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R-SEPARATING SETS

R. E. Gomory, T. C. Hu and J. M. Yohe

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ABSTRACT

Two points a and b in a set S are said to be r-connected if there is a finite sequence of points $a = p_0, p_1, \ldots, p_n = b$ with $p_i \in S$ and the distance $\rho(p_i, p_{i+1}) \leq r$, $i = 0, \ldots, n-1$. In this paper we deal with sets C whose removal from the plane R_2 r-separates two points a and b in $R_2 - C$. More precisely, we shall study the structure of r-separating sets containing no proper r-separating subsets.

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INTRODUCTION

The definition of r-connectedness is a familiar one; see, for example, Newman [1]. Two points a and b in a set S are said to be r-connected if there is a finite sequence of points $a = p_0, p_1, \ldots, p_n = b$ with $p_i \in S$ and the distance $\rho(p_i, p_{i+1}) \leq r$, $i = 0, \ldots, n-1$. In this paper we will develop properties related to r-connectedness. We will deal mainly with the notion of r-separation (two points in a set S are r-separated if they are not r-connected) and with planar r-separating sets, which are, roughly, sets C whose removal from the plane R_2 r-separates two points in $R_2 - C$. The prototype of such sets might be the annulus of Figure 1 which separates a from b. However, much more complicated r-separating sets are also possible. (Figure 2).

Of course, r-separating sets, as described, can have few interesting properties since almost any sufficiently large set will do. However, the sets shown in Figures 1 and 2 have an additional property: they are irreducible; i.e., each one contains no r-separating proper subset. It is the irreducible r-separating sets, which have a very detailed structure, that will be described below. Some subjects related to r-separating sets are outlined in [2].

In Part I we will develop the general properties of irreducible r-separating sets. Among other things, we will prove that their boundaries are always unions of Jordan curves. With this established we will be able, in Part II, to exhibit a

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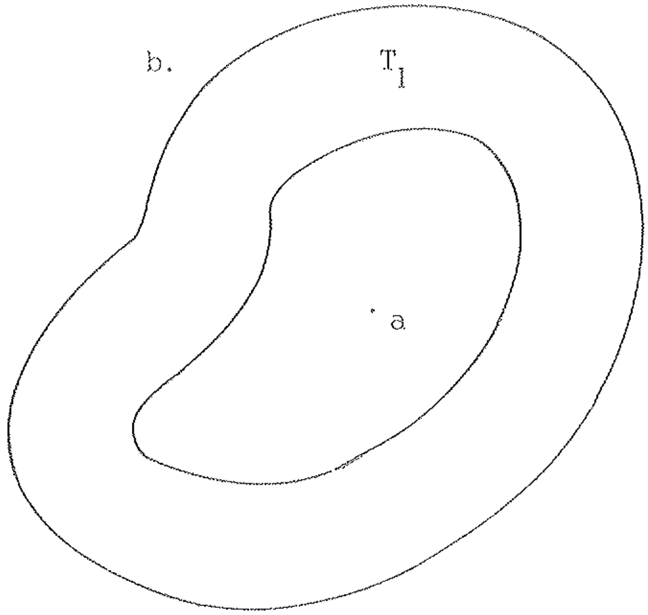

Figure 1

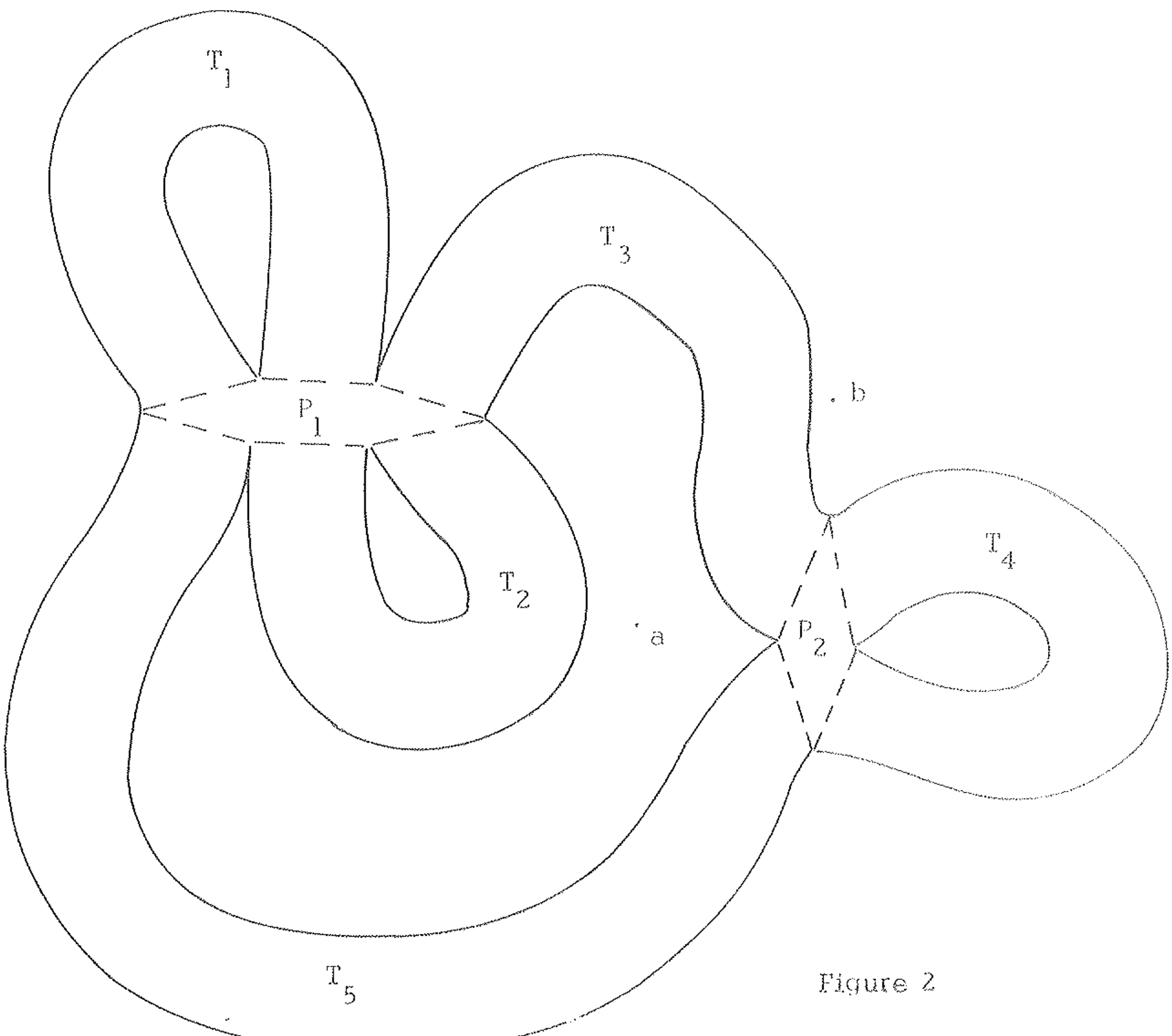

Figure 2

much more detailed structure. We will show, roughly, that all irreducible r-separating sets consist of simple tube-like sections of width r (such as the T_i in Figure 2) hooked together by polyhedra each having an even number of sides of length r. (such as the P_i in Figure 2).

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PART I

We now turn to more exact definitions. Using $\rho(p, q)$ for the Euclidean distance of two points in the plane R_2, we will say that the sequence of points $p_0, p_1, \ldots, p_n$ forms an r-chain from p_0 to p_n if $\rho(p_i, p_{i+1}) \leq r$.
We will say that a and b are r-connected in a set S if there is an r-chain with the properties

(i) $p_0 = a$, $p_n = b$, and

(ii) $p_i \in S$ for all i.

If p, q are two points of R_2, then by $\overline{pq}$ we will mean the line segment from p to q. If $p_0, \ldots, p_n$ is an r-chain, we will also use r-chain to denote the path consisting of $\bigcup_{i=0}^{n-1} \overline{p_i p_{i+1}}$. The context should resolve any ambiguities.
We will say that a set $C \subset R_2$ r-separates a and b if

(i) C is closed and bounded,

(ii) a and b are not r-connected in $R_2 - C$.

If C r-separates a and b then we will use A to denote the set of points which can be r-chained to a in $R_2 - C$, and we will define B analogously.

If x is a set, we use $F(x)$ to denote the frontier, or boundary, of x.

We will be able to see in retrospect that condition (i) of this definition does not meaningfully affect the structure of the separating sets, but it does facilitate the analysis.

An irreducible r-separating set C is defined as one that contains no r-separating set as a proper subset. It is the structure of these irreducible sets that will be analyzed.

Theorem 1. Every r-separating set contains an irreducible r-separating set.

Theorem 1 follows by routine arguments from the following easily established lemma:

Lemma 1. If $\mathfrak{C}$ is any collection of nested r-separating sets, i.e., for any $C, C' \in \mathfrak{C}$ we have either $C' \subset C$ or $C \subset C'$, then $C^* = \bigcap_{C \in \mathfrak{C}} C$ is an r-separating set.

A further useful property of C is given by Theorem 2, which we state without proof:

Theorem 2. If p belongs to the irreducible r-separating set C, then $\rho(p, A) \leq r$ and $\rho(p, B) \leq r$.

Evidently the set A of points r-connected to a in $R_2 - C$ has the property that any two points in A can be connected by an r-chain with vertices lying in A. We will call a set with this property an r-connected set. Theorem 3 is, then, somewhat analogous to the Jordan curve theorem in that it asserts that the removal of an irreducible r-separating set separates the plane into two r-connected open sets.

In order to prove Theorem 3, we need two results that are used repeatedly in what follows. These are given here as Lemmas 2 and 3.

Lemma 2. Let $\overline{p_1 \, p_2}$ and $\overline{q_1 \, q_2}$ be two intersecting closed line segments, both of length $\leq r$. Then for some one of the four end points, say p_1, either

(1) p_1 is an intersection point and it coincides with either q_1 or q_2, or

(2) the distances $\rho(p_1, q_1)$ and $\rho(p_1, q_2)$ are both $< r$.

In either case, the distances $\rho(p_1, q_1)$ and $\rho(p_1, q_2)$ are $\leq r$ so that p_1 is within distance r of all the end points.

Proof: Assume (by relabeling, if necessary) that p_1 is an end point nearest the point of intersection. The conclusion follows easily from elementary geometrical considerations.

The purpose of the next lemma is to enable us to deal with r-chains that do not cross themselves. An r-chain $p_0, p_1, \ldots, p_n$ is said to cross itself if the path consisting of the union of the segments $\overline{p_i p_{i+1}}$ $i = 0, \ldots, n-1$ is not an arc.

Lemma 3. If there is an r-chain $p = p_0, p_1, \ldots, p_n = q$ from a point p to a point q, then there is an r-chain from p to q, whose vertices are a subset of the original p_i, which does not cross itself.

The proof of this lemma is elementary, and we omit it.

Theorem 3. If C is an irreducible r-separating set which r-separates a from b then $R_2 - C = A \cup B$, where A is the r-component of $R_2 - C$ containing a and B is the r-component of $R_2 - C$ containing b.

Proof: We will show that if $D = R_2 - (A \cup B \cup C)$ is not empty, then A and D are r-connected. To do this, we will produce a simple closed curve J which consists of an arc in D together with an r-chain K in $A \cup C$, such that there are points in both components of $R_2 - J$ which can be r-chained to b in $B \cup F(C)$. We will then obtain our contradiction from the fact that this r-chain must cross J.

Let $p \in D$, and let x_1, x_2 be points of C which do not lie on the same line through p. Let q_1 be the first point of $R_2 - D$ encountered in traversing the segment from p to x_1, and let q_2 be defined analogously (see Figure 3).

Since D is interior to $D \cup C$, q_1 and q_2 must belong to C. Therefore, by Theorem 2, $\rho(q_1, A) \leq r$ and so there is a point q_1' on $F(A)$ with $\rho(q_1, q_1') \leq r$. Since q_1 is also on the frontier of D, we must also have $\rho(q_1, q_1') \geq r$, so $\rho(q_1, q_1') = r$. Since q_1' is in $F(A)$, there is certainly a point q_1'' of A within distance r and from there an r-chain of points of A leading to a.

In exactly the same way there is a q_2' on $F(A)$ with $\rho(q_2, q_2') = r$ and from it an r-chain to a.

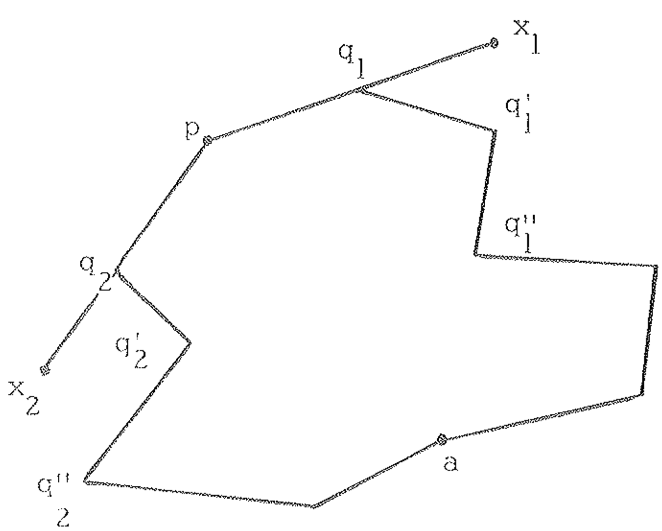

Figure 3

Consequently, there is an r-chain $q_1, q_1', q_1'', \ldots, a, \ldots, q_2'', q_2', q_2$. Using Lemma 3, we will replace this r-chain by a non-self-intersecting r-chain of the form $q_1 = p_0, \ldots, p_n = q_2$, where $n \geq 2$.

In order to get a non-self-intersecting r-chain of the desired form, we first show that $\overline{q_1q_1'}$ and $\overline{q_2q_2'}$ do not cross each other. Note that q_1 and q_2 belong to $F(D)$ and q_1' and q_2' belong to $F(A)$. Both $\overline{q_1q_1'}$ and $\overline{q_2q_2'}$ are of length r. If $\overline{q_1q_1'}$ and $\overline{q_2q_2'}$ cross each other, then according to Lemma 2, one endpoint of one of the segments will be within r of both endpoints of the other; hence we will have, say, $\rho(q_1, q_2') < r$. But $\rho(q_1, q_2') \geq r$ since $q_1 \in F(D)$ and $q_2' \in F(A)$. The contradiction proves that the two segments do not cross.

Now to obtain the non-self-crossing r-chain, we first apply Lemma 3 with $p_0 = q_1$ and $p_n = q_2'$ and then (since the fact that $\rho(q_1, x) > r$ for $x \in A$ guarantees that q_1' will be in the resulting chain) apply Lemma 3 with $p_0 = q_1'$ and $p_n = q_2$. Since $\overline{q_1q_1'}$ and $\overline{q_2q_2'}$ do not cross each other, we will get a non-self-intersecting chain K from q_1 to q_2 which contains q_1' and q_2'. (Note that we may have $q_1' = q_2'$).

We now show that the modified r-chain K, together with the arc $\overline{q_1pq_2}$, forms a simple closed curve J. To do this, we must show that the modified r-chain does not intersect $\overline{q_1pq_2}$ except in the points q_1, q_2. For any segment $\overline{p_ip_{i+1}}$, $i \neq 0, \ldots n-1$ at least one of the endpoints belongs to A and hence is at a distance greater than r from any point of $\overline{D}$. Since the length of the segment is $\leq r$, it is impossible for any such segment to intersect $\overline{q_1 p q_2}$. The segments $\overline{p_0 p_1}$ and $\overline{p_{n-1} p_n}$ intersect $\overline{q_1 p q_2}$ in q_1 and q_2 respectively, but lie entirely in C.

The vertices of the simple closed curve J all belong to $\overline{D}$ or $\overline{A}$, and any point of J either lies in D or has one of these vertices within a distance $< r$. Consequently no point of B can lie on J. The point b itself must be in one of the two domains R_I and R_0 into which J divides the plane. We can suppose, without loss of generality, that $b \in R_0$, the outer domain. Returning now to our original point $p \in D$ let us construct a straight line from p into R_I, the interior domain (Figure 4)

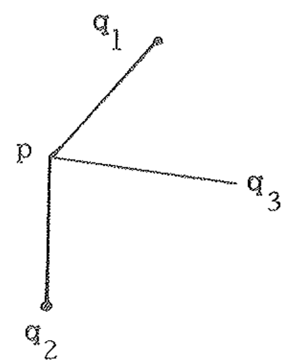

Figure 4

Let q_3 be the first point on this line not in D. q_3 must still be in R_I, for if the line were all D material up to its intersection with J, it would bring D material too close to the points $p_1, \ldots, p_{n-1}$, which are in $\overline{A}$. Since q_3 is also in C, there is a point p_1' of $F(B)$ with $\rho(q_3, p_1') \leq r$, and hence there is an r-chain $q_3 = p_0', p_1', \ldots, p_m' = b$ from q_3 to b with all p_i', $i \geq 1$, belonging to $\overline{B}$.

Since $q_3 \in R_I$ and $b \in R_0$, the chain from q_3 to b must cross J. Moreover, all of $\overline{q_1 p q_2}$ is in $\overline{D}$, so the chain cannot intersect $\overline{q_1 p q_2}$ since in the contrary case we would have a point $d \in D$ such that $\rho(d, \overline{B}) < r$. Thus there must be a segment, say $\overline{p'_i p'_{i+1}}$ of the chain from q_3 to b and a segment, say $\overline{p_j p_{j+1}}$ of the chain from q_1 to q_2 which interesect. The intersection is not a common vertex, since all p'_i belong to $\overline{B}$ except p'_0 which does not lie on J, and no vertex of J belongs to $\overline{B}$.

By Lemma 2, then, there is one endpoint of one of the segments at a distance of less than r from both endpoints of the other segment. The "close" endpoint cannot belong to $\overline{p_j p_{j+1}}$, since if it did there would be a point of $\overline{A}$ or of $\overline{D}$ (namely p_j or p_{j+1}) at a distance less than r from $p'_{i+1} \in \overline{B}$. Thus the "close" endpoint must be p'_j or p'_{j+1}. However, this point cannot belong to $\overline{B}$ by the same reasoning as above. The only remaining choice is $p'_0 = q_3$. But then by Lemma 2, $d(q_3, A) < r$ and hence, since there is clearly an r-chain from p to q_3 in D, we see that p can be r-chained to a in $D \cup A$. The contradiction proves the theorem.

We now know that an irreducible C splits R_2 into two r-connected sets, A and B, with nothing left over. We next approach the problem of obtaining more detailed properties of C. This is done by a detour. We will prove something about A and B that will establish $F(A)$ and $F(B)$ as well behaved; this will then give information on $F(C)$. Then with $F(C)$ established as well behaved it will become possible to establish more detailed properties of C.

An important property of the sets A and B which prevents them from becoming too unruly is given in Theorem 4.

<u>Theorem 4.</u> Let p_1 and p_2 be points of one component of A with $\rho(p_1, p_2) \leq r$. With radius r draw two circular arcs α_1 and α_2 through both points. Each arc should be $\leq \pi r/3$ in length. Then there is a path P^* from p_1 to p_2 that lies entirely in the closed sector bounded by α_1 and α_2 and consists only of points of A . (Figure 5).

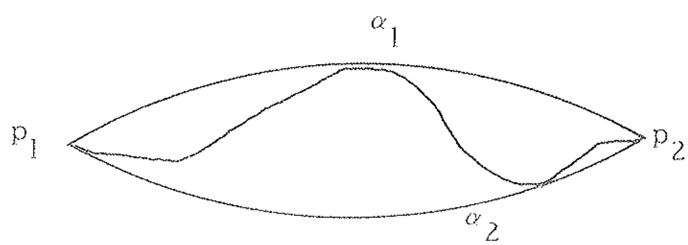

Figure 5

Proof: Since p_1 and p_2 belong to the same component of the open set A, there is some simple path P from p_1 to p_2 in A (Figure 6). This path can and will be taken to consist of straight line segments of length $\leq r$.

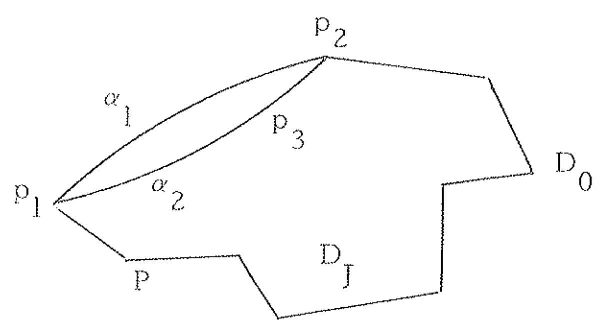

Figure 6

We will proceed to analyze a special case from which the general theorem can be deduced.

We consider the case in which P does not touch the sector bounded by α_1 and α_2 except at p_1 and p_2. We can assume, without loss of generality, that b lies in the outer domain D_0 of the Jordan curve J formed from P and the segment $\overline{p_1p_2}$.

Either α_1 or α_2 must lie inside J. Let us assume that the arc lying inside is α_2 as in Figure 6. Since all of α_2 lies within r of p_1, α_2 can consist only of points of A or of C. If there are no C points on α_2, then the theorem holds; so let us suppose there is a point $p_3 \in C \cap \alpha_2$. Since $p_3 \in C$, there is a segment of length $< r$ connecting p_3 to a point q of B or of $F(B)$. Thus there is an r-chain $p_3 = q_0, q_1, \ldots, q_n = b$. Just as in the proof of Theorem 3, this r-chain must cross J. If a segment of this chain crossed a segment of J other than $\overline{p_1p_2}$, we would have a point of A (namely, the intersection point) at a distance $< r$ from $\overline{B}$. Thus $\overline{p_3q}$ must cross $\overline{p_1p_2}$ as shown in Figure 7.

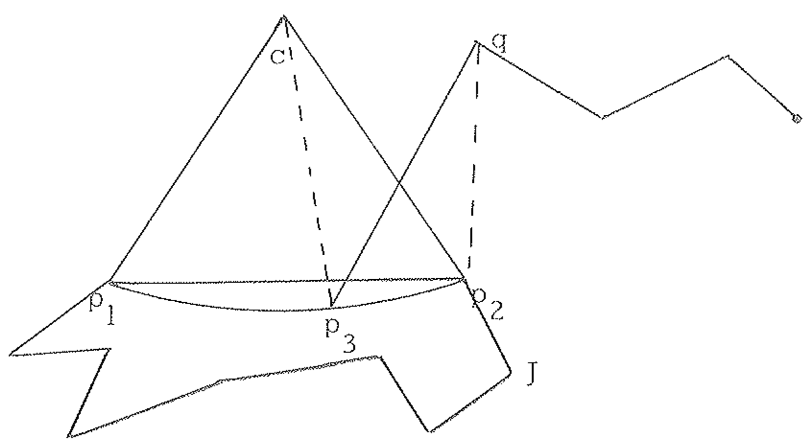

Figure 7

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Let c be the center of the circle of which α_2 is an arc. q cannot lie in or on the isosceles triangle p_1cp_2, for then its distance from p_1 and p_2 would be r or less, but p_1 and p_2, as interior points of A, have distance $> r$ from $B \cup F(B)$. So q lies outside p_1cp_2 and the edge $\overline{p_3q}$ intersects either the edge $\overline{p_1c}$ or the edge $\overline{p_2c}$. Let us assume it is $\overline{p_2c}$. Then, applying Lemma 2 to the edges of $\overline{p_3q}$ and $\overline{p_2c}$, we find that case (1) cannot hold since neither q nor p_3 can coincide with c or p_2, so case (2) is the only possibility. But $\overline{p_3c}$ has length exactly r and $\overline{qp_2}$ must have length $\geq r$. So none of the four vertices c, q, p_2 or p_3 can be within distance $< r$ of the two vertices of the opposite edge as described in case (2). So case (2) also cannot apply. Thus the existence of such a q contradicts Lemma 2 and we conclude that $p_3 \notin C$.

Since we have shown that $\alpha_2 \subset A$, it forms the P^* of our theorem in the special case we have been considering.

We next turn to the general case in which P intersects α_1 or α_2 (Figure 8).

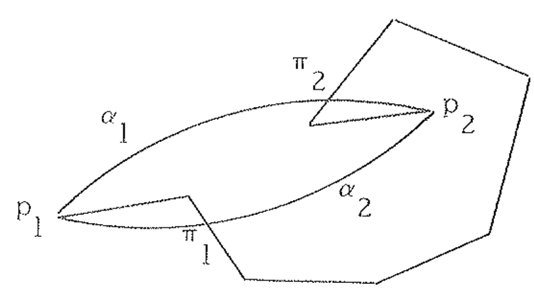

Figure 8

In this case P^* is formed by using the segmented path P up to the point π_1 where it first leaves the sector. All P is in A so in particular π_1 and the point of next return to the sector π_2 are in A . By connecting π_1 and π_2 by arcs of radius r we create the situation of the special case with π_1 and π_2 playing the role of p_1 and p_2 . Consequently, one of these new arcs α'

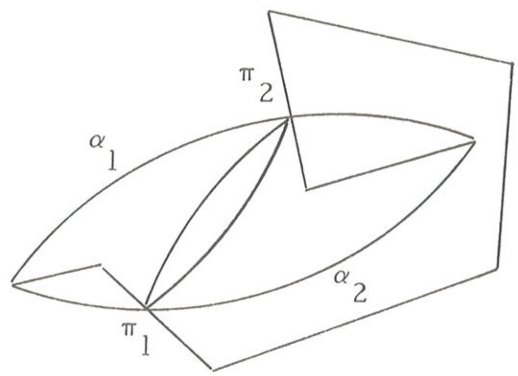

Figure 9

is in A and carries P^* up to π_2 . If P does not go out of the sector again, then P^* is completed with the remainder of P; otherwise, the process is repeated. Since P consists of a finite number of straight line segments, there will only be a finite number of intersections with α_1 and α_2 . Therefore, after a finite number of repetitions p_2 will be reached and the path P^* fully constructed. This establishes the theorem.

We are now in a position to say something about F(A) and F(B) and hence about F(C) which is $F(A) \cup F(B)$.

Theorem 5: The boundary of each component of A or B is a simple closed curve.

Proof: The proof is based on a converse to the Jordan curve theorem. A version by Newman [p. 166] asserts that if a domain in Z_2 (the two dimensional projective plane) is simply connected and uniformly locally connected, then its frontier is a simple closed curve, a point, or null. We will proceed to show that the components of A and B meet the conditions of this theorem.

We first discuss the question of uniform local connectedness. A set S is said to be uniformly locally connected, if for any $\varepsilon > 0$ and any point $p \in S$, there is a $\delta(\varepsilon)$, independent of p, such that if $\rho(y_1, p)$ and $\rho(y_2, p)$ are both $< \delta/2$ and y_1 and $y_2 \in S$, then y_1 and y_2 are joined by a connected subset of S of diameter $< \varepsilon$. Theorem 4 shows that the components of A and B are uniformly locally connected; for if we take a point p in a component of A, and y_1 and y_2 within distance δ, then the path P^* of Theorem 4 provides the connected component and actually lies within a circle of radius $\delta = \varepsilon$.

We next turn to simple connectedness and consider a component of A_0 of A. If we draw any simple closed curve in A_0, all of $B \cup C$ must be either in its outer domain R_0 or its inner domain R_I. First note that any attempt to split B between R_0 and R_I leads to the usual difficulties with some connecting chain crossing J which is in A. So B lies in one domain only, say R_I. If even one point of C were to lie in R_0, it would require a point of B within distance r, which again is impossible because of the intervening $J \subset A$; so $C \subset R_I$ also and $(B \cup C)$ must lie entirely in R_0 or entirely in R_I.

If $B \cup C$ is entirely in R_I, then every point in R_0 belongs to A_0. (Figure 10)

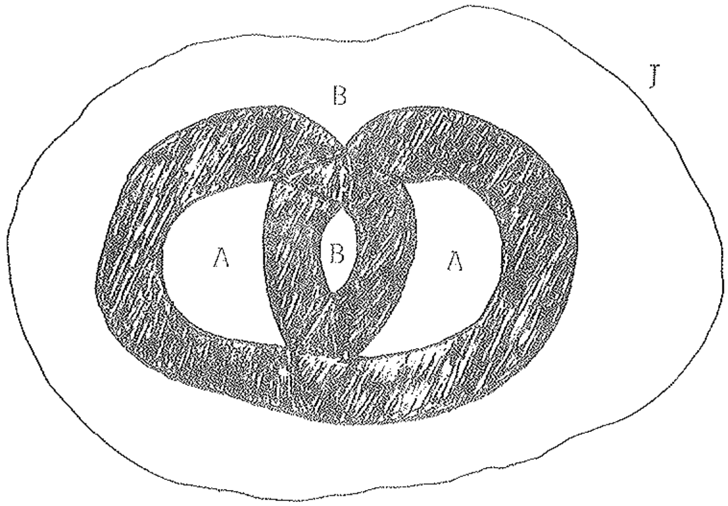

Figure 10

As R_0 then takes in the entire plane, except for a single bounded region, this can occur for only one component of $A \cup B$. Also, since we have confined ourselves to bounded C, this outer area must belong to $A = A \cup A_0$ and therefore, this case does occur for exactly one component. To apply the theorem to the unbounded component, we take as the corresponding set in Z_2 the line at infinity plus A_0. J can now be contracted to a point in this set. If A_0 is the bounded component and we draw a J with $B \cup C$ in its R_0, then J can be contracted to a point in R_I which contains only points of A. In all other cases, if we are dealing with the bounded components, $(B \cup C)$ will necessarily be in R_0; so J can be contracted in R_I; so A_0 or, more precisely, the corresponding set in Z_2, is simply connected.

Thus, the theorem applies and each component of A (or B) has as its frontier a simple closed curve or a point or the null set. Possibilities other than the curve are ruled out by choosing a point $p \in A_0$ and two distinct points q_1 and q_2 in the complement A'_0 of A_0 such that p, q_1, and q_2 are not collinear. Each of $\overline{pq_1}$ and $\overline{pq_2}$ must contain a point of $F(A_0)$ which, therefore, has at least two points. This establishes the theorem.

Part II

With something now established about the regularity of A, B and C and their boundaries, we turn to a more complete analysis of the structure of C.

Let us define a connector to be a closed segment of length r connecting F(A) and F(B). Then it is easy to prove:

Theorem 6. For every point p on F(A) (or F(B)) there exists at least one connector with p as one of its ends and the other end a point of F(B) (or F(A)). Furthermore, all points on the connector other than the endpoints belong to C - F(C).

Note that this theorem does not imply that every point of C must be on a connector.

Lemma 4. Two connectors $\overline{p_1 q_1}$ and $\overline{p_2 q_2}$ with distinct end points can have no points in common.

Proof: If two connectors have a point of intersection other than an end point, then it follows from Lemma 2 that there exists a point $p_1 \in F(A)$ with $\rho(p_1, p_2) < r$ and $\rho(p_1, q_2) < r$. This contradicts the assumption that A and B are r-separated.

Let $C_1 \subset C$ be the set of all points of C which lie on a connector, and let $C_2 = C - C_1$.

Lemma 5. C_1 is closed and C_2 is open.

The proof of Lemma 5 is elementary, and we omit it.

Lemma 6. If $\overline{p_0 q_0}$ is a connector, and $x_0 \in \text{int}\ \overline{p_0 q_0}$ is an interior point of C_1, then all interior points of $\overline{p_0 q_0}$ are interior points of C_1.

Proof: (See Figure 11) since x_0 is interior to C_1, There exists an $\epsilon > 0$ such that $N_\epsilon(x_0) \subset C_1$. Suppose $y \epsilon$ int $\overline{p_0 q_0}$ is not an interior point of C_1. Then there exists a sequence $\{y_i\} \rightarrow y$ with $y_i \notin C_1$; we may suppose that all of these points lie in one of the two half-planes determined by the line containing $\overline{p_0 q_0}$, say H_0.

Pick $x_1 \epsilon N_\epsilon(x_0) \cap H_0$. Then $\overline{x_0 x_1} \subset N_\epsilon(x_0)$; we parameterize $\overline{x_0 x_1}$ by α, $0 \leq \alpha \leq 1$. For each x_α we have a connector $\overline{p_\alpha q_\alpha}$ such that $\overline{p_\alpha q_\alpha} \cap \overline{x_0 x_1} = x_\alpha$.

Now there exists a δ such that $N_\delta(y) \cap (\{p_0, q_0\} \cup N_\epsilon(x) \cup \overline{p_1 q_1} = \phi$. Moreover, we want δ so small that every connector which intersects $N_\delta(y)$ must run through $N_\epsilon(x_0)$.

This choice of δ guarantees that no point of $N_\delta(y)$ may belong to A or B, since this would contradict r-separation. Hence all $y_i \epsilon N_\delta(y)$ are C_2-points.

Pick any such y_i, and from it draw a perpendicular line to $\overline{p_0 q_0}$. Let the first connector this line hits be $\overline{p_{\alpha_0}, q_{\alpha_0}}$; similarly determine $\overline{p_{\alpha_1} q_{\alpha_1}}$ (these connectors must be $\overline{p_\alpha q_\alpha}$'s, since in the contrary case we will contradict r-separation).

Now $\overline{p_{\alpha_0} q_{\alpha_0}}$ and $\overline{p_{\alpha_1} q_{\alpha_1}}$ may not cross; moreover, no connectors are sandwiched between these two. But then the gap between these connectors near y_i cannot be closed by the time we reach $\overline{x_0 x_1}$. The contradiction proves the lemma.

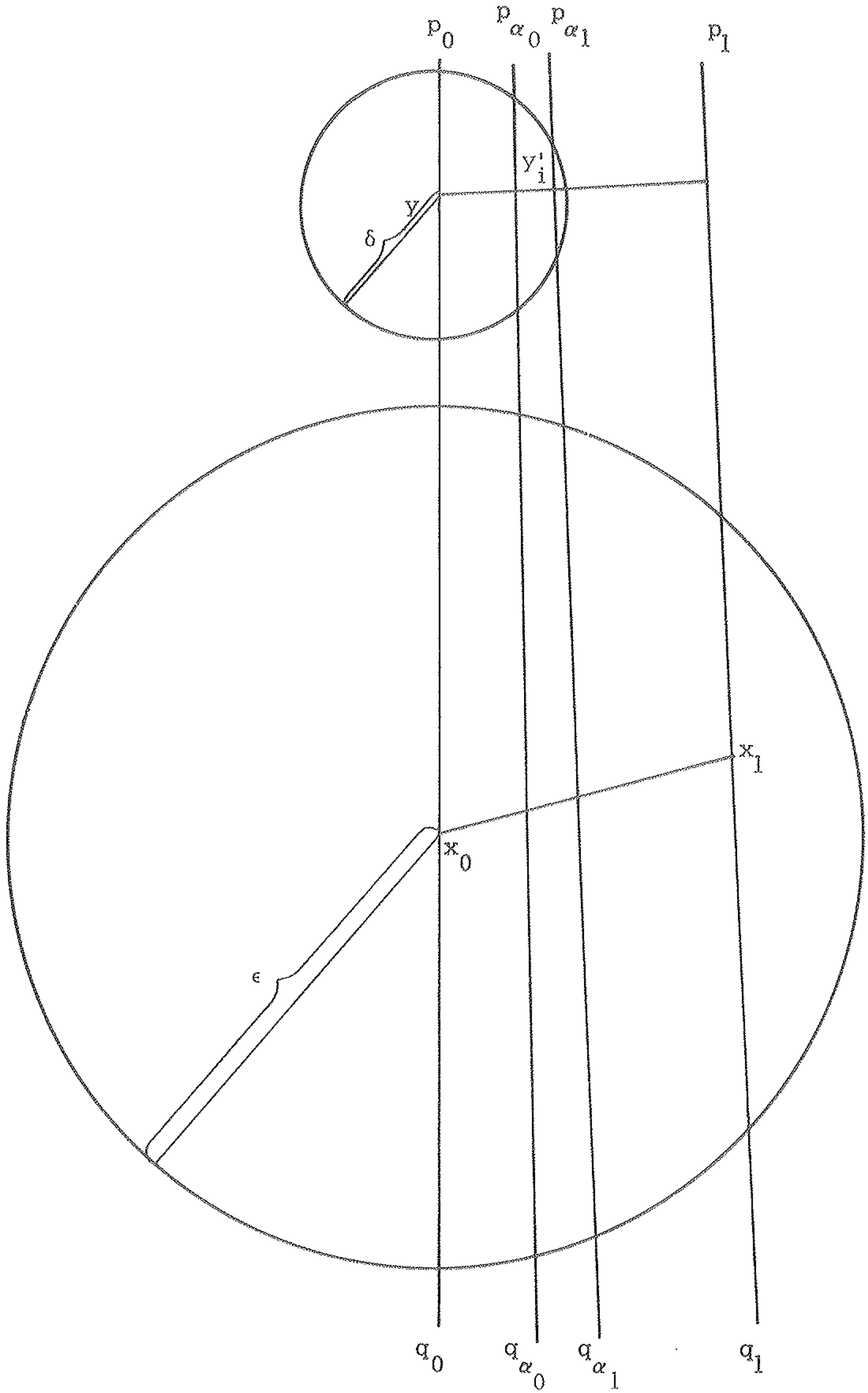

Figure 11

Lemma 7. Let $\tilde{C}$ be a component of C_2. Then each point of $F(\tilde{C})$ lies on a connector which lies on $F(\tilde{C})$.

Proof: We first show that if a connector has an interior point on $F(\tilde{C})$ then the entire connector lies on $F(\tilde{C})$. By Lemma 6, we know that such a connector lies on $F(C_2)$. If a connector were to lie partially, but not completely, on $F(\tilde{C})$, then, since $F(\tilde{C}) \subset F(C_2) \subset C_1$, we would have two connectors which cross, contradicting Lemma 4.

Now suppose there is an $x \in F(\tilde{C})$ which does not belong to any connector which lies on $F(\tilde{C})$. Then x is the endpoint of a connector; moreover, there is a neighborhood N_x of x such that any connector which intersects N_x contains no boundary point of $\tilde{C}$ on its interior. For, supposing that no such neighborhood exists, we can find a sequence of points converging to x such that each belongs to a connector containing a boundary point of $\tilde{C}$ on its interior, and therefore lying on $F(\tilde{C})$. These connectors converge to a connector containing x; this connector also lies on $F(\tilde{C})$.

We may suppose without loss of generality that $F(C) \cap N_x$ is an arc. Thus we can find a neighborhood N_x of x such that N_x is divided in two pieces by an arc on $F(C_2)$, where one of these pieces is in, say, A and the other is in C; moreover, each point p of $(C_1 - F(C)) \cap N_x$ is an interior point of $C - \tilde{C}$. Since

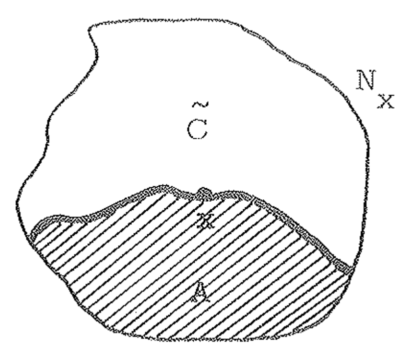

Figure 12

components of C_2 are open, we can conclude that each point of $(C - \tilde{C} - F(C)) \cap N_x$ an interior point of $C - \tilde{C}$, or that $F(\tilde{C}) \cap N_x \subset F(C)$. Then we must have $F(\tilde{C}) \cap N_x = F(C) \cap N_x$. But this is impossible, since this implies x does not belong to a connector.

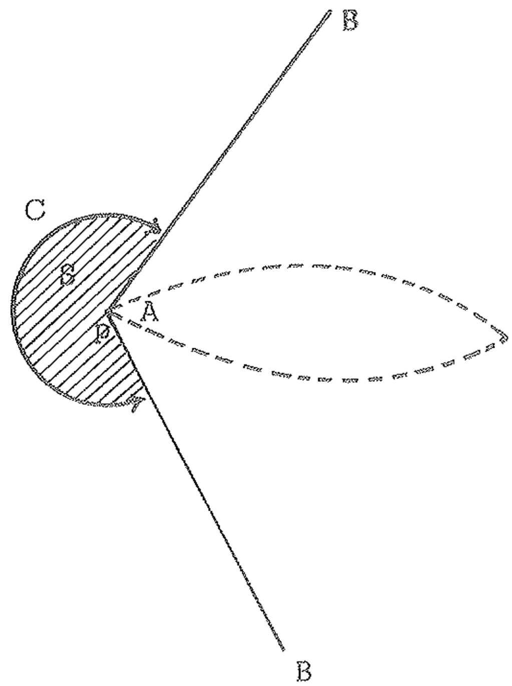

Figure 13

Lemma 8. If $p \in F(A)$ belongs to two connectors K_1 and K_2, and there exists an $\varepsilon > 0$ such that $N_\varepsilon(p) \cap C$ contains a sector S bounded by both $K_1 \cap C$ and $K_2 \cap C_2$, then the angle at the vertex of S is $\leq \pi$. (See Figure 13).

Proof: Suppose $p \in F(A)$ and the angle is greater than π. Then all points in a neighborhood of p will be within a distance r of B, contradicting the assumption that C r-separates A from B.

Lemma 9. Each component of C_2 is convex.

Proof: Suppose x, y belong to the same component $\tilde{C}$ of C_2, but $\overline{xy} \not\subset \tilde{C}$. Let $x', y' \in \overline{xy}$ be such that $\overline{x'y'} \cap \tilde{C} = \varphi$. There is a path α on $F(\tilde{C})$ which connects x' and y' such that the interior of the simple closed curve $\overline{x'y'} \cup \alpha$ lies outside of $\tilde{C}$. The path α consists of connectors (and portions of connectors) by Lemma 7; it follows that there must be a pair of adjacent connectors K_i, K_{i+1}, an $\varepsilon > 0$, and a sector S of $N_\varepsilon(p) \cap C$

such that the angle at the vertex of S is greater than π. This contradicts Lemma 8. (See Figure 14).

Lemma 10. The closure of each component of C_2 is a convex polyhedron with an even number of sides each of length r.

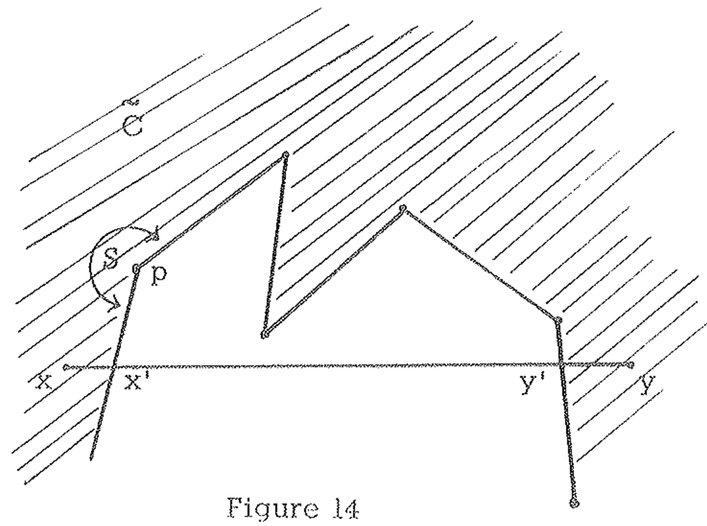

Figure 14

Proof: Since C is compact, the closure of a component $\tilde{C}$ of C_2 is closed and bounded. By Lemma 7, the boundary of $\tilde{C}$ consists of connectors, each of which is of length r. By Lemma 9, $\tilde{C}$ is convex and hence so is its closure. Hence $\tilde{\tilde{C}}$ can have only finitely many sides. Since vertices of $F(\tilde{C})$ are alternately in A and B, it follows that there must be an even number of sides.

Definition. A nondegenerate tube of width r is the closure of a differentiable embedding h of $[0,1] \times (-r/2, r/2)$ or $S^1 \times (-r/2, r/2)$ into R_2 which is

(a) an isometry on the second factor,

(b) such that for each $\alpha \in [0,1]$, $\{\alpha\} \times (-r/2, r/2)$ is normal to $h([0,1] \times \{0\})$, and

(c) each point of $\overline{h([0,1] \times (-r/2, r/2))} - h([0,1] \times (-r/2, r/2))$ is a boundary point of $R_2 - h([0,1] \times (-r/2, r/2))$.

A <u>tube of width r</u> is either a nondegenerate tube of width r or a connector.

<u>Theorem 7.</u> If C is an irreducible r-separating set, then $C = C_1 \cup C_2$, where

(a) the closure of each component of $C_1 - F(C)$ is a tube of width r which has one boundary component on A and the other on B

(b) each component of C_2 is the interior of a convex polyhedron with an even number of sides, each of length r, which intersects $F(C)$ only in its vertices.

<u>Proof:</u> Each point of C_1 lies on a connector by the definition of C_1. If $\tilde{C}_1$ is a component of $C_1 - F(C)$, then $\tilde{C}_1$ consists either of the interior of a single connector, in which case there is nothing to prove, or, since no two connectors may intersect except in a point of $F(C)$, an interval (or circle) of connectors. Certainly since each connector has length r, we may parameterize the interior of each connector via an isometry of $(-r/2, r/2) \to R_2$. By deciding in advance to map the positive side of this interval toward A, say, we may assure the possibility of a cohesive array of intervals. That these mappings may actually be extended to a differentiable embedding of $[0,1]$ (or S) $\times (-r, r)$ follows from the smoothness of the center line (the radius of curvature of the center line must be $\geq r/2$ at each point) and the fact that the isometries match up there.

The normality condition follows from the fact that $\rho(a, B) = r$ for all $a \in \text{Bd}\, A$.

The properties of components of C_2 follow from Lemma 10, Lemma 7 and the fact that interior points of connectors are also interior points of C. The theorem is proved.

We note that we may actually have an infinite number of C_2-components (Figure 15) and that these components may get arbitrarily near one another; note also that a connector which is on $F(C_2)$ need not be on the boundary of any component of C_2. We also observe that isolated connectors are indeed possible (Figure 16).

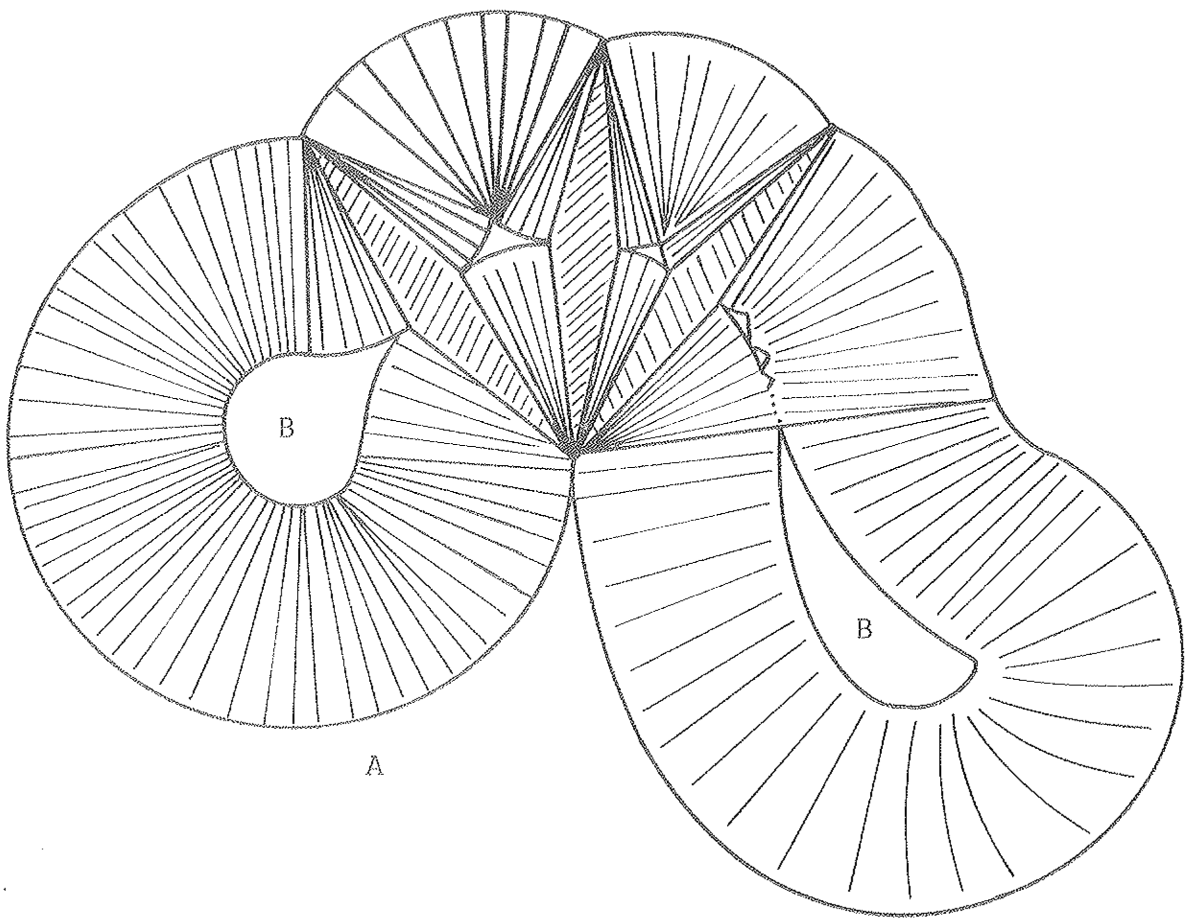

Figure 15

An r-separating set with an infinite number of C_2-components

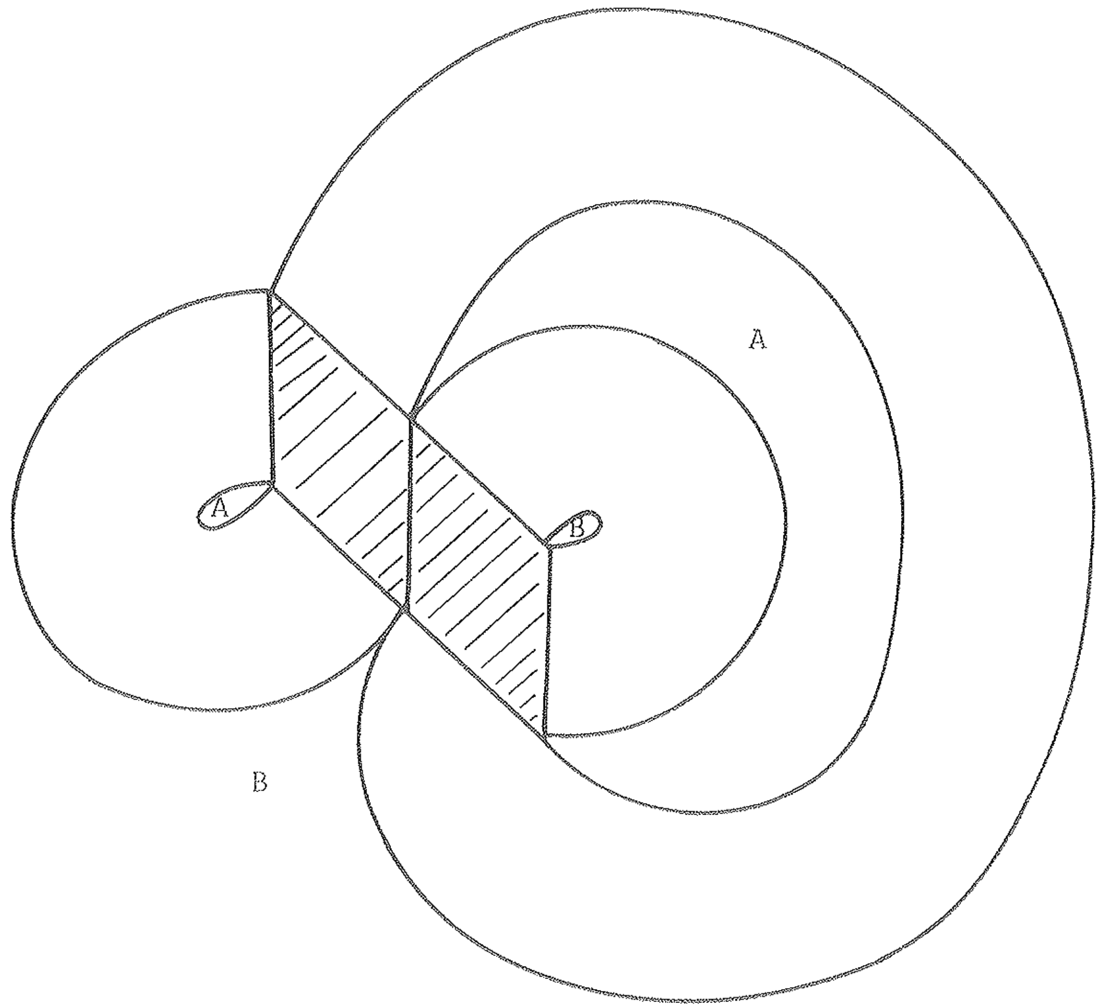

Figure 16

An irreducible r-separating set with an isolated connector

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13. ABSTRACT

Two points a and b in a set S are said to be r-connected if there is a finite sequence of points $a = p_0, p_1, \ldots, p_n = b$ with $p_i \in S'$ and the distance $\rho(p_i, p_{i+1}) \le r$, $i = 0, \ldots, n-1$. In this paper we deal with planar r-separating sets C whose removal from the plane R_2 r-separates two points a and b in $R_2 - C$. More precisely, we shall study the structure of irreducible r-separating subset.

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