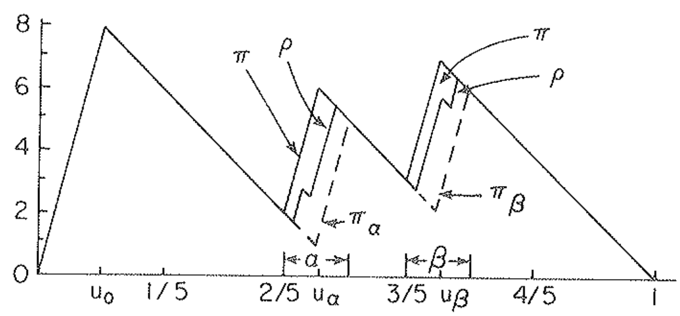

Figure 5.

Proof: We will consider only the case previously considered; that is $\alpha = (g_i, u_{i+1})$ so that the left end-point of α is in G_n. Figure 3 is this case, but Figure 4 is not. The case $\alpha = (u_i, g_i)$ is similar and will not be considered.

First, we will show that neither α nor β is a subinterval of $(0, g_1)$ or $(g_{n-1}, 1)$. Since α and β have an element of G_n as left end-point, if either was a subinterval of $(0, g_1)$, then it would have to be $(0, u_1)$. However, this interval is of the first type as was remarked before lemma IV.15. Hence, the only possibility is that α or β is (g_{n-1}, u_n). We will now exclude that possibility.

Corollary IV.18 says that π_α is extreme and, hence, subadditive. We will show that π is, then, linear on $(g_{n-1}, 1)$ with a slope $-\pi^-$, and, hence, neither α nor β could be (g_{n-1}, u_n). To see that π is linear on $(g_{n-1}, 1)$, recall that $\pi_\alpha(g_i) + \pi_\alpha(u_o - g_i) =$ $\pi(g_i) + \pi(u_o - g_i) = 1 = \pi(u_o)$ and that π_α and π are decreasing on

$(u_o - g_i,\ u_o - u_i)$ because of the shape of π in β. Hence,

$\pi_\alpha(g_i) + \pi_\alpha(u_o - u_i) = \pi_\alpha(R(u_o))$. But, $\pi_\alpha(u_\alpha) = \pi_\alpha(g_i) - \pi^-(|u_\alpha - g_i|)$,

and by subadditivity,

$$\pi_\alpha(u_\alpha) + \pi_\alpha(u_o - u_i) \geq \pi_\alpha(R(u_o) - (u_\alpha - g_i)), \text{ or}$$

$$\pi_\alpha(g_i) - \pi^-(|u - g_i|) + \pi_\alpha(u_o - u_i) \geq \pi_\alpha(R(u_o) - (u_\alpha - g_i)), \text{ or}$$

$$\pi_\alpha(R(u_o)) - \pi^-(|u_\alpha - g_i|) \geq \pi_\alpha(R(u_o) - (u_\alpha - g_i)).$$

By π_α having only two slopes, the reverse inequality also holds, and, hence, π_α is decreasing on the entire interval $(R(u_o),\ R(u_o) + g_1)$. This fact and subadditivity imply that π_α is decreasing on the entire interval $(g_{n-1}, 1)$, completing the proof that neither α nor β is a subinterval of $(g_{n-1}, 1)$ or $(0, g_1)$.

To return to the proof of the theorem, by lemma IV.15 we can prove that ρ is a valid inequality by showing $\rho(u) + \rho(u_o - u) \geq 1$ and $\rho(u) + \rho(v) \geq \rho(u+v)$ for u and v in $\alpha \cup \beta$. The first inequality is obvious from the construction of ρ. What remains is to establish $\rho(u) + \rho(v) \geq \rho(u+v)$ for u and v in $\alpha \cup \beta$.

There are two cases: (i) u,v both in α (or both in β), and (ii) $u \in \alpha$ and $v \in \beta$.

In case (i), we only consider u,v both in α since both in β is exactly similar. By ρ being continuous with the same two slopes as π_α,

$$\rho(u) + \rho(v) \geq \pi_\alpha(u) + \pi_\alpha(v).$$

By π_α being valid, and hence extreme,

$$\pi_\alpha(u) + \pi_\alpha(v) \geq \pi\ (u+v).$$

If $u + v \in I - (\alpha \cup \beta)$, then $\pi_\alpha(u+v) = \rho(u+v)$ so $\rho(u) + \rho(v) \geq \rho(u+v)$. If $u + v \in \beta$, then $\pi\ (u+v) = \pi(u+v) \geq \rho(u+v)$ so, again, $\rho(u) + \rho(v) \geq \rho(u+v)$. The third subcase, $u + v \in \alpha$, is excluded by α not being a subinterval of $(0,g_1)$ or $(g_{n-1},1)$. For any α which is a subinterval of (g_i,g_{i+1}), but not $(0,g_1)$ or $(g_{n-1},1)$, $u+v \notin \alpha$ when $u \in \alpha$ and $v \in \alpha$.

Next, we consider case (ii), $u \in \alpha$ and $v \in \beta$. Here, there are two subcases: $|v| \geq |u_o-u|$, and $|v| < |u_o-u|$. Consider, first, $|v| \geq |u_o-u|$. Since v and u_o-u are both in β, $|v-(u_o-u)| \leq |R(u_o)-u_o|$ and

$$\rho(u+v) = \rho(u_o + (v - (u_o-u)))$$

$$= 1 - \pi^-(|v - (u_o-u)|).$$

Hence, we need only show

$$\rho(u) + \rho(v) \geq 1 - \pi^-(|v - (u_o-u)|).$$

But ρ has only two slopes, so

$$\rho(v) \geq \rho(u_o-u) - \pi^-(|v - (u_o-u)|), \text{ and}$$

$$\rho(u) + \rho(v) \geq \rho(u) + \rho(u_o-u) - \pi^-(|v - (u_o-u)|)$$

$$\geq 1 - \pi^-(|v - (u_o-u)|),$$

completing the proof in this subcase.

Consider now $|v| < |u_o-u|$. In a similar way, we can now show that

$$\rho(u+v) = 1 - \pi^+(|(u_o-u) - v|), \text{ and}$$

$$\rho(v) \geq \rho(u_o-u) - \pi^+(|(u_o-u) - v|).$$

Hence, as before

$$\rho(u) + \rho(v) \geq \rho(u) + \rho(u_o - u) - \pi^+(|(u_o - u) - v|)$$

$$= 1 - \pi^+ (|(u_o - u) - v|) = \rho(u+v).$$

Hence, ρ is a valid inequality for $P(I,u_o)$. To show that it is an extreme valid inequality, we need only remark that $\rho(u) + \rho(u_o - u) = 1$ and apply corollary IV.13. The theorem is, thus, proven.

The development here can be extended to the case where there are several intervals of the second type. However, its present form suffices to show an exponential rate of growth for some of the polyhedra $P(G_n, g)$ of [3]. We show this fact by means of an example.

Consider the group G_n for $n = 20K$, $K \geq 1$, and let $u_o = 1/10 \in G_n$. We said that the function ρ in figure 5 gives an extreme valid inequality for $P(I,u_o)$. The same is true for a great many functions ρ. In figure 6 we illustrate the intervals α and β from figure 5.

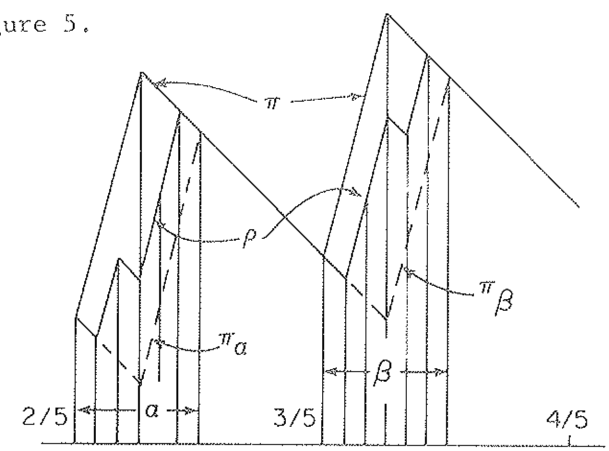

Figure 6.

Let us restrict ρ to be straight lines with breaks at points k/20K. In figure 6, K = 3, and we are perfectly free to let ρ have slope π^+ or $-\pi^-$ in the 3 intervals (8/20, 25/60), (25/60, 26/60), (26/60,9/20). The only restriction on ρ here is that it must have slope π^+ on as many intervals between 8/20 and 10/20 as it has slope π^-. Since ρ has been determined on 8/20 to 10/20, it is given on 12/20 to 14/20 by $\pi(u) + \pi(u_o - u) = 1$. In general, there will be K intervals between 8/20 and 9/20 on which ρ can have either slope. Thus, there are at least 2^K such functions ρ. By theorem IV.14 each one is a face for the problem $P(G_{20K}, 1/10)$. In fact, there are more that 2^K functions; there are

$$\frac{(2K)!}{K!\ K!}$$

such functions ρ. This number results from the fact that we can choose any K of the 2K intervals between 8/20 and 10/20 for ρ to have slope π^+. As K becomes large, this number approaches

$$\frac{2^{2K}}{\sqrt{\pi K}}$$

by Stirling's approximation to n!.

There are an abundance of such examples from table 2. In particular, for n = 7, there are several similar cases. A similar construction works as long as there are two complimentary intervals α and β with π_α and π_β valid and provided the u_o and all other break-points of π fall on group elements.

<u>Appendix:</u> Computing the Faces for $P_{-}^{+}(G_n, u_o)$.

Let $G_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\}$ be the cyclic group of order n.

By theorem II.2B, the faces (extreme valid inequalities) of the convex hull of solutions to $P_{-}^{+}(G_n, u_o)$ are

$$\frac{1}{\pi_o}(\pi(g_1), \ldots, \pi(g_n), \pi^+, \pi^-)$$

where $(\pi(g_1), \ldots, \pi(g_n), \pi^+, \pi^-)$ are the extreme rays of

(1) $\pi(g_i) + \pi(g_j) \geq \pi(g_i + g_j)$,

(2) $\frac{1}{n}\pi^+ \geq \pi(g_1)$, $\frac{1}{n}\pi^- \geq \pi(g_{n-1})$,

(3) $\pi(L(u_o)) + \pi^+(|u_o - L(u_o)|) = \pi(R(u_o)) + \pi^-(|R(u_o) - u_o|) > 0$,

which satisfy, in addition,

$$(4) \begin{cases} \pi(g_i) + \pi(L(u_o) - g_i) = \pi(L(u_o)) \text{ or} \\ \pi(g_i) + \pi(R(u_o) - g_i) = \pi(R(u_o)), \end{cases}$$

where $\pi_o = \pi(L(u_o)) + \pi^+(|u_o - L(u_o)|)$. Recall that $|L(u_o)| \leq |u_o| \leq |R(u_o)|$ for $L(u_o)$, $R(u_o)$ in G_n. For consistency of the above expressions, take $g_o = 0$, $g_n = F(\frac{n}{n}) = 0$ and $\pi(g_o) = \pi(g_n) = 0$. We do not include the constraints $\pi(g_i) \geq 0$ since they are, in fact, implied by (1) in this case.

Let A denote the matrix whose columns correspond to constraints (1); that is $a_{ij} = a_{kj} = 1$ and $a_{\ell j} = -1$ when $g_i + g_k = g_\ell$. Then, (1) is equivalent to $\pi A \geq 0$.

First, let us discuss the computation of the extreme rays of the cone $\pi A \geq 0$. The double description method [4] was used and is interesting theoretically in order to describe the resulting faces. We will describe, briefly, that method. The constraints (1) corresponding to columns of A are introduced successively. Initially, a cone having extreme rays $(1,0,\ldots,0)$, $(0,1,0,\ldots,0)$, $\ldots$, $(0,\ldots,0,1)$ is formed. As each constraint (1) is imposed, we find the extreme rays of the cone formed from the intersection of the previous cone and the half-space of solutions to the new inequality.

To elaborate, suppose that m - 1 of the constraints (1) have been imposed and the resulting extreme rays are $(\pi_i(g_1),\ldots,\pi_i(g_{n-1}))$, $i = 1,\ldots,K$. Consider the matrix

$$\begin{bmatrix} \pi_i(g_1),\ldots,\pi_1(g_{n-1}),\ \pi_1 A^1,\ldots,\pi_1 A^{m-1},\ \pi_1 A^m \\ \vdots \\ \pi_K(g_1),\ldots,\pi_K(g_{n-1}),\pi_K A^1,\ldots,\pi_K A^{m-1},\pi_K A^m \end{bmatrix}$$

where A^j denotes the jth column of the previously defined matrix A. Since each π_i was assumed to satisfy the first m-1 constraints of $\pi A \geq 0$, every $\pi_i A^j \geq 0$ for $i = 1,\ldots,K$ and $j = 1,\ldots,m-1$. The extreme rays of the cone formed by the intersection of the old cone having extreme rays $\pi_1,\ldots,\pi_K$ and the half-space $\pi A^m \geq 0$ will be among the vectors $\pi_1,\ldots,\pi_K$ satisfying $\pi_i A^m \geq 0$ and the linear combinations:

$$\pi = \alpha(\pi_i(g_1),\ldots,\pi_i(g_n)) + \beta(\pi_K(g_1),\ldots,\pi_K(g_n)),$$

$\alpha = \pi_K A^m > 0$ and $\beta = -\pi_i A^m > 0$. However, some of the vectors π formed in this way will not be extreme rays and can be identified by the fact that the j, $1 \leq j \leq m$, for which $\pi A^j = 0$ are a subset of the j similarly defined for another such vector and the ℓ for which $\pi(g_\ell) = 0$ and a subset of the similarly defined ℓ for the other such vector.

At this point, it is easy to see that the conditions (4) can be imposed throughout the above described process. Eventually, the π's not satisfying (4) will be discarded. But if an extreme ray π is formed at some iteration of the double description method and if π violates (4), then any extreme ray formed from π will also violate (4).

The effect of expanding the dimension of the cone by two and imposing conditions (2) is to adjoin two new extreme rays $(0,\ldots,0,1,0)$ and $(0,\ldots,0,1)$. Every other extreme ray will have $\pi(g_1) > 0$ and $\pi^+ = n\pi(g_1)$, and $\pi(g_{n-1}) > 0$ and $\pi^- = n\pi(g_{n-1})$.

The extreme rays of (1), (2), and (4) are called the <u>subadditive rays</u>. They are listed in table 1 for n = 1,2,3,4,5,6, and 7. In addition, the matrix A and the values πA^j are given in the columns following the π's. The last two columns correspond to the constraints $\pi^+ \geq \pi(g_1)/n$ and $\pi^- \geq \pi(g_{n-1})/n$.

For $n \geq 2$ and for each subadditive ray π_k, let the matrix α^k of 0's and 1's be defined by

$$\alpha^k_{ij} = \begin{cases} 1 \text{ if } \pi_k(g_j) + \pi_k(g_i - g_j) = \pi_k(g_i) \\ 0 \text{ otherwise.} \end{cases}$$

Then the condition (4) is equivalent to

$$\alpha^k_{i-1,j} = 1 \text{ or } \alpha^k_{ij} = 1 \text{ for all } j.$$

These α^k are useful in determining whether (4) holds for a particular extreme ray when (3) is imposed later. They are included in table 1.

Table 1. Subadditive Rays

n = 1

	π^+	π^-
1	1	0
2	0	1

n = 2 (A = ∅)

	$\pi(\frac{1}{2})$	π^+	π^-		
1	1	2	2	0	0
2	0	1	0	1	0
3	0	0	1	0	1

$$\alpha^1 = \begin{bmatrix}1\end{bmatrix} \quad \alpha^2 = \begin{bmatrix}1\end{bmatrix} \quad \alpha^3 = \begin{bmatrix}1\end{bmatrix}$$

n = 3

	$\pi(\frac{1}{3})$	$\pi(\frac{2}{3})$	π^+	π^-	$\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$			
1	1	2	3	6	0	3	0	0
2	2	1	6	3	3	0	0	0
3	0	0	1	0	0	0	1	0
4	0	0	0	1	0	0	0	1

$$\alpha^1 = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} \quad \alpha^2 = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \quad \alpha^3 = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} \quad \alpha^4 = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

n = 4

	$\pi(\frac{1}{4})$	$\pi(\frac{2}{4})$	$\pi(\frac{3}{4})$	π^{+}	π^{-}	$\begin{matrix}2\\-1\\0\end{matrix}$	$\begin{matrix}1\\1\\-1\end{matrix}$	$\begin{matrix}-1\\1\\1\end{matrix}$	$\begin{matrix}0\\-1\\2\end{matrix}$		
1	1	2	1	4	4	0	2	2	0	0	0
2	3	2	1	12	4	4	4	0	0	0	0
3	1	2	3	4	12	0	0	4	4	0	0
4	1	0	1	4	4	2	0	0	2	0	0
5	0	0	0	1	0	0	0	0	0	1	0
6	0	0	0	0	1	0	0	0	0	0	1

$$\alpha^1 = \begin{bmatrix}1&0&0\\1&1&1\\0&0&1\end{bmatrix} \quad \alpha^2 = \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix} \quad \alpha^3 = \begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix} \quad \alpha^4 = \begin{bmatrix}1&1&1\\0&1&0\\1&1&1\end{bmatrix} \quad \alpha^5 = \begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} \quad \alpha^6 = \begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

n = 5

$$\begin{bmatrix} 2 & 1 & 1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 2 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & -1 & 1 & 0 & 1 & 2 \end{bmatrix}$$

	$\pi(\frac{1}{5})$	$\pi(\frac{2}{5})$	$\pi(\frac{3}{5})$	$\pi(\frac{4}{5})$	π^+	π^-	$(2,-1,0,0)^T$	$(1,1,-1,0)^T$	$(1,0,1,-1)^T$	$(0,2,0,-1)^T$	$(-1,1,0,1)^T$	$(-1,0,2,0)^T$	$(0,-1,1,1)^T$	$(0,0,-1,2)^T$		
1	2	4	6	3	10	15	0	0	5	5	5	10	5	0	0	0
2	3	6	4	2	15	10	0	5	5	10	5	5	0	0	0	0
3	4	3	2	6	20	30	5	5	0	0	5	0	5	10	0	0
4	6	2	3	4	30	20	10	5	5	0	0	0	5	5	0	0
5	4	3	2	1	20	5	5	5	5	5	0	0	0	0	0	0
6	1	2	3	4	5	20	0	0	0	0	5	5	5	5	0	0
7	2	4	1	3	10	15	0	5	0	5	5	0	0	5	0	0
8	3	1	4	2	15	10	5	0	5	0	0	5	5	0	0	0
9	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0
10	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1

$$\alpha^1 = \begin{bmatrix} 1&0&0&0\\1&1&0&0\\1&1&1&1\\0&0&0&1 \end{bmatrix} \quad \alpha^2 = \begin{bmatrix} 1&0&0&0\\1&1&1&1\\0&0&1&1\\0&0&0&1 \end{bmatrix} \quad \alpha^3 = \begin{bmatrix} 1&0&1&0\\0&1&0&0\\0&0&1&0\\1&1&1&1 \end{bmatrix} \quad \alpha^4 = \begin{bmatrix} 1&1&1&1\\0&1&0&0\\0&0&1&0\\0&1&0&1 \end{bmatrix} \quad \alpha^5 = \begin{bmatrix} 1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1 \end{bmatrix} \quad \alpha^6 = \begin{bmatrix} 1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1 \end{bmatrix}$$

$$\alpha^7 = \begin{bmatrix} 1&0&1&0\\1&1&1&1\\0&0&1&0\\0&0&1&1 \end{bmatrix} \quad \alpha^8 = \begin{bmatrix} 1&1&0&1\\0&1&0&0\\1&1&1&1\\0&1&0&1 \end{bmatrix} \quad \alpha^9 = \begin{bmatrix} 1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1 \end{bmatrix} \quad \alpha^{10} = \begin{bmatrix} 1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1 \end{bmatrix}$$

$n = 6$

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 & 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

	$\pi(\frac{1}{6})$	$\pi(\frac{2}{6})$	$\pi(\frac{3}{6})$	$\pi(\frac{4}{6})$	$\pi(\frac{5}{6})$	π^+	π^-														
1	1	2	3	4	5	6	30	0	0	0	0	0	0	6	6	6	6	6	6	0	0
2	5	4	3	2	1	30	6	6	6	6	6	6	6	0	0	0	0	0	0	0	0
3	1	2	3	4	2	6	12	0	0	0	3	0	3	3	6	3	6	3	0	0	0
4	2	4	3	2	1	12	6	0	3	3	3	6	6	3	3	0	0	0	0	0	0
5	1	2	3	2	1	6	6	0	0	2	2	2	4	2	4	2	2	0	0	0	0
6	1	2	3	1	2	6	12	0	0	3	0	3	3	3	3	3	0	0	3	0	0
7	2	1	3	2	1	12	6	3	0	3	3	0	3	0	3	3	3	0	0	0	0
8	2	1	0	2	1	12	6	3	3	0	3	0	0	0	0	0	3	3	0	0	0
9	1	2	0	1	2	6	12	0	3	0	0	3	0	3	0	0	0	3	3	0	0
10	1	0	1	0	1	6	6	2	0	2	0	0	0	0	0	2	0	0	2	0	0
11	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

$$\alpha^1 = \begin{bmatrix} 1&0&0&0&0 \\ 1&1&0&0&0 \\ 1&1&1&0&0 \\ 1&1&1&1&0 \\ 1&1&1&1&1 \end{bmatrix} \quad \alpha^2 = \begin{bmatrix} 1&1&1&1&1 \\ 0&1&1&1&1 \\ 0&0&1&1&1 \\ 0&0&0&1&1 \\ 0&0&0&0&1 \end{bmatrix} \quad \alpha^3 = \begin{bmatrix} 1&0&0&0&0 \\ 1&1&0&0&0 \\ 1&1&1&0&0 \\ 1&1&1&1&1 \\ 0&0&0&0&1 \end{bmatrix} \quad \alpha^4 = \begin{bmatrix} 1&0&0&0&0 \\ 1&1&1&1&1 \\ 0&0&1&1&1 \\ 0&0&0&1&1 \\ 0&0&0&0&1 \end{bmatrix} \quad \alpha^5 = \begin{bmatrix} 1&0&0&0&0 \\ 1&1&0&0&0 \\ 1&1&1&1&1 \\ 0&0&0&1&1 \\ 0&0&0&0&1 \end{bmatrix} \quad \alpha^6 = \begin{bmatrix} 1&0&0&0&0 \\ 1&1&0&1&0 \\ 1&1&1&1&1 \\ 0&0&0&1&0 \\ 1&0&0&1&1 \end{bmatrix}$$

$$\alpha^7 = \begin{bmatrix} 1&1&0&0&1 \\ 0&1&0&0&0 \\ 1&1&1&1&1 \\ 0&1&0&1&1 \\ 0&0&0&0&1 \end{bmatrix} \quad \alpha^8 = \begin{bmatrix} 1&1&1&1&1 \\ 0&1&1&0&1 \\ 0&0&1&0&0 \\ 1&1&1&1&1 \\ 0&1&1&0&1 \end{bmatrix} \quad \alpha^9 = \begin{bmatrix} 1&0&1&1&0 \\ 1&1&1&1&1 \\ 0&0&1&0&0 \\ 1&0&1&1&0 \\ 1&1&1&1&1 \end{bmatrix} \quad \alpha^{10} = \begin{bmatrix} 1&1&1&1&1 \\ 0&1&0&1&0 \\ 1&1&1&1&1 \\ 0&1&0&1&0 \\ 1&1&1&1&1 \end{bmatrix} \quad \alpha^{11} = \begin{bmatrix} 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \end{bmatrix} \quad \alpha^{12} = \begin{bmatrix} 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \end{bmatrix}$$

n = 7

									2	1	1	1	1	0	0	0
									-1	1	0	0	0	2	1	1
									0	-1	1	0	0	0	1	0
									0	0	-1	1	0	-1	0	1
									0	0	0	-1	1	0	-1	0
	$\pi(\frac{1}{7})$	$\pi(\frac{2}{7})$	$\pi(\frac{3}{7})$	$\pi(\frac{4}{7})$	$\pi(\frac{5}{7})$	$\pi(\frac{6}{7})$	π^+	π^-	0	0	0	0	-1	0	0	-1
1	12	3	8	6	4	9	84	63	21	7	14	14	7	0	7	0
2	9	4	6	8	3	12	63	84	14	7	7	14	0	0	7	0
3	6	12	4	3	9	8	42	56	0	14	7	0	7	21	7	7
4	8	9	3	4	12	6	56	42	7	14	7	0	14	14	0	7
5	4	8	12	9	6	3	28	21	0	0	7	7	7	7	14	14
6	3	6	9	12	8	4	21	28	0	0	0	7	7	0	7	14
7	10	6	2	5	8	4	70	28	14	14	7	7	14	7	0	7
8	4	8	5	2	6	10	28	70	0	7	7	0	0	14	7	0
9	5	10	8	6	4	2	35	14	0	7	7	7	7	14	14	14
10	2	4	6	8	10	5	14	35	0	0	0	0	7	0	0	7
11	8	2	10	4	5	6	56	42	14	0	14	7	7	0	7	0
12	6	5	4	10	2	8	42	56	7	7	0	14	0	0	7	7
13	8	2	3	4	5	6	56	42	14	7	7	7	7	0	0	0
14	6	5	4	3	2	8	42	56	7	7	7	7	0	7	7	0
15	4	8	5	2	6	3	28	21	0	7	7	0	7	14	7	7
16	3	6	2	5	8	4	21	28	0	7	0	0	7	7	0	7
17	5	3	8	6	4	2	35	14	7	0	7	7	7	0	7	7
18	2	4	6	8	3	5	14	35	0	0	0	7	0	0	7	7
19	6	5	4	3	2	1	42	7	7	7	7	7	7	7	7	7
20	1	2	3	4	5	6	7	42	0	0	0	0	0	0	0	0
21	3	6	2	5	1	4	21	28	0	7	0	7	0	7	7	7
22	4	1	5	2	6	3	28	21	7	0	7	0	7	0	0	0
23	2	4	6	1	3	5	14	35	0	0	7	0	0	7	7	0
24	5	3	1	6	4	2	35	14	7	7	0	7	7	0	0	7
25	8	9	3	4	5	6	56	42	7	14	7	7	7	14	7	7
26	6	5	4	3	9	8	42	56	7	7	7	0	7	7	0	0
27	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
28	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0

-1	0	-1	0	-1	0	0	0	0	0
1	0	0	-1	0	-1	0	0	0	0
0	2	1	1	0	0	-1	-1	0	0
0	0	0	0	2	1	1	0	-1	0
0	0	1	0	0	1	0	2	1	-1
1	1	0	1	0	0	1	0	1	2

0	7	0	14	0	7	7	0	7	14	0	0
7	0	0	14	7	7	14	0	7	21	0	0
14	0	7	0	0	0	7	14	14	7	0	0
7	0	7	0	0	7	7	21	14	0	0	0
7	21	14	7	14	7	0	0	0	0	0	0
7	14	14	7	21	14	7	7	0	0	0	0
0	0	0	0	0	7	7	14	7	0	0	0
14	0	7	7	0	0	7	7	14	14	0	0
7	14	7	0	7	0	0	0	0	0	0	0
7	7	14	7	14	14	7	14	7	0	0	0
0	14	7	14	0	7	0	0	7	7	0	0
7	0	0	7	14	7	14	0	0	14	0	0
0	0	0	7	0	7	7	7	7	7	0	0
7	0	0	7	0	0	7	0	7	14	0	0
7	7	7	0	0	0	0	7	7	0	0	0
7	0	7	0	7	7	7	14	7	0	0	0
0	14	7	7	7	7	0	0	0	0	0	0
7	7	7	7	14	7	7	0	0	7	0	0
0	7	0	0	0	0	0	0	0	0	0	0
7	0	7	7	7	7	7	7	7	7	0	0
7	0	0	0	7	0	7	0	0	7	0	0
0	7	7	7	0	7	0	7	7	0	0	0
7	7	7	7	0	0	0	0	7	7	0	0
0	0	0	0	7	7	7	7	0	0	0	0
7	0	0	0	0	0	7	7	7	7	0	0
7	0	7	7	0	7	7	14	14	7	0	0
0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	1

$$\alpha^{1} = \begin{bmatrix} 1&1&1&1&1&1 \\ 0&1&0&0&0&0 \\ 0&0&1&0&1&0 \\ 0&1&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&1&0&1&0&1 \end{bmatrix} \quad \alpha^{2} = \begin{bmatrix} 1&0&1&0&1&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&1&0 \\ 0&1&0&1&0&0 \\ 0&0&0&0&1&0 \\ 1&1&1&1&1&1 \end{bmatrix} \quad \alpha^{3} = \begin{bmatrix} 1&0&0&1&0&0 \\ 1&1&1&1&1&1 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 1&0&0&1&1&0 \\ 0&0&1&0&0&1 \end{bmatrix} \quad \alpha^{4} = \begin{bmatrix} 1&0&0&1&0&0 \\ 0&1&1&0&0&1 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 1&1&1&1&1&1 \\ 0&0&1&0&0&1 \end{bmatrix}$$

$$\alpha^{5} = \begin{bmatrix} 1&0&0&0&0&0 \\ 1&1&0&0&0&0 \\ 1&1&1&1&1&1 \\ 0&0&0&1&1&1 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&1 \end{bmatrix} \quad \alpha^{6} = \begin{bmatrix} 1&0&0&0&0&0 \\ 1&1&0&0&0&0 \\ 1&1&1&0&0&0 \\ 1&1&1&1&1&1 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&1 \end{bmatrix} \quad \alpha^{7} = \begin{bmatrix} 1&1&1&1&1&1 \\ 0&1&1&0&0&1 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&1&1&0&1&1 \\ 0&0&1&0&0&1 \end{bmatrix} \quad \alpha^{8} = \begin{bmatrix} 1&0&0&1&0&0 \\ 1&1&0&1&1&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 1&0&0&1&1&0 \\ 1&1&1&1&1&1 \end{bmatrix}$$

$$\alpha^{9} = \begin{bmatrix} 1&0&0&0&0&0 \\ 1&1&1&1&1&1 \\ 0&0&1&1&1&1 \\ 0&0&0&1&1&1 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&1 \end{bmatrix} \quad \alpha^{10} = \begin{bmatrix} 1&0&0&0&0&0 \\ 1&1&0&0&0&0 \\ 1&1&1&0&0&0 \\ 1&1&1&1&0&0 \\ 1&1&1&1&1&1 \\ 0&0&0&0&0&1 \end{bmatrix} \quad \alpha^{11} = \begin{bmatrix} 1&1&0&1&0&1 \\ 0&1&0&0&0&0 \\ 1&1&1&1&1&1 \\ 0&1&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&1&0&1&0&1 \end{bmatrix} \quad \alpha^{12} = \begin{bmatrix} 1&0&1&0&1&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&1&0 \\ 1&1&1&1&1&1 \\ 0&0&0&0&1&0 \\ 1&0&1&0&1&1 \end{bmatrix}$$

$$\alpha^{13} = \begin{bmatrix} 1&1&1&1&1&1 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&1&0&1&0&0 \\ 0&1&1&0&1&0 \\ 0&1&1&1&0&1 \end{bmatrix} \quad \alpha^{14} = \begin{bmatrix} 1&0&1&1&1&0 \\ 0&1&0&1&1&0 \\ 0&0&1&0&1&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 1&1&1&1&1&1 \end{bmatrix} \quad \alpha^{15} = \begin{bmatrix} 1&0&0&1&0&0 \\ 1&1&1&1&1&1 \\ 0&0&1&1&0&1 \\ 0&0&0&1&0&0 \\ 1&0&0&1&1&1 \\ 0&0&0&0&0&1 \end{bmatrix} \quad \alpha^{16} = \begin{bmatrix} 1&0&0&0&0&0 \\ 1&1&1&0&0&1 \\ 0&0&1&0&0&0 \\ 1&0&1&1&0&0 \\ 1&1&1&1&1&1 \\ 0&0&1&0&0&1 \end{bmatrix}$$

$$\alpha^{17} = \begin{bmatrix} 1&1&0&0&0&1\\ 0&1&0&0&0&0\\ 1&1&1&1&1&1\\ 0&1&0&1&1&1\\ 0&0&0&0&1&1\\ 0&0&0&0&0&1 \end{bmatrix} \quad \alpha^{18} = \begin{bmatrix} 1&0&0&0&0&0\\ 1&1&0&0&0&0\\ 1&1&1&0&1&0\\ 1&1&1&1&1&1\\ 0&0&0&0&1&0\\ 1&0&0&0&1&1 \end{bmatrix} \quad \alpha^{19} = \begin{bmatrix} 1&1&1&1&1&1\\ 0&1&1&1&1&1\\ 0&0&1&1&1&1\\ 0&0&0&1&1&1\\ 0&0&0&0&1&1\\ 0&0&0&0&0&1 \end{bmatrix} \quad \alpha^{20} = \begin{bmatrix} 1&0&0&0&0&0\\ 1&1&0&0&0&0\\ 1&1&1&0&0&0\\ 1&1&1&1&0&0\\ 1&1&1&1&1&0\\ 1&1&1&1&1&1 \end{bmatrix}$$

$$\alpha^{21} = \begin{bmatrix} 1&0&1&0&1&0\\ 1&1&1&1&1&1\\ 0&0&1&0&1&0\\ 1&0&1&1&1&1\\ 0&0&0&0&1&0\\ 1&0&1&0&1&1 \end{bmatrix} \quad \alpha^{22} = \begin{bmatrix} 1&1&0&1&0&1\\ 0&1&0&0&0&0\\ 1&1&1&1&0&1\\ 0&1&0&1&0&0\\ 1&1&1&1&1&1\\ 0&1&0&1&0&1 \end{bmatrix} \quad \alpha^{23} = \begin{bmatrix} 1&0&0&1&0&0\\ 1&1&0&1&1&0\\ 1&1&1&1&1&1\\ 0&0&0&1&0&0\\ 1&0&0&1&1&0\\ 1&1&0&1&1&1 \end{bmatrix} \quad \alpha^{24} = \begin{bmatrix} 1&1&1&0&1&1\\ 0&1&1&0&0&1\\ 0&0&1&0&0&0\\ 1&1&1&1&1&1\\ 0&1&1&0&1&1\\ 0&0&1&0&0&1 \end{bmatrix}$$

$$\alpha^{25} = \begin{bmatrix} 1&0&1&1&1&0\\ 0&1&1&1&1&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&1&0&0&1 \end{bmatrix} \quad \alpha^{26} = \begin{bmatrix} 1&0&0&1&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 1&1&1&1&1&0\\ 0&1&1&1&0&1 \end{bmatrix} \quad \alpha^{27} = \begin{bmatrix} 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1 \end{bmatrix} \quad \alpha^{28} = \begin{bmatrix} 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1 \end{bmatrix}$$

The faces of P(G,g) from [3] are included among the subadditive rays and correspond to rays having a row of 1's in the associated α matrix. In addition, there are subadditive rays which are not faces of P(G,g) but can be associated with a pair of elements g_i, g_{i+1} of G such that $\alpha_{ij} = 1$ or $\alpha_{i+1,j} = 1$ for all j. For n = 1,2,3,4,5, and 6 there are no such subadditive rays, but for n = 7 there are two given in table 1: rays 25 and 26 which are actually reflections of each other.

By the reflection of $(\pi(g_1),\ldots,\pi(g_{n-1}), \pi^+, \pi^-)$ is meant $(\pi_R(g_1),\ldots, \pi_R(g_{n-1}), \pi_R^+, \pi_R^-)$ given by

$$\pi_R(g_i) = \pi(g_{n-i}),\ i = 1,\ldots,n-1,$$
$$\pi_R^+ = \pi^-,$$
$$\pi_R^- = \pi^+.$$

Every reflection of a subadditive ray is a subadditive ray, and every reflection of a face for $P_-^+(G_n,u_o)$ is a face of $P_-^+(G_n,1-u_o)$. For that reason, in listing the faces of $P_-^+(G_n,u_o)$ we only consider $0 \le |u_o| \le R(\frac{1}{2})$. The only reflections listed are, then, for odd n and $L(\frac{1}{2}) \le |u_o| \le R(\frac{1}{2})$.

The faces (except for scaling by π_o) of $P_-^+(G_n,u_o)$ are among the extreme rays of the cone formed by intersecting the cone having extreme rays the subadditive rays with the hyperplane.

$$\pi(L(u_o))+\pi^+(|u_o-L(u_o)|) = \pi(R(u_o))+\pi^-(|R(u_o)-u_o|).$$

In addition, the faces must be minimal; that is, satisfy (4). Any subadditive ray in the hyperplane will be a face. The other faces are among the linear combinations

$$(\pi(g_1),\ldots,\pi(g_n),\pi^+,\pi^-) = \lambda_1(\pi_i(g_1),\ldots,\pi_i(g_n),\pi_i^+,\pi_i^-) + \lambda_2(\pi_k(g_1),\ldots,\pi_k(g_n),\pi_k^+,\pi_k^-),$$

where

$$\lambda_1 = \pi_k(L(u_o))+\pi_k^+(|u_o-L(u_o)|)-\pi_k(R(u_o))-\pi_k^-(|R(u_o)-u_o|) > 0,$$

$$\lambda_2 = -(\pi_i(L(u_o))+\pi_i^+(|u_o-L(u_o)|)-\pi_i(R(u_o))-\pi_i^-(|R(u_o)-u_o|)) > 0.$$

These linear combinations must satisfy (4) in order to be a face and must be extreme. As before, in order to be extreme the set of inequality restrictions (1) and (2) which hold with equality must not be a subset of the similar set for another potential face. Satisfying (4) is equivalent to $\alpha_{ij} = 1$ or $\alpha_{o+1,j} = 1$ for all j where $L(u_o) = g_i$ and $R(u_o) = g_{i+1}$.

The faces for $n = 1,\ldots,7$ are given in table 2. In each case, it was directly verified that the face remained extreme for all u_o in the indicated subinterval of I. The values λ_1, λ_2, and the scaling were then computed for a general value of u_o in the subinterval.

We are now in a position to discuss in general the "persistence" of faces as u_o varies. In addition, extremality for certain potential faces will be discussed.

<u>Proposition 1</u> For any $P_-^+(G_n,u_o)$, $u_o \notin G_n$, a face is given by

$$\pi(g) = 0, g \in G;\ \pi^+ = \frac{1}{|u_o-L(u_o)|};\ \pi^- = \frac{1}{|R(u_o)-u_o|}.$$

Proof: This face is obtained by the linear combination

$$\lambda_1(0,\ldots,0,1,0) + \lambda_2(0,\ldots,0,0,1)$$

of the two subadditive rays associated with π^+ and π^-, where $\lambda_1 = |u_o - L(u_o)|$ and $\lambda_2 = + |R(u_o)-u_o|$. The scaling factor is $\pi_o = |u_o - L(u_o)|$.

Condition (4) is seen to be trivially satisfied. Extremality is true because every one of (1) is satisfied with equality. No other potential face can make that claim.

Proposition 2 If $(\pi(g_1),\ldots,\pi(g_{n-1}))$ is a face of $P(G_n,g_i)$, $g_i \in G_n$, $g_i \neq 0$, then

$$\pi' = (\pi(g_1),\ldots,\pi(g_{n-1}),\frac{\pi(g_i)+n\pi(g_{n-1})(|g_i-u_o|)-\pi(g_{i-1})}{|u_o-g_{i-1}|},n\pi(g_{n-1}))$$

is a face for $P_-^+(G_n,u_o)$ whenever $|g_{i-1}| < |u_o| \leq |g_i|$ and the scaling factor is $\pi_o = \pi(g_i) + n\pi(g_{n-1})(|g_i-u_o|)$. Also,

$$\pi'' = (\pi(g_1),\ldots,\pi(g_{n-1}),n\pi(g_1),\frac{\pi(g_i)+n\pi(g_1)(|u_o-g_i|)-\pi(g_{i+1})}{|g_{i+1}-u_o|})$$

is a face for $P_-^+(G_n,u_o)$ whenever $|g_i| \leq |u_o| < |g_{i+1}|$ and the scaling factor is $\pi_o = \pi(g_i) + n\pi(g_1)(|u_o-g_i|)$.

Proof: The first assertion will be proved, and the second is similar. The π' given in the first assertion is a linear combination of subadditive rays:

$$(\pi(g_1),\ldots,\pi(g_{n-1}),n\pi(g_1),n\pi(g_{n-1})) + \lambda_2(0,\ldots,0,1,0)$$

where

$$\lambda_2 = -(\pi(g_{i-1}) + n\pi(g_1)(|u_o - g_{i-1}|) - \pi(g_i) - n\pi(g_{n-1})(|g_i - u_o|)).$$

We know $\lambda_2 > 0$ by $|g_{i-1}| \leq |u_o| \leq |g_i|$ and $\pi(g_{i-1}) + \pi(g_1) = \pi(g_i)$ because π is a face for $P(G_n, g_i)$ (See [3], theorem 18). By $\pi(g_{i-1}) + \pi(g_i) = \pi(g_i)$ and $\pi(g_{n-1}) > 0$,

$$\lambda_2 > -\pi(g_{i-1}) - n\pi(g_1)(|u_o - g_{i-1}|) + \pi(g_i)$$

$$> -\pi(g_{i-1}) - \pi(g_1) + \pi(g_i) = 0.$$

Clearly, (4) is satisfied, by theorem 18 of [3] which says $\pi(g_j) + \pi(g_i - g_j) = \pi(g_i)$ all j. Hence, $\alpha_{ij} = 1$ all j.

Extremality remains to be proven. For convenience, let us distinguish the subadditive rays

$e^+ = (0,\ldots,0,1,0)$, and

$e^- = (0,\ldots,0,0,1)$.

The face given in proposition 1 will be denoted, e and it satisfies neither condition (2) with equality. The potential faces for $P_-^+(G_n, u_o)$ will, then, be of three types:

<u>type 1</u>: $e = \lambda_1 e^+ - \lambda_2 e^-$;

<u>type 2</u>: a linear combination of two subadditive rays, one of which is e^+ or e^- and the other is neither e^+ nor e^-;

<u>type 3</u>: a linear combination of two subadditive rays, neither of which is e^+ or e^-.

Include in type 3 any subadditive ray which satisfies (3) so by itself may be a face for $P_-^+(G_n,u_o)$. The following gives the manner in which conditions (2) are satisfied for the three types:

<u>type 1</u>: $\pi^+ > n\pi(g_1)$ and $\pi^- > n\pi(g_{n-1})$;

<u>type 2</u>: one of $\pi^+ - n\pi(g_1)$, $\pi^- - n\pi(g_{n-1})$ is zero, the other positive;

<u>type 3</u>: $\pi^+ = n\pi(g_1)$ and $\pi^- = n\pi(g_{n-1})$.

This classification will be useful in proving the next two propositions, as well as this one.

To return to proving extremality, here, recall that we must show that the conditions (1) and (2) satisfied with equality are not a subset of the same conditions for some other potential face. By the above, we need only make this comparison with potential faces of type 2 or 3. Consider the subaddivity conditions (2) satisfied with equality for π'. They are the same as for $(\pi(g_1)\ldots,\pi(g_{n-1}))$, which is a subadditive ray, and, therefore, they are not a subset of the conditions (2) satisfied with equality for any other subadditive ray. But the conditions (2) satisfied with equality for a potential face of type 2 or 3 are a subset of the conditions satisfied with equality for some subadditive ray. Hence, extremality is proven.

We have actually proven, as well, the following proposition.

<u>Proposition 3</u> If $(\pi(g_1),\ldots,\pi(g_{n-1}))$ is a subadditive ray with $\alpha_{ij} = 1$ or $\alpha_{i+1,j} = 1$ for all j and some fixed i and if u_π is defined by

$$\pi(g_i) + n\pi(g_1)(|u_\pi - g_i|) = \pi(g_{i+1}) + n\pi(g_{n-1})(|g_{i+1} - u_\pi|),$$

then

$$(\pi(g_1),\ldots,\pi(g_{n-1}),\frac{\pi(g_{i+1})+n\pi(g_{n-1})(|g_{i+1}-u_o|)-\pi(g_i)}{|u_o-g_i|},n\pi(g_{n-1}))$$

is a face for $P_-^+(G_n,u_o)$ whenever $|g_i| \le |u_o| \le |u_\pi|$, and

$$(\pi(g_1),\ldots,\pi(g_{n-1}),n\pi(g_1),\frac{\pi(g_i)+n\pi(g_1)(|u_o-g_i|)-\pi(g_{i+1})}{|g_{i+1}-u_o|})$$

is a face for $P_-^+(G_n,u_o)$ whenever $|u_\pi| \le |u_o| \le |g_{i+1}|$.

The scaling factor is

$\pi_o = \max\,\{\pi(g_i)+n\pi(g_1)(|u_o-g_i|),\ \pi(g_{i+1})+n\pi(g_{n-1})(|g_{i+1}-u_o|)\}$.

If we consider subadditive rays which are not faces for any $P(G_n,g)$, then the faces described in proposition 3 do not occur until $n = 7$. For $n = 7$ and $u_o \varepsilon\ (\frac{1}{7},\frac{2}{7})$, face number 13 is of this case.

So far, three kinds of faces have been described. Referring to the classification of potential faces in proposition 2, we see that every potential face of type 1 or 2 which satisfies (4) and can be formed, is extreme and therefore is a face. The situation for potential faces of type 3 is more complicated and is summarized below.

<u>Proposition 4</u> Let π_1 and π_2 be subadditive rays such that the product $\alpha_{ij}^1\ \alpha_{ij}^2 = 1$ or $\alpha_{i+1,j}^1\ \alpha_{i+1,j}^2 = 1$ for all j and some fixed i, and let $|g_i| \le |u_o| \le |g_{i+1}|$ so that

$$\lambda_2 = \pi_1(g_i)+n\pi_1(g_1)(|u_o-g_i|)-\pi_1(g_{i+1})-n\pi_1(g_{n-1})(|g_{i+1}-u_o|),$$

$$\lambda_1 = -\pi_2(g_i)-n\pi_2(g_1)(|u_o-g_i|)+\pi_2(g_{i+1})-n\pi_2(g_{n-1})(|g_{i+1}-u_o|)$$

satisfy $\lambda_2 > 0$ and $\lambda_1 > 0$. If

$$\pi = \lambda_1(\pi_1(g_1),\ldots, \pi_1(g_{n-1}),n\pi_1(g_1),n\pi_1(g_{n-1}))$$
$$+ \lambda_2(\pi_2(g_1),\ldots,\pi_2(g_{n-1}),n\pi_2(g_1),n\pi_2(g_{n-1})),$$

properly scaled, is a face for $P_-^+(G_n,u_o)$, then the same construction gives a face for $P_-^+(G_n,u_o)$ for every $u_o, |g_i| \le |u_o| \le |g_{i+1}|$, such that λ_1 and λ_2 are both non-negative.

Proof: The proposition says that every potential face of type 3 which is a face for some u_o is a face for any other u_o provided only that it can be formed. The only thing to be proven is that extremality persists for u_o in the interval for which the face can be formed.

In order to prove this persistence, we resort to theorem III.4. That theorem can be says that when u_o is varied within an interval (g_i,g_{i+1}), the vertices $t' = (t(g_1),\ldots,t(g_n),s^+,s^-)$ of the convex hull of solutions to $P_-^+(G_n,u_o)$ only change in the s^+ and s^- components. Only one of s^+, s^- is positive for a vertex and the one which is positive varies linearly in $|_uo|$ for $|g_i| \le |u_o| \le |g_{i+1}|$.

If π is a face for some u_o in (g_i,g_{i+1}), then it must satisfy

$$\pi(g_1)t(g_1)+\ldots+\pi(g_{n-1})t(g_{n-1})+ \pi^+s^+ + \pi^-s^- = \pi_o$$

for n+1 linearly independent vertices t'. Here, that means

$$(5)\quad \lambda_1 a_1+\lambda_2 a_2+(\lambda_1\pi_1{}^+ + \lambda_2\pi_2{}^+)s^+ +(\lambda_1\pi_1{}^- + \lambda_2\pi_1{}^-)s^-$$

$$= \lambda_1(\pi_1(g_i)+\pi_1{}^+(|u_o-g_i|))+ \lambda_2(\pi_2(g_i)+\pi_2{}^+(|u_o-g_i|),$$

where

$$a_1 = \sum_{j=1}^{n-1}\pi_1(g_j)t(g_j),$$

$$a_2 = \sum_{j=1}^{n-1}\pi_2(g_j)t(g_j),$$

$$\pi_1{}^+ = n\pi_1(g_1),\ \pi_1{}^- = n\pi_1(g_{n-1}),$$

$$\pi_2{}^+ = n\pi_2(g_1),\ \pi_2{}^- = n\pi_2(g_{n-1})$$

are all constant as u_o changes. Let us consider a particular vertex with, say, $s^+ > 0$ and $s^- = 0$. From theorem III.4, either $s^+ = |u_o-g_i|$ or $s^+ = |u_o-g_i| + k/n$ for $k \geq 1$. In the latter case, $t(g_1) = 0$. The case $s^+ = |u_o-g_i|$ will be treated and the other one is similar. In this case,(5) reduces to

$$\lambda_1 a_1 + \lambda_2 a_2 = \lambda_1\pi_1(g_i) + \lambda_2\pi_2(g_i).$$

But subadditivity implies $a_1 \geq \pi_1(g_i)$ and $a_2 \geq \pi_2(g_i)$ because $\sum g_j t(g_j) = g_i$ for any vertex t' with $s^+ = |u_o-g_i|$. Hence, (5) implies $a_1 = \pi_1(g_i)$ and $a_2=\pi_2(g_i)$. In order for (5) to continue to hold as u_o varies, we need only know that

$$(\lambda_1\pi_1{}^+ + \lambda_2\pi_2{}^+)\ s^+ = (\lambda_1\pi_1{}^+ + \lambda_2\pi_2{}^+)|u_o-g_i|$$

which is clear from $s^+ = |u_o-g_i|$ for u_o in (g_i,g_{i+1}).

Table 2. Faces of the Convex Hull of Solutions to $P_{-}^{+}(G_n, u_o)$

n=1 $u_o(0,1)$, $|u| -x$

face	π^+	π^-
1	$\frac{1}{x}$	$\frac{1}{1-x}$

n = 2 $u_o \in (0,\frac{1}{2})$, $|u_o| = x$

face	$\pi(\frac{1}{2})$	π^+	π^-
1	$\frac{1}{2-2x}$	$\frac{1}{x}$	$\frac{1}{1-x}$
2	0	$\frac{1}{x}$	$\frac{2}{1-2x}$

n = 3 $u_o \in (0,\frac{1}{3})$, $|u_o| = x$

face	$\pi(\frac{1}{3})$	$\pi(\frac{2}{3})$	π^+	π^-
1	$\frac{2}{3-3x}$	$\frac{1}{3-3x}$	$\frac{1}{x}$	$\frac{1}{1-x}$
2	0	0	$\frac{1}{x}$	$\frac{1}{1-3x}$

$u_o \in (\frac{1}{3},\frac{2}{3})$, $|u_o| = x$

face	$\pi(\frac{1}{3})$	$\pi(\frac{2}{3})$	π^+	π^-
1	$\frac{1}{3x}$	$\frac{1}{3-3x}$	$\frac{1}{x}$	$\frac{1}{1-x}$
2	$\frac{1}{3x}$	$\frac{1}{6x}$	$\frac{1}{x}$	$\frac{6x-1}{4x-6x^2}$
3	$\frac{1}{6-6x}$	$\frac{1}{3-3x}$	$\frac{5-6x}{(1-x)(6x-4)}$	$\frac{1}{1-x}$
4	0	0	$\frac{3}{3x-1}$	$\frac{3}{2-3x}$

$n = 4 \qquad u_o \in (0, \frac{1}{4}), \quad |u_o| = x$

face	$\pi(\frac{1}{4})$	$\pi(\frac{2}{4})$	$\pi(\frac{3}{4})$	π^+	π^-
1	$\frac{3}{4-4x}$	$\frac{2}{4-4x}$	$\frac{1}{4-4x}$	$\frac{1}{x}$	$\frac{1}{1-x}$
2	$\frac{1}{2-4x}$	0	$\frac{1}{2-4x}$	$\frac{1}{x}$	$\frac{1}{1-2x}$
3	0	0	0	$\frac{1}{x}$	$\frac{4}{1-4x}$

$u_o \in (\frac{1}{4}, \frac{1}{2}), \quad |u_o| = x$

face	$\pi(\frac{1}{4})$	$\pi(\frac{2}{4})$	$\pi(\frac{3}{4})$	π^+	π^-
1	$\frac{1}{4x}$	$\frac{2}{4-4x}$	$\frac{1}{4-4x}$	$\frac{1}{x}$	$\frac{1}{1-x}$
2	$\frac{1}{4x}$	0	$\frac{1}{4x}$	$\frac{1}{x}$	$\frac{2}{1-2x}$
3	$\frac{1}{4-4x}$	$\frac{1}{2-2x}$	$\frac{1}{4-4x}$	$\frac{3-4x}{-1+5x-4x^2}$	$\frac{1}{1-x}$
4	$\frac{1}{4x}$	$\frac{1}{6x}$	$\frac{1}{12x}$	$\frac{1}{x}$	$\frac{6x-1}{3x-6x^2}$
5	0	0	0	$\frac{4}{4x-1}$	$\frac{2}{1-2x}$

. $n = 5$ $\qquad u_o \in (0, \frac{1}{5}), \quad |u_o| = x$

faces	$\pi(\frac{1}{5})$	$\pi(\frac{2}{5})$	$\pi(\frac{3}{5})$	$\pi(\frac{4}{5})$	π^+	π^-
1	$\frac{4}{5-5x}$	$\frac{3}{5-5x}$	$\frac{2}{5-5x}$	$\frac{1}{5-5x}$	$\frac{1}{x}$	$\frac{1}{1-x}$
2	$\frac{3}{5-10x}$	$\frac{1}{5-10x}$	$\frac{3}{10-20x}$	$\frac{2}{5-10x}$	$\frac{1}{x}$	$\frac{2}{1-2x}$
3	0	0	0	0	$\frac{1}{x}$	$\frac{5}{1-5x}$

$u_o \in (\frac{1}{5}, \frac{2}{5}), \quad |u_o| = x$

faces	$\pi(\frac{1}{5})$	$\pi(\frac{2}{5})$	$\pi(\frac{3}{5})$	$\pi(\frac{4}{5})$	π^+	π^-
1	$\frac{1}{5x}$	$\frac{3}{5-5x}$	$\frac{2}{5-5x}$	$\frac{1}{5-5x}$	$\frac{1}{x}$	$\frac{1}{1-x}$
2	$\frac{1}{5x}$	$\frac{2+5x}{5x(3-5x)}$	$\frac{1}{10x}$	$\frac{-1+10x}{10x(3-5x)}$	$\frac{1}{x}$	$\frac{-1+10x}{2x(3-5x)}$
3	$\frac{3}{10-10x}$	$\frac{3}{5-5x}$	$\frac{2}{5-5x}$	$\frac{1}{5-5x}$	$\frac{7-10x}{2(1-x)(5x-1)}$	$\frac{1}{1-x}$
4	$\frac{2}{10-15x}$	$\frac{4}{10-15x}$	$\frac{1}{10-15x}$	$\frac{3}{10-15x}$	$\frac{8-15x}{(5x-1)(2-3x)}$	$\frac{3}{2-3x}$
5	$\frac{1}{5x}$	$\frac{3}{20x}$	$\frac{1}{10x}$	$\frac{1}{20x}$	$\frac{1}{x}$	$\frac{20x-3}{8x-20x^2}$
6	$\frac{1}{5x}$	$\frac{1}{15x}$	$\frac{1}{10x}$	$\frac{2}{15x}$	$\frac{1}{x}$	$\frac{15x-1}{x(6-15x)}$
7	0	0	0	0	$\frac{5}{5x-1}$	$\frac{5}{2-5x}$

$u_o \in (\frac{2}{5},\frac{3}{5}), \quad |u_o| = x$

	$\pi(\frac{1}{5})$	$\pi(\frac{2}{5})$	$\pi(\frac{3}{5})$	$\pi(\frac{4}{5})$	π^+	π^-
1	$\frac{1}{5x}$	$\frac{2}{5x}$	$\frac{2}{5(1-x)}$	$\frac{1}{5(1-x)}$	$\frac{1}{x}$	$\frac{1}{1-x}$
2	$\frac{1}{5x}$	$\frac{2}{5x}$	$\frac{1}{10x}$	$\frac{3}{10x}$	$\frac{1}{x}$	$\frac{10x-1}{6x-10x^2}$
3	$\frac{3}{10-10x}$	$\frac{1}{10-10x}$	$\frac{2}{5-5x}$	$\frac{1}{5-5x}$	$\frac{9-10x}{(1-x)(10x-4)}$	$\frac{1}{1-x}$
4	$\frac{1}{5x}$	$\frac{2}{5x}$	$\frac{4}{15x}$	$\frac{2}{15x}$	$\frac{1}{x}$	$\frac{15x-4}{x(9-15x)}$
5	$\frac{2}{15-15x}$	$\frac{4}{15-15x}$	$\frac{2}{5-5x}$	$\frac{1}{5-5x}$	$\frac{11-15x}{(1-x)(15x-6)}$	$\frac{1}{1-x}$
6	0	0	0	0	$\frac{5}{5x-2}$	$\frac{5}{3-5x}$

$n = 6 \qquad u_o \in (0,\frac{1}{6}), \quad |u_o| = x$

faces	$\pi(\frac{1}{6})$	$\pi(\frac{2}{6})$	$\pi(\frac{3}{6})$	$\pi(\frac{4}{6})$	$\pi(\frac{5}{6})$	π^+	π^-
1	$\frac{5}{6-6x}$	$\frac{4}{6-6x}$	$\frac{3}{6-6x}$	$\frac{2}{6-6x}$	$\frac{1}{6-6x}$	$\frac{1}{x}$	$\frac{1}{1-x}$
2	$\frac{2}{3-6x}$	$\frac{1}{3-6x}$	0	$\frac{2}{3-6x}$	$\frac{1}{3-6x}$	$\frac{1}{x}$	$\frac{2}{1-2x}$
3	$\frac{1}{2-6x}$	0	$\frac{1}{2-6x}$	0	$\frac{1}{2-6x}$	$\frac{1}{x}$	$\frac{3}{1-3x}$
4	0	0	0	0	0	$\frac{1}{x}$	$\frac{6}{1-6x}$

$u_o \in (\frac{1}{6},\frac{2}{6})$, $|u_o| = x$

faces	$\pi(\frac{1}{6})$	$\pi(\frac{2}{6})$	$\pi(\frac{3}{6})$	$\pi(\frac{4}{6})$	$\pi(\frac{5}{6})$	π^+	π^-
1	$\frac{1}{6x}$	$\frac{4}{6-6x}$	$\frac{3}{6-6x}$	$\frac{2}{6-6x}$	$\frac{1}{6-6x}$	$\frac{1}{x}$	$\frac{1}{1-x}$
2	$\frac{1}{6x}$	$\frac{1}{3-6x}$	0	$\frac{1}{6x}$	$\frac{1}{3-6x}$	$\frac{1}{x}$	$\frac{2}{1-2x}$
3	$\frac{2}{6-6x}$	$\frac{4}{6-6x}$	$\frac{3}{6-6x}$	$\frac{2}{6-6x}$	$\frac{1}{6-6x}$	$\frac{4-6x}{(1-x)(6x-1)}$	$\frac{1}{1-x}$
4	$\frac{1}{6x}$	0	$\frac{1}{6x}$	0	$\frac{1}{6x}$	$\frac{1}{x}$	$\frac{3}{1-3x}$
5	$\frac{1}{6x}$	$\frac{2}{24x-54x^2}$	$\frac{3-9x}{24x-54x^2}$	$\frac{1}{24x-54x^2}$	$\frac{-1+9x}{24x-54x^2}$	$\frac{1}{x}$	$\frac{-1+9x}{4x-9x^2}$
6	$\frac{1}{6x}$	$\frac{1}{12x}$	0	$\frac{1}{12x}$	$\frac{1}{12x}$	$\frac{1}{x}$	$\frac{12x-1}{4x-12x^2}$
7	$\frac{1}{6x}$	$\frac{4}{30x}$	$\frac{3}{30x}$	$\frac{1}{30x}$	$\frac{1}{30x}$	$\frac{1}{x}$	$\frac{15x-2}{5x-15x^2}$
8	0	0	0	0	0	$\frac{6}{6x-1}$	$\frac{3}{1-3x}$

$u_o \in (\frac{2}{6},\frac{3}{6})$, $|u_o| = x$

faces	$\pi(\frac{1}{6})$	$\pi(\frac{2}{6})$	$\pi(\frac{3}{6})$	$\pi(\frac{4}{6})$	$\pi(\frac{5}{6})$	π^+	π^-
1	$\frac{1}{6x}$	$\frac{2}{6x}$	$\frac{3}{6-6x}$	$\frac{2}{6-6x}$	$\frac{1}{6-6x}$	$\frac{1}{x}$	$\frac{1}{1-x}$
2	$\frac{1}{6x}$	$\frac{2}{6x}$	0	$\frac{1}{6x}$	$\frac{2}{6x}$	$\frac{1}{x}$	$\frac{2}{1-2x}$
3	$\frac{1}{6x}$	$\frac{2}{6x}$	$\frac{3}{6x(4-6x)}$	$\frac{1}{6x}$	$\frac{6x-1}{6x(4-6x)}$	$\frac{1}{x}$	$\frac{6x-1}{x(4-6x)}$
4	$\frac{5-6x}{6(1-x)(6x-2)}$	$\frac{1}{6-6x}$	$\frac{3}{6-6x}$	$\frac{2}{6-6x}$	$\frac{1}{6-6x}$	$\frac{5-6x}{(1-x)(6x-2)}$	$\frac{1}{1-x}$
5	$\frac{1}{6-6x}$	$\frac{2}{6-6x}$	$\frac{3}{6-6x}$	$\frac{2}{6-6x}$	$\frac{1}{6-6x}$	$\frac{2-3x}{(1-x)(3x-1)}$	$\frac{1}{1-x}$
6	$\frac{1}{4-6x}$	0	$\frac{1}{4-6x}$	0	$\frac{1}{4-6x}$	$\frac{3}{3x-1}$	$\frac{3}{2-3x}$
7	$\frac{1}{6x}$	$\frac{2}{6x}$	$\frac{3}{12x}$	$\frac{1}{6x}$	$\frac{1}{12x}$	$\frac{1}{x}$	$\frac{4x-1}{2x-4x^2}$
8	$\frac{1}{9-12x}$	$\frac{2}{9-12x}$	$\frac{1}{3-4x}$				
9	0	0	0	0	0	$\frac{3}{3x-1}$	$\frac{2}{1-2x}$

$n = 7 \qquad u_o \in (\frac{3}{7},\frac{4}{7}), \qquad |u_o| = x$

faces	$\pi(\frac{1}{7})$	$\pi(\frac{2}{7})$	$\pi(\frac{3}{7})$	$\pi(\frac{4}{7})$	$\pi(\frac{5}{7})$	$\pi(\frac{6}{7})$	π^+	π^-
1	$\frac{1}{7x}$	$\frac{2}{7x}$	$\frac{3}{7x}$	$\frac{3}{7(1-x)}$	$\frac{2}{7(1-x)}$	$\frac{1}{7(1-x)}$	$\frac{1}{x}$	$\frac{1}{1-x}$
2	$\frac{1}{7x}$	$\frac{2}{7x}$	$\frac{3}{7x}$	$\frac{12-7x}{14x(5-7x)}$	$\frac{3}{14x}$	$\frac{14x-3}{14x(5-7x)}$	$\frac{1}{x}$	$\frac{14x-3}{2x(5-7x)}$
3	$\frac{11-14x}{14(1-x)(7x-2)}$	$\frac{3}{14(1-x)}$	$\frac{7x+5}{14(1-x)(7x-2)}$	$\frac{3}{7(1-x)}$	$\frac{2}{7(1-x)}$	$\frac{1}{7(1-x)}$	$\frac{11-14x}{2(1-x)(7x-2)}$	$\frac{1}{1-x}$
4	$\frac{2-3x}{21x(1-x)-4}$	$\frac{x}{21x(1-x)-4}$	$\frac{2(1-x)}{21x(1-x)-4}$	$\frac{2x}{21x(1-x)-4}$	$\frac{1-x}{21x(1-x)-4}$	$\frac{3x-1}{21x(1-x)-4}$	$\frac{7(2-3x)}{21x(1-x)-4}$	$\frac{7(3x-1)}{21x(1-x)-4}$
5	$\frac{4}{28x}$	$\frac{8}{28x}$	$\frac{12}{28x}$	$\frac{9}{28x}$	$\frac{6}{28x}$	$\frac{3}{28x}$	$\frac{1}{x}$	$\frac{28x-9}{4x(4-7x)}$
6	$\frac{3}{28(1-x)}$	$\frac{6}{28(1-x)}$	$\frac{9}{28(1-x)}$	$\frac{12}{28(1-x)}$	$\frac{8}{28(1-x)}$	$\frac{4}{28(1-x)}$	$\frac{19-28x}{4(1-x)(7x-3)}$	$\frac{1}{1-x}$
7	$\frac{5}{7(5x-1)}$	$\frac{3}{7(5x-1)}$	$\frac{8}{7(5x-1)}$	$\frac{6}{7(5x-1)}$	$\frac{4}{7(5x-1)}$	$\frac{2}{7(5x-1)}$	$\frac{5}{5x-1}$	$\frac{}{(5x-1)(4-7x)}$
8	$\frac{2}{7(4-5x)}$	$\frac{4}{7(4-5x)}$	$\frac{6}{7(4-5x)}$	$\frac{8}{7(4-5x)}$	$\frac{3}{7(4-5x)}$	$\frac{5}{7(4-5x)}$	$\frac{22-35x}{(4-5x)(7x-3)}$	$\frac{5}{4-5x}$
9	$\frac{8}{14(4x-1)}$	$\frac{2}{14(4x-1)}$	$\frac{10}{14(4x-1)}$	$\frac{4}{14(4x-1)}$	$\frac{5}{14(4x-1)}$	$\frac{6}{14(4x-1)}$	$\frac{4}{4x-1}$	$\frac{28x-9}{(4-7x)(4x-1)}$
10	$\frac{6}{14(3-4x)}$	$\frac{5}{14(3-4x)}$	$\frac{4}{14(3-4x)}$	$\frac{10}{14(3-4x)}$	$\frac{2}{14(3-4x)}$	$\frac{8}{14(3-4x)}$	$\frac{19-28x}{(3-4x)(7x-3)}$	$\frac{4}{3-4x}$
11	$\frac{2}{14x}$	$\frac{4}{14x}$	$\frac{6}{14x}$	$\frac{1}{14x}$	$\frac{3}{14x}$	$\frac{5}{14x}$	$\frac{\frac{1}{2}}{x}$	$\frac{14x-1}{2x(4-7x)}$
12	$\frac{5}{14x}$	$\frac{3}{14x}$	$\frac{1}{14x}$	$\frac{6}{14x}$	$\frac{4}{14x}$	$\frac{2}{14x}$	$\frac{13-14x}{2(1-x)(7x-3)}$	$\frac{1}{1-x}$
13	0	0	0	0	0	0	$\frac{7}{7x-3}$	$\frac{7}{4-7x}$

$n = 7$

$u_o \in (\frac{1}{7}, \frac{3}{14}) \qquad x = |u_o|$

faces	$\pi(\frac{1}{7})$	$\pi(\frac{2}{7})$	$\pi(\frac{3}{7})$	$\pi(\frac{4}{7})$	$\pi(\frac{5}{7})$	$\pi(\frac{6}{7})$	π^+	π^-
1	$\frac{1}{7x}$	$\frac{5}{7(1-x)}$	$\frac{4}{7(1-x)}$	$\frac{3}{7(1-x)}$	$\frac{2}{7(1-x)}$	$\frac{1}{7(1-x)}$	$\frac{1}{x}$	$\frac{1}{1-x}$
2	$\frac{1}{7x}$	$\frac{2+21x}{14x(4-7x)}$	$\frac{3+7x}{14x(4-7x)}$	$\frac{1}{14x}$	$\frac{14x-1}{7x(4-7x)}$	$\frac{14x-1}{14x(4-7x)}$	$\frac{1}{x}$	$\frac{14x-1}{2x(4-7x)}$
3	$\frac{5}{14(1-x)}$	$\frac{5}{7(1-x)}$	$\frac{4}{7(1-x)}$	$\frac{3}{7(1-x)}$	$\frac{2}{7(1-x)}$	$\frac{1}{7(1-x)}$	$\frac{9-14x}{2(1-x)(7x-1)}$	$\frac{1}{1-x}$
4	$\frac{1}{7x}$	$\frac{4+7x}{21x(3-7x)}$	$\frac{2}{21x}$	$\frac{1+14x}{21x(3-7x)}$	$\frac{1}{21x}$	$\frac{21x-2}{21x(3-7x)}$	$\frac{1}{x}$	$\frac{21x-2}{3x(3-7x)}$
5	$\frac{4}{7(2-3x)}$	$\frac{8}{7(2-3x)}$	$\frac{5}{7(2-3x)}$	$\frac{2}{7(2-3x)}$	$\frac{6}{7(2-3x)}$	$\frac{3}{7(2-3x)}$	$\frac{10-21x}{(2-3x)(7x-1)}$	$\frac{3}{2-3x}$
6	$\frac{3}{14(1-2x)}$	$\frac{6}{14(1-2x)}$	$\frac{2}{14(1-2x)}$	$\frac{5}{14(1-2x)}$	$\frac{1}{14(1-2x)}$	$\frac{4}{14(1-2x)}$	$\frac{11-28x}{2(1-2x)(7x-1)}$	$\frac{4}{2(1-2x)}$
7	$\frac{3}{14(1-2x)}$	$\frac{6}{14(1-2x)}$	$\frac{2}{14(1-2x)}$	$\frac{3}{28(1-2x)}$	$\frac{9}{28(1-2x)}$	$\frac{4}{14(1-2x)}$	$\frac{11-28x}{2(1-2x)(7x-1)}$	$\frac{4}{2(1-2x)}$
8	$\frac{6}{42x}$	$\frac{5}{42x}$	$\frac{4}{42x}$	$\frac{3}{42x}$	$\frac{2}{42x}$	$\frac{1}{42x}$	$\frac{1}{x}$	$\frac{42x-5}{6x(2-7x)}$
9	$\frac{10}{70x}$	$\frac{6}{70x}$	$\frac{2}{70x}$	$\frac{5}{70x}$	$\frac{8}{70x}$	$\frac{4}{70x}$	$\frac{1}{x}$	$\frac{35x-3}{5x(2-7x)}$
10	$\frac{8}{56x}$	$\frac{2}{56x}$	$\frac{3}{56x}$	$\frac{4}{56x}$	$\frac{5}{56x}$	$\frac{6}{56x}$	$\frac{1}{x}$	$\frac{28x-1}{4x(2-7x)}$

faces

11	$\frac{8}{56x}$	$\frac{2}{56x}$	$\frac{2}{21x}$	$\frac{4}{56x}$	$\frac{1}{21x}$	$\frac{6}{56x}$	$\frac{1}{x}$	$\frac{28x-1}{4x(2-7x)}$
12	0	0	0	0	0	0	$\frac{7}{7x-1}$	$\frac{7}{2-7x}$
13	$\frac{8}{21(1-2x)}$	$\frac{9}{21(1-2x)}$	$\frac{3}{21(1-2x)}$	$\frac{4}{21(1-2x)}$	$\frac{5}{21(1-2x)}$	$\frac{6}{21(1-2x)}$	$\frac{13-42x}{3(1-2x)(7x-1)}$	$\frac{2}{1-2x}$
14	$\frac{1}{7x}$	$\frac{6-7x}{14x(5-14x)}$	$\frac{9-35x}{14x(5-14x)}$	$\frac{1}{14x}$	$\frac{7x-1}{14x(5-14x)}$	$\frac{28x-3}{14x(5-14x)}$	$\frac{1}{x}$	$\frac{28x-3}{2x(5-14x)}$
15	$\frac{1}{7x}$	$\frac{3}{7(1-2x)}$	$\frac{1}{7x(1-2x)}$	$\frac{1}{14x}$	$\frac{1-3x}{7x(1-2x)}$	$\frac{2}{7(1-2x)}$	$\frac{1}{x}$	$\frac{2}{1-2x}$

$u_o \in (\frac{3}{14},\frac{2}{7})$

1-12
as above

13	$\frac{8}{56x}$	$\frac{9}{56x}$	$\frac{3}{56x}$	$\frac{4}{56x}$	$\frac{5}{56x}$	$\frac{6}{56x}$	$\frac{1}{x}$	$\frac{56x-9}{8x(2-7x)}$
14	$\frac{1}{7x}$	$\frac{3}{7(1-2x)}$	$\frac{1}{7(1-2x)}$	$\frac{6x-1}{7x(1-2x)}$	$\frac{1-3x}{7x(1-2x)}$	$\frac{2}{7(1-2x)}$	$\frac{1}{x}$	$\frac{2}{(1-2x)}$
15	$\frac{1}{7x}$	$\frac{3}{7(1-2x)}$	$\frac{1}{7(1-2x)}$	$\frac{1}{14x}$	$\frac{8x-1}{14x(1-2x)}$	$\frac{2}{7(1-2x)}$	$\frac{1}{x}$	$\frac{2}{1-2x}$

$n = 7$

$u_o \in (0, \frac{1}{7}), \quad x = |u_o|$

faces	$\pi(\frac{1}{7})$	$\pi(\frac{2}{7})$	$\pi(\frac{3}{7})$	$\pi(\frac{4}{7})$	$\pi(\frac{5}{7})$	$\pi(\frac{6}{7})$	π^+	π^-
1	$\frac{6}{7(1-x)}$	$\frac{5}{7(1-x)}$	$\frac{4}{7(1-x)}$	$\frac{3}{7(1-x)}$	$\frac{2}{7(1-x)}$	$\frac{1}{7(1-x)}$	$\frac{1}{x}$	$\frac{1}{1-x}$
2	$\frac{5}{7(1-2x)}$	$\frac{3}{7(1-2x)}$	$\frac{1}{7(1-2x)}$	$\frac{5}{14(1-2x)}$	$\frac{4}{7(1-2x)}$	$\frac{2}{7(1-2x)}$	$\frac{1}{x}$	$\frac{2}{1-2x}$
3	$\frac{4}{7(1-3x)}$	$\frac{1}{7(1-3x)}$	$\frac{3}{14(1-3x)}$	$\frac{2}{7(1-3x)}$	$\frac{5}{14(1-3x)}$	$\frac{3}{7(1-3x)}$	$\frac{1}{x}$	$\frac{3}{1-3x}$
4	$\frac{4}{7(1-3x)}$	$\frac{1}{7(1-3x)}$	$\frac{5}{14(1-3x)}$	$\frac{2}{7(1-3x)}$	$\frac{3}{14(1-3x)}$	$\frac{3}{7(1-3x)}$	$\frac{1}{x}$	$\frac{3}{1-3x}$
5	0	0	0	0	0	0	$\frac{1}{x}$	$\frac{7}{1-7x}$

$n = 7$

$u_o \in (\frac{2}{7},\frac{3}{7})$, $x = |u_o|$

faces	$\pi(\frac{1}{7})$	$\pi(\frac{2}{7})$	$\pi(\frac{3}{7})$	$\pi(\frac{4}{7})$	$\pi(\frac{5}{7})$	$\pi(\frac{6}{7})$	π^+	π^-
1	$\frac{1}{7x}$	$\frac{2}{7x}$	$\frac{4}{7(1-x)}$	$\frac{3}{7(1-x)}$	$\frac{2}{7(1-x)}$	$\frac{1}{7(1-x)}$	$\frac{1}{x}$	$\frac{1}{1-x}$
2	$\frac{4}{28x}$	$\frac{8}{28x}$	$\frac{5}{28x}$	$\frac{2}{28x}$	$\frac{6}{28x}$	$\frac{3}{28x}$	$\frac{1}{x}$	$\frac{28x-5}{4x(3-7x)}$
3	$\frac{6}{42x}$	$\frac{12}{42x}$	$\frac{4}{42x}$	$\frac{3}{42x}$	$\frac{9}{42x}$	$\frac{8}{42x}$	$\frac{1}{x}$	$\frac{21x-2}{3x(3-7x)}$
4	$\frac{3}{21x}$	$\frac{6}{21x}$	$\frac{2}{21x}$	$\frac{5}{21x}$	$\frac{1}{21x}$	$\frac{4}{21x}$	$\frac{1}{x}$	$\frac{21x-2}{3x(3-7x)}$
5	$\frac{5}{35x}$	$\frac{10}{35x}$	$\frac{8}{35x}$	$\frac{6}{35x}$	$\frac{4}{35x}$	$\frac{2}{35x}$	$\frac{1}{x}$	$\frac{35x-8}{5x(3-7x)}$
6	$\frac{4}{21(1-x)}$	$\frac{8}{21(1-x)}$	$\frac{12}{21(1-x)}$	$\frac{9}{21(1-x)}$	$\frac{6}{21(1-x)}$	$\frac{3}{21(1-x)}$	$\frac{13-21x}{3(1-x)(7x-2)}$	$\frac{1}{1-x}$
7	$\frac{5}{14(1-x)}$	$\frac{3}{14(1-x)}$	$\frac{8}{14(1-x)}$	$\frac{6}{14(1-x)}$	$\frac{4}{14(1-x)}$	$\frac{2}{14(1-x)}$	$\frac{11-14x}{2(1-x)(7x-2)}$	$\frac{1}{1-x}$
8	$\frac{1}{7x}$	$\frac{2}{7x}$	$\frac{2(6-7x)}{7x(11-21x)}$	$\frac{16-35x}{7x(11-21x)}$	$\frac{6-7x}{7x(11-21x)}$	$\frac{21x-4}{7x(11-21x)}$	$\frac{1}{x}$	$\frac{21x-4}{x(11-21x)}$
9	$\frac{1}{7x}$	$\frac{2}{7x}$	$\frac{3+7x}{14x(4-7x)}$	$\frac{1}{14x}$	$\frac{3}{14x}$	$\frac{14-1}{14x(4-7x)}$	$\frac{1}{x}$	$\frac{14x-1}{2x(4-7x)}$
10	$\frac{8}{14(2-3x)}$	$\frac{2}{14(2-3x)}$	$\frac{10}{14(2-3x)}$	$\frac{4}{14(2-3x)}$	$\frac{5}{14(2-3x)}$	$\frac{6}{14(2-3x)}$	$\frac{13-21x}{(2-3x)(7x-2)}$	$\frac{3}{2-3x}$
11	$\frac{2}{7(3-5x)}$	$\frac{4}{7(3-5x)}$	$\frac{6}{7(3-5x)}$	$\frac{1}{7(3-5x)}$	$\frac{3}{7(3-5x)}$	$\frac{5}{7(3-5x)}$	$\frac{17-35x}{(3-5x)(7x-2)}$	$\frac{5}{3-5x}$
12	0	0	0		0	0	$\frac{7}{7x-2}$	$\frac{7}{3-7x}$

REFERENCES

1. Dantzig, G.B., *Linear Programming and Extensions*, Princeton University Press, Princeton, New Jersey, 1963.

2. Gomory, R.E., "An Algorithm for Integer Solutions to Linear Programs", in R.L. Graves and P. Wolfe (eds.), *Recent Advances in Mathematical Programming*, McGraw-Hill Book Company, Inc., New York, 1963, pp.269-302.

3. Gomory, R.E., "Some Polyhedra Related to Combinatorial Problems", *Linear Algebra and its Applications*, Vol. 2 (1969), pp.451-558.

4. Motzin, T.S., H. Raiffa, G.L. Thompson, and R.M. Thrall, "The Double Description Method", in H.W. Kuhn and A.W. Tucker (eds.), *Contributions to the Theory of Games*, Vol. II Annals of Mathematics Study No.28, Princeton University Press, Princeton, New Jersey, 1953, pp.51-73.