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A Ricardo Model with Economies of Scale*

RALPH E. GOMORY†

Alfred P. Sloan Foundation, 630 Fifth Avenue, New York, New York 10111-0242

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This paper extends the analysis of the classical n-good two-country Ricardo model of international trade to the case where the production functions have economies of scale. It addresses the technical difficulties inherent in economies of scale by new integer programming and linear programming methods. The analysis reveals the existence of a well defined region which fills in solidly with equilibrium points as the number of goods becomes large. New economic conclusions follow from the ability to analyze these large models, among them that, in the presence of economies of scale, considerable conflict exists between the interests of the two trading partners. *Journal of Economic Literature* Classification Numbers: C61, C62, C68, F12. © 1994 Academic Press, Inc.

INTRODUCTION

There are significant and unavoidable technical difficulties in working with large models having scale economies, and this paper represents a direct attack on those difficulties. The techniques introduced here enable us to deal directly with two-country models having *large* numbers of traded goods and, consequently, very large numbers of equilibria. Our model is directly analogous to the classical Ricardo model but with economies of scale in place of the classical linear or diseconomies assumptions.

We show that, contrary to what one might expect, the many equilibria do not occur just anywhere. Rather they lie densely in a clearly delineated region of a graph of utility versus relative national income which we describe. This region of equilibria has a characteristic shape that persists across many different models and has significant economic consequences. We also provide algorithms that, even for large problems, select from the large array of equilibria those that tend to maximize utility for each country.

* A summary of some of the results of this paper appeared as Gomory [2]. This paper is a revised version of C. V. Starr Economic Research Report RR 92-04 and has been greatly improved by suggestions from William J. Baumol.

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Recent work has contributed greatly to the understanding of the theoretical consequences of scale economies for trade theory. This literature has generally worked with small models, typically two countries and two goods. The results of this paper are complementary to this work. There are significant features of a scale economies world that only appear in the larger models and require the analysis of regions of equilibria. There are also features that emerge from the analysis of the individual equilibria in the smaller models which cannot be deduced from the study of equilibrium regions.

Summary of Results

1. In the presence of economies of scale of the type we specify below, the set of equilibrium solutions can be described by a *region* in which these equilibrium points lie, and which they tend to *fill up*. The equilibrium points tend to fill up the solution region in the sense that, given any arbitrarily selected point, P, in the region, then, with a sufficient number of commodities traded, an equilibrium point will appear within any preselected distance, however small, from P.

2. We give simple and rapid algorithms for obtaining from among the very large number of possible equilibria those equilibria that are very good for one country or the other. We see that the best possible equilibrium for one country is usually poor for the other. The algorithms also show that the two countries' interests are less opposed when their demand structures are similar, and more opposed when they are dissimilar.

3. We give a simple algorithm for calculating the boundaries of the region. This allows the location and shape of the region to be obtained rapidly even for large models.

4. The characteristic shape of the region of equilibria has the following implications:

(a) The region always contains a large subregion of equilibria that are advantageous and often strongly advantageous to Country 1 relative to autarky. However, the region also contains a subregion at whose equilibria Country 1 receives *less* utility than it would in autarky. This region can sometimes be substantial in size, especially for the larger trading partner.

(b) The region always contains a central subregion, often a very large one, within which the interests of the trading partners are generally opposed. At the center of this subregion is an area of equilibria with utility above autarky for both countries as in the classical model. However, moving to the right in this subregion, which means that Country 1 captures more and more industries, generally results in significant further increases in utility for Country 1 accompanied by significant losses in utility for Country 2.

(c) As Country 1 captures an ever larger share of export industries from Country 2 there comes a point beyond which any further acquisition of industries by Country 1 is disadvantageous to *both* Country 1 and Country 2.

Statements (4a)–(4c) apply, with obvious changes, to Country 2. These results cannot be obtained from very small models because the subregions referred to will often be completely empty of equilibria. Experience indicates that with six or more industries the regions are already reasonably populated.

Assumptions of the Model and of the Related Literature

The economic literature has employed at least three different models of the nature of scale economies. The first of these, which we use here, assumes that firms are perfectly competitive, that they operate as individual entities under constant returns to scale, and that the scale economies are produced by externalities that benefit the firms within a single industry in a given country. Under such circumstances prices are set at levels that yield zero profits. Examples of work using this approach include Kemp [8] and Ethier [1]. The assumptions of this model also match quite well the author's direct observations of industries containing small numbers of large firms which are very competitive with each other. These firms have significant internal economies at low levels of output, but at the higher levels at which they actually operate the internal economies of scale have been realized and the firms' cost structure tends to be linear.

The second widely used scale economies model assumes them to be internal to the firm, while the third model, like the first, assumes perfect competition and externalities. Externalities however are generated by the industry's output world wide.

This literature, as well as the work of Helpman and Krugman [7] and that of Grossman and Helpman [6], has greatly enlarged our understanding of scale economies. The results of this paper are consistent with and complementary to this existing work.

1. Some Basic Properties and the Basic Graph

The outcomes from a typical model are illustrated by Fig. 1 which is a type of graph we use repeatedly. Figure 1 plots Cobb–Douglas utility on the right vertical axis against normalized national income Z_1 for Country 1 on the horizontal axis. By normalized national income Z_j of Country j we mean $Z_j = Y_j/(Y_1 + Y_2)$ where Y_j is the national income of Country j. Each dot in the figure represents an equilibrium point. The large dots are outcomes in which only one of the two countries is a producer for each good;

these are the perfectly specialized equilibria which play a special role in this theory. The exchange rate w_1/w_2, or equivalently the ratio of the wages in the two countries corresponding to the normalized national incomes, is plotted on the top horizontal line. The utility obtained by Country 1 in a state of autarky is marked by the horizontal bar on the right. The utility of Country 2, the larger country, can be read from the left vertical axis and the normalized national income of Country 2 is $Z_2 = 1 - Z_1$. To avoid confusion, only the outline of the region of perfectly specialized equilibria has been plotted for Country 2, not the utility for Country 2 of the individual equilibria. The utility of each country is normalized separately so that the greatest utility it attains is 1.

There are several aspects of Fig. 1 worth noting. First we see the large number of equilibria, more than 20,000, present even in this nine-industry model. Second, the equilibria form an array of points with a rather definite shape which is in fact characteristic of many models. Third, the upper edge of the array of outcomes is rather well defined. In the figure it is marked by a dotted line. The equilibria near this boundary are the ones that do relatively well for Country 1 for any given Z_1. It is this boundary line, and the equilibria near it, that we compute by simple and rapid calculations in Section 3. Fourth, there is a lower boundary as well as an upper boundary to the array of perfectly specialized equilibrium points; this lower boundary can also be computed easily.

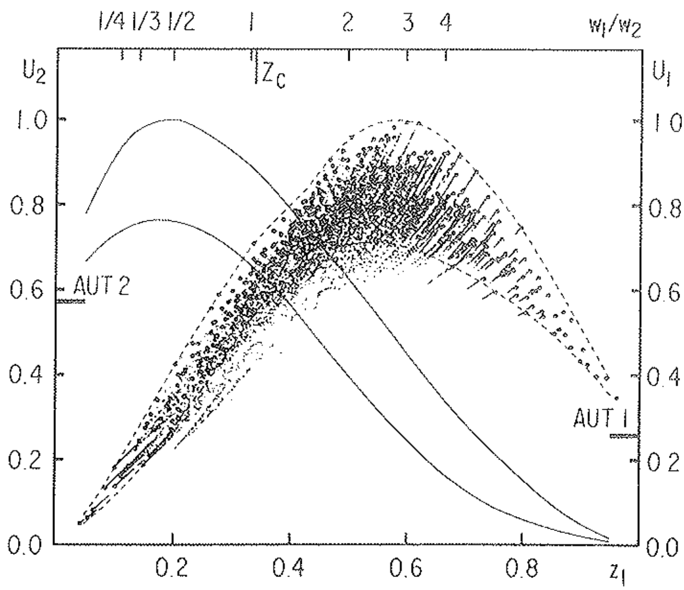

FIG. 1. 9 products.

The data tables on which Fig. 1 and all other figures are based appear in Gomory [3].

2. Existence of Solutions

In this model the production functions $f_{i,j}$ for good i in Country j always have economies of scale. The Cobb–Douglas utility function, or its logarithm, is used throughout, so for Country j ($j = 1, 2$) we have utility U_j given by

$$U_j = \Pi_i\, y_{i,j}^{d_{i,j}} \text{ and } u_j = \ln U_j = \sum_i d_{i,j} \ln y_{i,j}, \qquad d_{i,j} > 0, \sum_i d_{i,j} = 1,$$

with $y_{i,j}$ the quantity of the ith good obtained by Country j. It is a well known consequence of this choice of utility function that Country j spends a constant fraction $d_{i,j}$ of its national income Y_j on good i, for all prices p_i.

For any pattern of production using the labor-input production functions $f_{i,j}$, a zero-profit pricing equilibrium is a price vector p_i, wage rates w_j, and an allocation $l_{i,j}$ of each country's entire labor supply, L_j, among the industries in which it participates such that the supply of the ith good is equal to its demand and each active industry makes a profit of zero. So for each i

$$p_i \sum_j f_{i,j}(l_{i,j}) = \sum_j d_{i,j} Y_j = \sum_j d_{i,j} w_j L_j \text{ and } p_i f_{i,j}(l_{i,j}) = w_j l_{i,j}. \tag{2.1}$$

We now make two assumptions about the production functions $f_{i,j}$:

A1. Aside from a possible initial interval in which $f_{i,j}(l_{i,j})$ is zero, average productivity $f_{i,j}(l_{i,j})/l_{i,j}$ is continuous and strictly increasing.

A2. Each country is autarky produces a positive quantity of all goods; i.e., $f_{i,j}(d_{i,j} L_j) > 0$.

Theorem 2.1. *Under these assumptions, there is a zero-profit pricing equilibrium for* any *pattern of specialization in which each of the two countries is the* sole *producer of at least one of the goods in which it specializes. At all these equilibria each industry assigned to each country will produce positive quantities of output.*

The proof of this theorem is given in Appendix 2-1 of Gomory [3]. Since the theorem allows three possibilities for each industry, two possible sole producers or both producing, we would expect something on the order of 3^n equilibria. The actual formula is $3^n - 2^{n+1} + 1$. This explains the many equilibria present in Fig. 1.

To ensure stability we make a third assumption about our production functions.

A3. $\lim f(l_{i,j})/l_{i,j} = 0$ as $l_{i,j} \to 0$.

A3 asserts that a non-producer's unit cost for very small $l_{i,j}$ is arbitrarily large. If Country j is a non-producer of good i at some equilibrium point, A3 ensures that the non-producing industry would earn a negative profit in the immediate neighborhood of that equilibrium. Conditions A1–A3 are satisfied for all production functions of the form $f(l) = el^{\alpha}$ with $\alpha > 1$, as well as by any production function that satisfies the increasing average productivity condition A1, and is zero for an initial interval consistent with A2. It does not hold for the Ricardo case el^{α} with $\alpha = 1$, but it does hold if el is preceded by a short interval of zero output.

3. Boundaries of the Region of Equilibria

We now introduce variables $x_{i,j}$ that determine the pattern of production and play a key role in the analysis. At any equilibrium point we have the conditions (2.1). Together these imply that, for each good, expenditure equals wages, so

$$d_{i,1} Y_1 + d_{i,2} Y_2 = w_1 l_{i,1} + w_2 l_{i,2}.$$

We now define $x_{i,j}$ to be the fraction of the total expenditure on the ith product that is spent on product made in Country j. So

$$x_{i,1}(d_{i,1} Y_1 + d_{i,2} Y_2) = w_1 l_{i,1} \qquad \text{and} \qquad x_{i,2}(d_{i,1} Y_1 + d_{i,2} Y_2) = w_2 l_{i,2}. \tag{3.1a, b}$$

By definition $0 \leqslant x_{i,j} \leqslant 1$ and $x_{i,1} + x_{i,2} = 1$. We sometimes refer to the collection of $x_{i,j}$ as the assignment x.

We usually use normalized national incomes $Z_j = Y_j/(Y_1 + Y_2)$. Clearly $Z_1 + Z_2 = 1$ and $0 \leqslant Z_i \leqslant 1$. Since there are no profits, the national incomes of the two countries are $Y_j = w_j L_j$, so $Z_1/Z_2 = Y_1/Y_2 = (w_1/w_2)(L_1/L_2)$, which shows that Z_1/Z_2 is proportional to the wage ratio. We also use Z without a subscript to denote the two-vector $Z = (Z_1, Z_2)$. In terms of these normalized national incomes, (3.1a) and (3.1b) become

$$x_{i,1}(d_{i,1} Z_1 + d_{i,2} Z_2) = l^*_{i,1} Z_1 \qquad \text{and} \qquad x_{i,2}(d_{i,1} Z_1 + d_{i,2} Z_2) = l^*_{i,2} Z_2. \tag{3.2a, b}$$

Here the $l^*_{i,j}$ are normalized labor variables, $l^*_{i,j} = l_{i,j}/Lj$ representing the fraction of the labor force in Country j employed in making product i. One of the conditions for equilibrium is that the assignment of labor provided by the $x_{i,j}$ is in fact a partition of the entire labor force; i.e., that $\sum_i l^*_{i,j} = 1$. We can sum (3.2a) and (3.2b), obtaining the identities

$$\sum_i l^*_{i,1} = \sum_i \frac{1}{Z_1}(d_{i,1}Z_1 + d_{i,2}Z_2)\,x_{i,1} \quad \text{and} \quad \sum_i l^*_{i,2} = \sum_i \frac{1}{Z_2}(d_{i,1}Z_1 + d_{i,2}Z_2)\,x_{i,2}. \tag{3.3a, b}$$

The n.a.s.c. for assignment x to provide a partition of the labor force is that x and the normalized national income Z make $\sum_i l^*_{i,j} = 1$. This is equivalent to

$$\sum_i (d_{i,1}Z_1 + d_{i,2}Z_2)\,x_{i,1} = Z_1 \qquad \text{and} \qquad \sum_i (d_{i,1}Z_1 + d_{i,2}Z_2)\,x_{i,2} = Z_2. \tag{3.4a, b}$$

Equations (3.4a) and (3.4b) are in fact linearly dependent and therefore we need only one of them. This dependence is a consequence of Walras Law, and can also be verified directly by adding the two equations. We refer to (3.4a) and (3.4b) as the zero excess labor condition. This condition links *any* assignment x to the Z value, $Z(x)$, required to satisfy (3.4a) and (3.4b). In economic terms $Z(x)$ gives the wage ratio w_1/w_2 at which the production pattern resulting from x exactly uses the labor of both countries.

Equilibrium Conditions and Integer x

We now look at the conditions that must be met for x to be an equilibrium point. For any x, whether it is an equilibrium x or not, the normalized national income $Z(x)$ required to satisfy the zero excess labor condition can be calculated from (3.4a) and (3.4b). x and $Z(x)$ then determine labor quantities $l^*_{i,j}$ from Eqs. (3.2a) and (3.2b) whose meaning is that expenditures must match wage bills. The $l^*_{i,j}$ in turn determine the amounts produced, $f_{i,j}(l_{i,j})$.

For this arbitrary x and its $Z(x)$ to determine an equilibrium point, only one more condition must hold. As in (2.1) there must be a price p_i for the ith good at which the value of the goods produced matches the wage bill: i.e., $p_i f_{i,j} = w_j l_{i,j}$ for $j = 1, 2$. We refer to this as the price condition $(3.P)$. When there is more than one producer both must have the same unit cost $f_{i,j}(l_{i,j})/w_j l_{i,j} = 1/p_i$. However, when there is only one producer, that producer's unit cost alone determines a p_i satisfying $(3.P)$ since for the non-producer we always have $p_i f_{i,j} = 0 = w_j l_{i,j}$.

If any x and $Z(x)$ satisfying (3.4a) and (3.4b) meet the price condition, supply equals wage bill equals demand, the total labor force is used, and x is an equilibrium point.

While most arbitrarily chosen x and their $Z(x)$ do not satisfy $(3.P)$, all integers x do. For integer x, i.e., $x_{i,j}=0, 1$, there is only one producer of each good so $(3.P)$ is always satisfied. Consequently *all integer x are equilibria.* They are of course the perfectly specialized equilibria. Condition A3 then gives these equilibria a certain degree of stability. We will also see that the perfectly specialized equilibria are the ones that largely determine the shape of the equilibrium region.

Utility and Linearized Utility

The logarithm of Cobb–Douglas utility is a sum of terms involving the quantity $y_{i,1}$ of the ith good Country 1 receives. The $y_{i,j}$ can be written as the product of two terms $F_{i,1}(Z)\,Q_i(x, Z)$, where $Q_i(x, Z)$ is the total quantity of the ith good produced in the world and $F_{i,1}(Z)$ is the fraction obtained by Country 1.

$$u_1(x, Z) = \ln U_1(x, Z) = \sum_i d_{i,1} \ln F_{i,1}(Z)\,Q_i(x, Z). \tag{3.5}$$

Since the goods are all sold at a world price, the fraction going to Country 1 is its expenditure as a fraction of world expenditure and the world quantity produced is the sum of the quantities produced in each country so

$$F_{i,1}(Z) = \frac{d_{i,1}Y_1}{d_{i,1}Y_1 + d_{i,2}Y_2} = \frac{d_{i,1}Z_1}{d_{i,1}Z_1 + d_{i,2}Z_2}$$

and

$$Q_i(x, Z) = q_{i,1}(x_{i,1}, Z) + q_{i,2}(x_{i,2}, Z).$$

The $q_{i,j}$ are defined by $q_{i,j}(x_{i,j}Z) = f_{i,j}(l_{i,j})$ where the labor quantity $l_{i,j}$ is determined by x and Z and is found from (3.2a) and (3.2b). Substituting for $F_{i,j}$ and $Q_{i,j}$ in (3.5) gives

$$u_1(x, Z) = \sum_i d_{i,1} \ln \frac{d_{i,1}Z_1}{d_{i,1}Z_1 + d_{i,2}Z_2} \{q_{i,1}(x_{i,1}, Z_1) + q_{i,2}(x_{i,2}, Z_2)\}. \tag{3.6}$$

This expression is complicated both in its dependence on the assignment x and on the normalized national incomes Z. In addition, for equilibrium x, Z and x are linked to each other through (3.4a) and (3.4b). This makes it difficult to compare the many different equilibria except by fully computing each one. Although useful and suggestive experiments along that line were done as part of this work, we take a different approach in what follows. We emphasize perfectly specialized equilibria and the simplifications that are possible with them.

If x_1 and x_2 are any variables constrained to be either 0 or 1, and if $x_1 = 0$ implies $x_2 = 1$ and vice versa, then we always have for any function $g(x_1, x_2)$ the tautology $g(x_1, x_2) = x_1 g(1, 0) + x_2 g(0, 1)$. x_1 and x_2 act as a switch between the two possible values of g. The individual terms $x_{i,1}$ and $x_{i,2}$ of an *integer* assignment x are of course variables of this type. Letting g be successively the individual terms of the sum (3.6) and using the tautology give an expression for utility that is valid for integer x. This is the linearized utility Lu_1.

$$\mathrm{Lu}_1(x, Z) = \sum_i \{x_{i,1} d_{i,1} \ln F_{i,1}(Z)\, q_{i,1}(1, Z) + x_{i,2} d_{i,1} \ln F_{i,1}(Z)\, q_{i,2}(1, Z)\}. \tag{3.L}$$

For integer x *only* we have $\mathrm{Lu}_1(x, Z) = u_1(x, Z)$. The merit of $\mathrm{Lu}_1(x, Z)$ is that for fixed Z it is linear in the variables x.

Boundary Calculation Preliminaries

Using the concept of boundary turns out to enormously simplify finding high utility equilibria and determining the shape of the equilibrium region in the (Z_1, U_1) plane. To find the upper boundary of the array of points (Z_1, U_1) corresponding to all perfectly specialized equilibria, we might think of defining a function $B_1(Z)$ to be the result of fixing Z and then maximizing $u_1(x, Z) = \mathrm{Lu}_1(x, Z)$ over all perfectly specialized equilibria x having that Z value, i.e., satisfying (3.4a) and (3.4b) for that Z. This is maximizing a linear expression Lu_1 over integer x satisfying a single linear constraint, a very simple integer programming problem. The collection of values $B_1(Z)$, computed in this way for each Z, would form an upper boundary for the perfectly specialized equilibria.

The maximization problem for each fixed Z has the following economic interpretation: Once Z is fixed, the expenditure in each country for the ith good is completely determined. Therefore the fraction $F_{i,j}$ of the total production of the ith good that goes to Country j is also fixed. The only way to improve the utility that Country j gets from the ith good is to increase Q_i. This can only be done, for integer x, by assigning its production entirely to the producer who is, at that Z, the more productive one. The labor constraint (3.4a) and (3.4b) prevents this assignment from being made simultaneously for every good, and the maximization problem is to find the best assignment possible subject to the labor constraint.

While this direction and motivation are fundamentally correct, there is still one difficulty to overcome: Precisely as written, Eqs. (3.4a) and (3.4b) will not usually have *any* solution in integer x for an arbitrary Z, much less many different x to maximize over. This reflects the economic fact that there are equilibria for certain Z only. To deal with this difficulty we need one more concept, the classical level. In Gomory [2] this was referred to as the Ricardo level.

The Classical Level

For any Z we can define an assignment $x^C(Z)$, which we call the classical assignment. The components of $x^C(Z)$ are defined by setting $x_{i,1} = 1$ if $q_{i,1}(1, Z) > q_{i,2}(1, Z)$ while otherwise $x_{i,1} = 0$. This is simply assigning the production of good i entirely to Country 1 if Country 1 is the cheaper producer at that Z, and otherwise assigning it entirely to Country 2.

For an arbitrary Z and its $x^C(Z)$ we usually do not have equality in (3.4a) and (3.4b). In fact, for Z with very small Z_1, which means a low wage in Country 1, the terms on the right in (3.3a), involving as they do Z_2/Z_1, will be very large, and the right side of (3.3a) will be greater than 1. In economic terms, if the wage is very low in Country 1 and production is assigned to the country that is the cheaper producer, the resulting demand for labor in Country 1 will outstrip the supply. Similarly, for any $x^C(Z)$ with large Z_1, which means a high wage in Country 1, the terms on the right in (3.4a) will be small, their total will be < 1, so demand for Country 1's labor will be less than the supply.

As wages increase, the individual terms on the right in (3.3a) only decrease, while the $x_{i,1}$ switch from 1 to 0, as Country 1 stops being the cheaper producer in industry after industry. Thus the demand for labor in Country 1 produced by $x^C(Z)$ decreases as Z_1 increases. It follows that there is a unique transition value of Z_1 which we call Z_C, the classical level. Z_C separates the Z_1 ($Z_1 < Z_C$) for which demand exceeds the labor supply from the Z_1 ($Z_1 > Z_C$) for which the demand is less than the labor supply. More formally we define the classical level to be $Z_C = \sup Z_1$ such that if $x = x^C(Z)$, $\sum_i (d_{i,1} Z_1 + d_{i,2} Z_2)\, x_{i,1} > Z_1$. For Country 2 the situation is reversed. Below the classical level the demand for Country 2's labor is less than the supply, if x^C is used, and *above* Z_C the demand exceeds the supply.

Since for any Z_1 we can easily determine if it is above or below Z_C, Z_C is easily calculated iteratively. In Figs. 1–3 and 5–7 the classical level is marked by a vertical bar.

The Boundary $B_I(Z)$

Earlier, in outlining a boundary calculation that involved maximizing over many equilibria x for a fixed Z, we encountered the difficulty that, instead of having many specialized equilibria, there were, for most Z, no equilibria at all. To deal with this difficulty we now relax (3.4b) to an inequality, which then has many solutions for any Z, and define $B_I(Z)$ by the integer programming problem

$$B_I(Z) = \operatorname*{Max}_x u_1(x, Z) = \operatorname*{Max}_x \mathrm{Lu}_1(x, Z) \qquad x \text{ integer},$$

$$\text{with} \quad \sum_i (d_{i,1} Z_1 + d_{i,2} Z_2)\, x_{i,2} \leqslant Z_2. \tag{3.7}$$

The inequality assumes the direction shown if Z is above the classical level ($Z_1 > Z_C$) and is reversed for Z below the classical level. This relaxation *allows underutilization of labor* in the country whose labor is *scarce*. Consequently maximizing utility for the given Z should push the inequality very close to equality as the attempt is made to use this valuable labor.

In (3.7) we have arbitrarily used the inequality form of (3.4b) as the constraint. Of course we could just as well have chosen (3.4a). When we need to refer to the inequality versions of (3.4a) or (3.4b) we designate them by i-(3.4a) and i-(3.4b). It is always assumed that these inequalities point in the proper directions.

Inequality i-(3.4b) only involves the variables $x_{i,2}$, but the objective function Lu_1 involves both $x_{i,1}$ and $x_{i,2}$. If we rewrite Lu_1 in terms of $x_{i,2}$ only, using $x_{i,1} + x_{i,2} = 1$ to eliminate $x_{i,1}$, we get

$$\mathrm{Lu}_1(x, Z) = \sum_i d_{i,1} \ln F_{i,1} q_{i,1}(1, Z) + \sum_i x_{i,2} d_{i,1} \ln \frac{q_{i,2}(1, Z)}{q_{i,1}(1, Z)} \tag{3.8}$$

so we can put the maximization problem (3.7) in a form that involves the $x_{i,2}$ only. If we use $P_1(Z)$ to denote the first sum in (3.8), which is a function of Z but not of $x_{i,2}$, and use $c_{i,2}(Z)$ to denote $d_{i,1} \ln(q_{i,2}(1, Z)/q_{i,1}(1, Z))$, we obtain

$$B_I(Z) = \operatorname*{Max}_x P_1(Z) + \sum_i x_{i,2} c_{i,2}(Z)$$

$$\text{with} \quad \sum_i (d_{i,1} Z_1 + d_{i,2} Z_2)\, x_{i,2} \leqslant Z_2 \text{ and } x_{i,2} \text{ integer}. \tag{3.9}$$

Equation (3.9) is our basic tool in dealing with equilibrium regions. In (3.9) both the objective function and the inequality are linear in x for fixed Z. While the coefficients of the $x_{i,2}$ in the inequality are always positive, the $c_{i,2}$ in the objective function can be either positive or negative. The sign of $c_{i,2}$ is determined by the ratio $q_{i,2}(Z)/q_{i,1}(Z)$, if $q_{i,2} > q_{i,1}$. If $q_{i,2} > q_{i,1}$ Country 2 is the cheaper producer of the world supply and $c_{i,2}$ will be positive. If Country 1 is the cheaper producer $c_{i,2}$ will be negative.

Computation of $B_l(Z)$

The $B_l(Z)$ defined by (3.9) can be computed by any integer programming technique. For a single inequality problem such as this, ordinary dynamic programming is very effective. This calculation is spelled out in Appendix 3-2 of Gomory [3]. The dynamic program gives actual integer solutions x so that we can compute the corresponding $Z(x)$ from (3.4b) and go on to compute utility, and hence fully describes the maximizing equilibrium for each Z_1. As Section 4 shows, the equilibria so attained will be arbitrarily close to the boundary as the problem size grows. In practice we find the resulting equilibria to be extremely close to the boundary even for very moderate sized problems. We can spell out the boundary curve by computing a series of Z_1 values from $Z_1 = 0$ to $Z_1 = 1$, and we get a whole series of boundary points and nearby equilibria. This entire calculation requires only minutes on a home personal computer (see Section 5 of Gomory [3]).

The Boundary $B(Z)$

There is an even easier calculation that gives a slightly weaker but extremely useful boundary curve, which we call $B(Z)$. To get $B(Z)$ we further relax the problem (3.9) by allowing *continuous* $x_{i,2}$. It is easily seen that with continuous variables the maximizing x will always satisfy the inequality in (3.9) as an equality, so in fact $B(Z)$ is given by maximizing $\mathrm{Lu}_1(x, Z)$ subject to (3.4b); i.e.,

$$B(Z) = \operatorname*{Max}_{x} \mathrm{Lu}_1(x, Z) = P_1(Z) + \sum_i x_{i,2} c_{i,2}$$

$$\text{subject to} \quad \sum_i (d_{i,1} Z_1 + d_{i,2} Z_2)\, x_{i,2} = Z_2. \qquad (3.9a)$$

In (3.9a) we are looking at a particularly simple linear programming problem with only one equation and upper bounds, $0 \leqslant x_{i,2} \leqslant 1$, on the variables $x_{i,2}$. The solution technique for such a special linear programming problem or "continuous knapsack problem" is particularly simple. Equation (3.9a) can be thought of as filling a space of length Z_2 with amounts $x_{i,2}$ (not necessarily integer) of goods. The ith good has length $d_{i,1} Z_1 + d_{i,2} Z_2$ and value $c_{i,2}$; the goal is to fill the space with the most valuable assortment of goods. The solution to such a problem is to put goods in the order of their value per unit length, which we call value density. The densest good is used first. When its turn comes the amount $x_{i,2}$ of each good is increased from zero until either the amount $x_{i,2} = 1$, or Eq. (3.4b) is satisfied; i.e., the space is used up, whichever occurs first. If the $x_{i,2}$ reaches 1, we start again with the next good in order of value density.

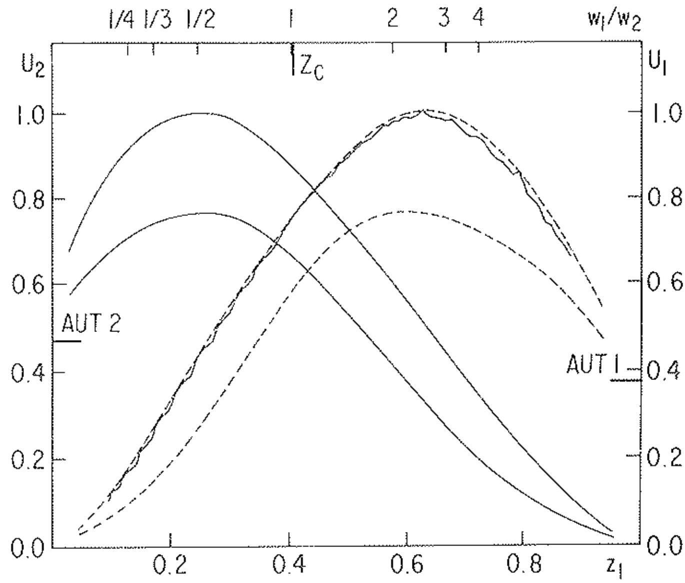

FIG. 2. $B_i(Z)$—17 products.

If for the kth good (3.4b) is satisfied for some value of $x_{k,2} < 1$, the current values for all variables $x_{i,2}$ are the optimizing solution. *Note that $x_{k,2}$ is the only variable that is non-integer in this solution.* The variables that preceded it in value density are 1, and those after it are 0. This calculation is then repeated for different Z_1 to get the boundary curve. It is the results of these simple calculations that appear as the upper boundary curves in Figs. 1 and 2.

This calculation too has an economic interpretation. As (3.2a) and (3.2b) show the length $d_{i,1}Z_1 + d_{i,2}Z_2$ is, for fixed Z_1, proportional to the amount of labor required to produce the ith good in Country 2 when Country 2 is the sole producer. As (3.8) shows, the expression for the value in (3.9a) represents the *change* in utility for Country 1 resulting from Country 2 becoming the producer instead of Country 1. So the algorithm selects industries in the order of value density; those which yield the greatest improvement in utility per labor hour are chosen first.

For two countries with identical demand structure, i.e., $d_{i,1} = d_{i,2}$, the length is simply $d_{i,1}$ and the density, or change in utility per labor hour, is $c_{i,2}/d_{i,1} = \ln(q_{i,2}(1, Z)/q_{i,1}(1, Z))$. Industry i chosen in the algorithm before industry j if $\ln(q_{i,2}(1, Z)/q_{i,1}(, Z)) > \ln(q_{j,2}(1, Z)/q_{j,1}(1, Z))$. In other words, industry i is chosen before industry j if it has greater *comparative advantage.* However, when the countries have dissimilar demand structures, the order of choice is determined by $c_{i,2}/(d_{i,1}Z_1 + d_{i,2}Z_2) = (d_{i,1}/(d_{i,1}Z_1 + d_{i,2}Z_2))$ $\ln(q_{i,2}/q_{i,1})$ which involves q_2/q_1 but also is influenced by the term

$d_{i,1}/(d_{i,1}Z_1 + d_{i,2}Z_2)$ which measures the relative importance of the ith good to Country 1.

So far we have discussed (3.9a) in terms of obtaining a boundary, not in terms of obtaining high utility equilibria. The optimizing x in (3.9a), being non-integer, is usually not itself an equilibrium point. However, equilibria can be obtained by rounding the single non-integer variable in x either up or down. This is dealing with an integer programming problem by the time-honored device of rounding.

Lower Boundaries

While the goal so far has been to find the upper boundary of the array of perfectly specialized equilibria, exactly the same methods will give us the *lower* boundary. If we minimize the objective functions in problems (3.9) and (3.9a), instead of maximizing, we get lower boundaries $\mathrm{BL}_I(Z)$ and $\mathrm{BL}(Z)$ corresponding to the two different relaxations. The second approach produced the lower boundaries seen in Figs. 1 and 2. All perfectly specialized equilibria are somewhere between these curves.

The Two Methods

We have described two methods of calculation, one with integer variables and the inequality i-(3.4b) and one with continuous variables and the equality (3.4b). Both generate boundary curves and also find actual equilibria near those curves.

$B_I(Z)$ is of course a tighter fit to the equilibrium points than $B(Z)$. However, for small problems, where the biggest differences exist, $B_I(Z)$ tends to be jagged. In larger models, such as the 17-industry model shown in Fig. 2, we see that the two boundary curves for Country 1, the dashed $B(Z)$ and the solid $B_I(Z)$, are much more alike. Both calculations can also give us points near their respective boundaries; the integer calculation does this automatically while the continuous calculation does this by rounding the non-integer variable up or down. Figure 3 represents a problem with 27 goods. It shows the $B(Z)$ from the continuous calculation together with the integer points obtained by the integer maximization calculation. From the more than 100 million specialized equilibria in the 27-good model, the calculation has produced the 75 shown in the figure that are sitting virtually on top of $B(Z)$. If we select from these the one that maximizes the utility of Country 1, we get a utility value that is within 1/6 of 1% of the highest point of $B(Z)$, so this equilibrium point is at least within 1/6 of 1% of the highest utility that can be obtained by Country 1 at any specialized equilibrium. In Section 5 we show that $B(Z)$ lies above *all* the equilibria as well as above the specialized ones, so this equilibrium point is close to the best utility that can be obtained at any equilibrium. It is also worth noting

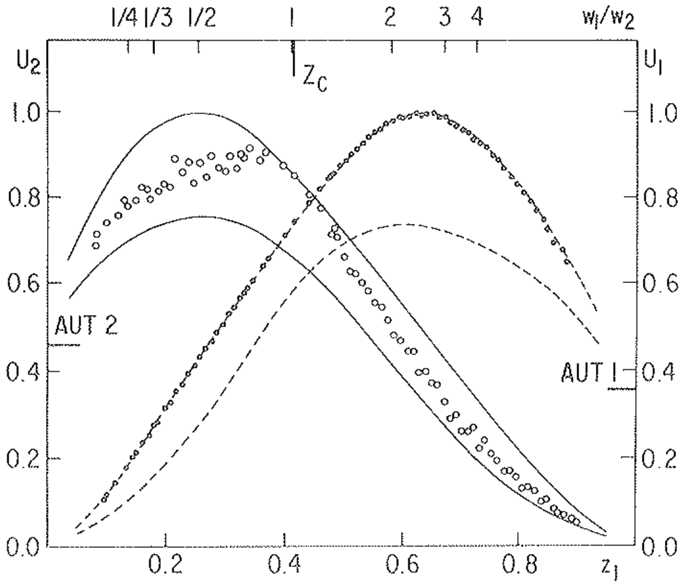

FIG. 3. Near boundary equilibria—27 products.

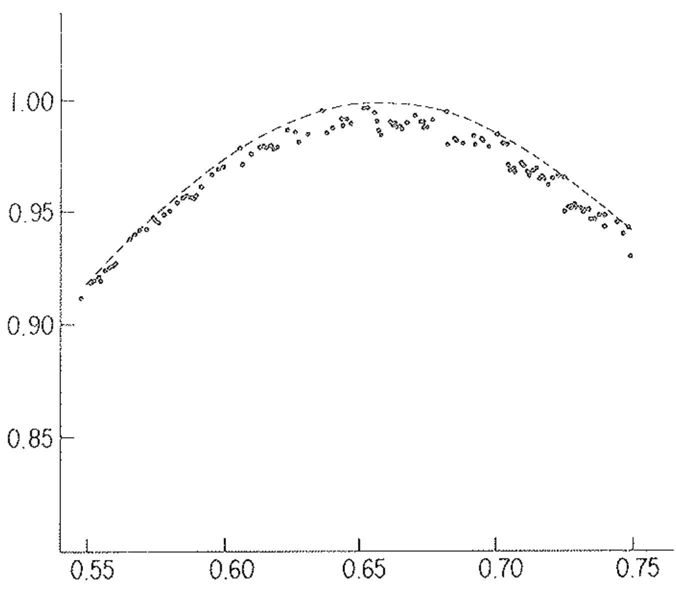

FIG. 4. Hump area—27 products.

that the *utility to Country 2 is low* for any of the points that are near maximal for Country 1. This is in line with statement 1 of the Introduction.

Both calculations appear to have their advantages. Boundary $B(Z)$ is smoother and sometimes easier to deal with theoretically. The integer calculation is capable of producing many more actual equilibria near $B(Z)$. A very good way of combining the strengths of both methods is to use them together as a sort of coarse and fine microscope. First, using the continuous method, we obtain an entire boundary, for example $B(Z)$, the upper boundary of Country 1 in Fig. 3. Then, in some narrower range of interest, for example near the maximizing hump of $B(Z)$, we compute the nearby integer points by using the integer calculation and a finer Z_1 grid. The result of doing this appears in Fig. 4 which represents the Country 1 hump area of Fig. 3, from $Z_1 = 0.55$ to $Z_1 = 0.75$, magnified by a factor of five. There are 119 equilibria computed in that range by the integer method while rounding the continuous method would produce 11. Using this technique it becomes possible to isolate the equilibria in a particular area, for example the equilibria between $Z_1 = 0.625$ and $Z_1 = 0.675$ at the very peak of the hump in Fig. 4 and examine them for their common characteristics.

4. Filling In

For any point $(Z_1, B(Z_1))$ on the boundary $B(Z)$ there is an x, with at most one non-integer component, that obtains the linearized utility value $B(Z_1)$ for that Z_1. This suggests that the equilibrium points obtained by rounding this x might be near the boundary in a many-industry model. Gomory [3] uses this reasoning to show that for large problems there are always equilibria near the boundary. Here we use the same basic idea to show that the entire region between the upper and lower boundaries fills in with equilibria for large problems. We now state this theorem; the proof is given in Appendix 1.

The δ that appears in the theorem is the largest of the demands $d_{i,j}$, while $g = \sum_i |d_{i,1} - d_{i,2}|$ measures the difference between the demand structures of the two countries. g is 0 for identical demands $d_{i,1} = d_{i,2}$, and $g = 1$ for "orthogonal demands" $d_{i,1} d_{i,2} = 0$. R and M are parameters that depend on the nature of the individual industries, and are explained in Appendix 1.

Theorem 4.1. *If (v, Z') is any point between $B(Z')$ and $B_L(Z')$, then, provided g is not 1, there is an integer equilibrium point x with $|Z_1(x) - Z_1'| \leqslant \delta/(1-g)$ and with $|v - u_1(x, Z(x))| \leqslant \delta(2R + M)/(1-g)$.*

If, in a many industry model, the largest individual industry is a small fraction of the total economy, while the individual industry characteristics remain within bounds that are not dependent on the number of industries,

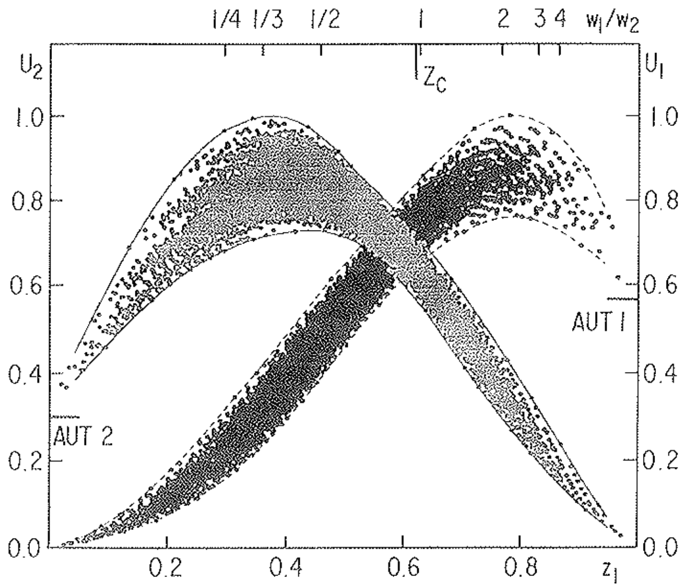

FIG. 5. Filling in.

Theorem 4.1 gives us an equilibrium near (Z_1, v). The fill in effect is already visible in Fig. 5 which plots all the perfectly specialized equilibria from a 13-product model. Because of the fill in effect we can discuss any part of the region between the upper and lower boundary curves with confidence that it will be populated with equilibria.

5. Non-Specialized Equilibria

So far we have worked entirely with integer solutions, that is to say with perfectly specialized equilibria. Considerable justification for this approach can be seen from the following theorem.

THEOREM 5.1. *Let x be any equilibrium solution, whether specialized or not. Let $Z(x)$ be the corresponding Z and $u_1(x, Z)$ the utility of x to Country* 1, *then*

$$u_1(x, Z) \leqslant B_I(Z) \leqslant B(Z).$$

So *all* the equilibrium points, not just the specialized ones, lie under the upper boundary curves. The proof of Theorem 5.1 is given in Appendix 2.

Non-specialized equilibria, however, can lie below the lower boundary curve as is clear from Fig. 1. These equilibria are numerous and have their own interesting properties. While non-specialized equilibria in the presence of economies of scale are *usually* unstable, there are stable non-specialized equilibria as well. This is discussed in considerable detail in Gomory [3].

6. The General Shape of the Region and a Special Case

The more than 80 numerical examples that have been examined all show the same characteristic regional shape that is seen in Figs. 1–3 and 5–7. These characteristics are, for Country 1, (1) a steady rise in the height of the upper and lower boundaries over the range from $Z_1 = 0$ to Z_C, (2) to the right of Z_C a height for the upper boundary that is always above the autarky level, (3) a point of maximum utility to the right of Z_C followed by a descent of the upper boundary to the autarky level, and (4) upper and lower boundaries that coincide at $Z_1 = 0$ and $Z_1 = 1$. These characteristics are not accidental but derive from the fundamental economics. We now sketch out the elements of a very rough economic rationale. A proper treatment is a key element of Gomory and Baumol [4].

At very low levels of relative national income Z_1 Country 1 is the producer of very few goods making up a small fraction of the total world demand. Since we are below the classical level Z_C, as we increase Z_1 the boundary algorithm can and will select as a new industry to add to those already in Country 1 one in which Country 1 is a cheaper producer than Country 2 is. The effects of this change are to increase Country 1's fraction $F_{i,1}$ of every good, (relative consumption) make the goods cheaper in the transferred industry, make the other goods made in Country 1 more expensive, and those made abroad less expensive. Since most goods are made abroad there is a net improvement in Country 1's utility. As more and more industries are added to Country 1, Z_1 increases and we finally reach Z_C. For $Z_1 > Z_C$ the industries added are ones in which Country 1 is the more expensive producer, and also more and more industries are in Country 1 and each addition makes their goods become still more expensive, while the goods made ever more cheaply abroad are ever fewer in number. At some point the total utility of these changes becomes negative, and the boundary turns down. Finally, as Country 1 becomes the producer of almost everything, Country 2 hardly matters to it either as a supplier or customer, and Country 1's utility approaches the autarky level. While this discussion is plausible, it remains to show that the various competing effects do play out in the way suggested. We do this in Gomory and Baumol [4].

The Regional Shape for Identical Countries

Competition among identical countries is non-trivial in this model and produces many different equilibrium outcomes. If we assume production functions of the form $e_{i,1}l^{\alpha} = e_{i,2}l^{\alpha}$ the problem simplifies. As in shown in Gomory [3] the upper and lower boundary curves coincide, and all the integer equilibria lie directly on a curve which is given exactly by the formula

$$U_1(Z) = U_1^A Z_1 \left\{ \left(\frac{1}{Z_1} \right)^{Z_1} \left(\frac{1}{(1-Z_1)} \right)^{(1-Z_1)} \right\}^{\alpha},$$

U_1^A the utility in autarky of Country 1.

This result is plotted in Fig. 6 for $\alpha = 1.5$.

Even in this special case, with the region collapsed to a single curve, the various regions mentioned in statements (4a)–(4c) of the Introduction are all plainly present in simple form. The subregions of (4a) that are respectively advantageous and disadvantageous relative to autarky can be read from the location of the horizontal autarky bars. Beyond the two humps are the two subregions (4c) that are relatively disadvantageous to both countries. The region of opposed interests, (4c) in the Introduction, is the curve from the point below the maximum for Country 2 up to Country 1's maximum,

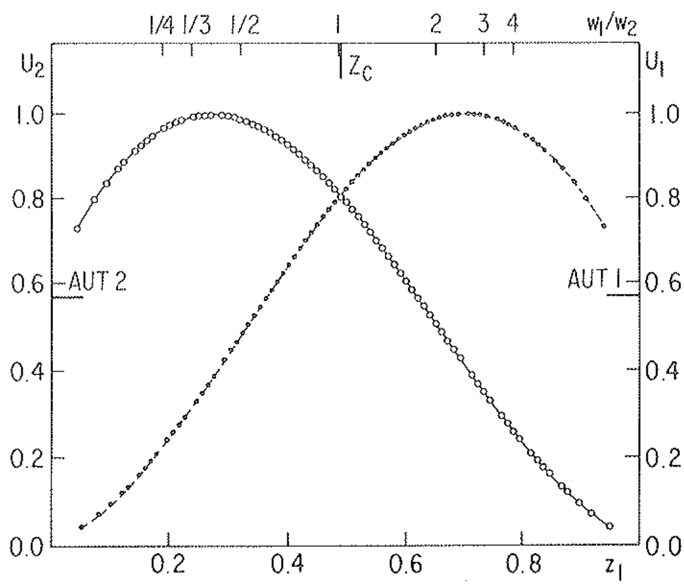

FIG. 6. Identical countries.

so the interests of the two countries are strictly opposed over that entire range, with the maximum for Country 1 being a rather poor outcome for Country 2.

7. General Properties of the Model

Economies of scale have two distinguishable effects. The first is an "impediment to entry" effect which gives a *producing* country an advantage over a *non-producing* one. These impediments to entry come from many sources aside from the obvious possibility of economies of scale in manufacturing. Examples are knowledge and expertise in the manufacturing process, the largely experience-born ability to design a manufacturable product, knowledge of and experience with marketing channels, knowledge of customer needs, and even knowledge of and being known to particular customers. Much knowledge can only be obtained by doing, and there will be a period of doing poorly through inexperience for any new entrant. In addition, especially in the case of industries in different countries, there is the question of infrastructure. If one industry is flourishing in Country 1, and non-existent in Country 2, a large part of the difficulty in entry will be to find the people or companies who can build plants of the proper type and supply parts, specialized instruments, and specialized support services. While some of this can be imported, some cannot, and working at a distance is often not the same as working close by. All of these factors and many more can make entry into a new industry a large sunk commitment now in exchange for a return that is both distant and inherently uncertain. In our model this aspect of economies of scale shows itself in assumption A3 as little output for the labor input.

The second aspect of economies of scale is the advantage that *larger* scale may give one producer over another when *both* are active in the industry. In this model this is reflected in the shape of the production functions for larger labor quantities.

The two aspects of economies of scale are quite separable; one can have, for example, a strong barrier to entry and weak large scale economies in a single production function. The roles of the two aspects in our models are quite different. The values of the production functions for large labor quantities enter into the coefficients of the objective function in the maximization problem and thus affect the location of the boundary and of the region. The shape of the production functions for low labor levels affects the degree of stability of the individual equilibrium points. If the production functions rise sharply near 0, the impediment to entry will be feeble and the stability of the equilibrium point will be weak. On the other hand, if the production functions are zero till near the autarky level of production and then jump rapidly up, we have an extremely strong impediment to entry and a strongly stable equilibrium point. In Gomory and Baumol [5] we show the results of modifying the algorithms of Section 3 to take into account the stability of the equilibrium points.

Identical and Non-Identical Demand Structures

Any production assignment x determines a $Z(x)$, and worldwide production quantities Q_i of each good. The utilities resulting from x are $U_1 = \Pi_i(F_{i,1}Q_i)^{d_{i,1}}$ and $U_2 = \Pi_i(F_{i,2}Q_i)^{d_{i,2}}$. For identical demands $d_{i,1} = d_{i,2}$ we have $F_{i,1} = Z_1$ and $F_{i,2} = Z_2$ so $U_1/U_2 = Z_1/Z_2$. Since $U_1 = (Z_1/Z_2)\, U_2$, once Z is fixed a production pattern that is good for one country is good for the other. So the conflict between the interests of the two countries is confined to the determination of Z.

This benign property of identical demands does not carry over to the non-identical demand case. There countries will put different weights on different elements of the utility, and the production plan that is best for one is generally not best for the other, even for fixed Z. This effect is clearly illustrated in Fig. 7, a 37-industry model. Using a fine grid we have computed a large number of maximizing equilibria around the hump area of Country 1 using the integer programming method. The corresponding utilities for Country 2 would be on its upper boundary in the identical demands case; here they have moved down sharply, even though the

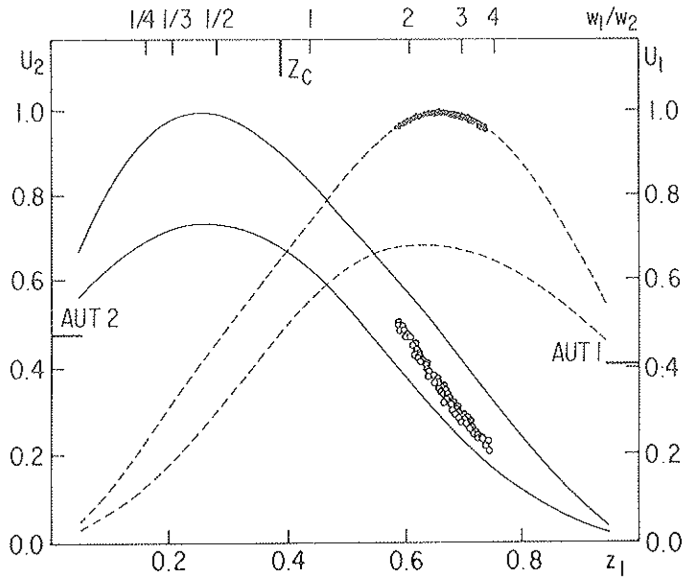

FIG. 7. Non-identical demands.

g-value of the model, $g = \frac{1}{2}\sum_i |d_{i,1} - d_{i,2}|$, is only 0.187 on a scale in which identical demands measure 0 and orthogonal demands measure 1.

The Effect of Country Size

In this model, as in those having diseconomies of scale, the relative sizes of the two trading partners matter. As is spelled out in Gomory [3], the smaller trading partner is much ore likely to benefit from trade than is the larger one. These size effects can be quite strong.

8. SUMMARY

As promised in the Introduction we have given algorithms for the selection of good equilibria, and for obtaining the boundaries of the region of equilibria. We have shown that the equilibrium region tends to fill up with equilibrium points as the problem becomes large. It is also evident that the characteristic regional shape discussed in Section 8 and illustrated in our figures supports the conclusions (4a)–(4c) of the Introduction. The theory we have developed tells us that the various subregions we have mentioned are in fact all well populated with equilibria for large problems. The general picture that emerges from this analysis is that gains from trade are possible but not automatic, and that there is a considerable range of conflict in the interests of the two trading partners.

In Gomory and Baumol [4] we advance this work in several directions. These include a rigorous treatment of the characteristic regional shape, some strong results on gains from trade, and an analysis of the case where some industries have economies and some have diseconomies.

APPENDIX 1

Proof of Theorem 4.1. For any choice of Z' we define an integer equilibrium point x to be *near equality* (n.e.) at Z' if x satisfies i-(3.4b) for $Z = Z'$ while increasing some component $x_{k,2}$ of x from 0 to 1 results in a new x that does not satisfy i-(3.4b). For n.e. equilibria we have:

LEMMA 4. *If x is n.e. then, if δ is the largest of the individual demands $d_{i,j}$ and g is given by $g = \frac{1}{2}\sum_i |d_{i,1} - d_{i,2}|$, then $|Z'_1 - Z_1(x)| \leqslant \delta/(1-g)$.*

Proof. Using first the zero excess labor inequality i-(3.4b) involving x and Z', and then the n.e. property on the component $x_{k,2}$, gives two inequalities.

$$\sum_{i \neq k} (d_{i,1} Z'_1 + d_{i,2} Z'_2)\, x_{i,2} \leqslant Z'_2 \leqslant \sum_{i \neq k} (d_{i,1} Z'_1 + d_{i,2} Z'_2)\, x_{i,2} + d_{k,1} Z'_1 + d_{k,2} Z'_2.$$

Since x is an equilibrium point (3.4b) gives

$$\sum_i (d_{i,1} Z_1(x) + d_{i,2} Z_2(x))\, x_{i,2} = Z_2(x).$$

Subtracting the equality from the inequalities, rearranging terms, and using $Z_1 + Z_2 = 1$ gives

$$0 \leqslant (Z_1' - Z_1(x)) \left\{1 - \sum_i (d_{i,2} - d_{i,1})\, x_{i,2}\right\} \leqslant Z_1' d_{k,1} + Z_2' d_{k,2} \leqslant \delta.$$

The largest possible value of the sum is g where

$$g = \sum_i (d_{i,1} - d_{i,2}),\ i \text{ such that } (d_{i,1} - d_{i,2}) > 0. \text{ So } 0 \leqslant |Z_1(x) - Z_1'| \leqslant \frac{\delta}{1-g}.$$

We still need to convert g to the more useful form given in the lemma. If we add up all the terms $(d_{i,1} - d_{i,2})$ for which $d_{i,1} < d_{i,2}$, we get a negative sum $-g'$. But $g + g' = \sum_i d_{i,1} - \sum_i d_{i,2} = 1 - 1 = 0$. So $g = g'$. It follows that $\sum_i |d_{i,2} - d_{i,1}| = g + g = 2g$ which proves Lemma 4.1.

Since we now know that $Z(x)$ is near Z' for any n.e. equilibrium, it follows that $u_1(x, Z(x))$ is near $u_1(x, Z')$ as long as the derivative of u_1 with respect to Z_1 is bounded. It is clear that it is bounded, in Appendix 4-2 of Gomory [3] we show in a straightforward but tedious way that the bound is given explicitly by

$$M(Z') = \frac{1}{Z_m'} \left\{2 + \frac{\alpha(Z')}{Z_m'}\right\}.$$

Here $Z_m' = \min(Z_1', Z_2')$, and $\alpha(Z') = \max \alpha_{i,j}(Z')$, with each $\alpha_{i,j} = f'_{i,j}(l_{i,j})\, l_{i,j}/f_{i,j}$, which is the ratio of marginal to average productivity evaluated at the labor level required to be sole producer. If we combine this bound with Lemma 4.1 we have

LEMMA 4.2. *If x is n.e. then $|u_1(x, Z') - u_1(x, Z(x))| < \delta M(Z^*)/1 - g$, where Z^* lies between $Z(x)$ and Z'.*

Linear programming enters with Lemma 4.3. As usual we assume $Z_1' > Z_C$.

LEMMA 4.3. *Let $B(Z')$ and $BL(Z')$ be the values of the upper and lower boundary curves for some Z'. Then for any intermediate value v, $BL(Z') \leqslant v \leqslant B(Z')$, there is a feasible solution x' to* (3.9a), *with at most two non-integer components, for which the value of* $\mathrm{Lu}_1(x, Z')$ *is v.*

Proof. Let us add to the maximization problem (3.9a) the linear constraint $Lu_1(x, Z') \leqslant v$. The problem now has two constraints, so the x that attains the maximum value, which is v, will have at most *two* variables that are neither 0 or 1; this is the x' of the lemma.

x' has two non-integer components $x'_{j,2}$ and $x'_{k,2}$. x' satisfies (3.4b) as an equality, so the integer point obtained by rounding up both $x'_{j,2}$ and $x'_{k,2}$ to 1 cannot satisfy i-(3.4b), while the x obtained by rounding them both down clearly does. *It follows that either x itself or one of the two x's obtained by rounding either the jth or kth component up has the n.e. property.* Therefore Lemma 4.1 applies to that x, as does Lemma 4.2. Consequently we have for this n.e. x

$$|Z'_1 - Z_1(x)| \leqslant \frac{\delta}{1-g} \qquad \text{and} \qquad |u_1(x, Z') - u_1(x, Z(x))| \leqslant \frac{\delta M}{1-g}.$$

The first inequality shows that this equilibrium x is near (Z', v) in its Z-coordinate, and the second will enable us to show that it is also near in its log utility value $u_1(x, Z(x))$ if we can bound the difference between $u_1(x, Z')$ and $v = u_1(x', Z')$. To do this we observe that the change in Lu_1 produced by changing the terms $x'_{j,2}$ and $x'_{k,2}$ cannot exceed $2\,\delta R$ where $R(Z) = \max_i |\ln q_{i,2}(1, Z(x))/q_{i,1}(1, Z(x))|$. Putting together these elements we have the regional fill in theorem:

THEOREM 4. *If (v, Z') is any point between $B(Z')$ and $B_L(Z')$, then, provided g is not 1, there is an integer equilibrium point x with $|Z_1(x) - Z'_1| \leqslant \delta/1-g$ and with $|v - u_1(x, Z(x))| \leqslant \delta(2R + M)/(1-g)$.*

APPENDIX 2

Proof of Theorem 5.1. The idea of the proof is to compare the utility at x, which for an intermediate x is not the same as the linearized utility, with the linearized utility of rounded versions of x. The key is the following lemma:

LEMMA 5.1. *Let x be an intermediate equilibrium point with associated national income $Z(x)$. Let $q_{i,1}(x_{i,1}, Z(x))$ and $q_{i,2}(x_{i,2}, Z(x))$ be the quantities of the ith good produced in the two countries. Then $q_{i,1}(x_{i,1}, Z(x)) + q_{i,2}(x_{i,2}, Z(x)) \leqslant \mathrm{Min}(q_{i,1}(1, Z(x)), q_{i,2}(1, Z(x)))$.*

This lemma states that *either* country, as the sole producer of good i, *at the demand and wage levels of the equilibrium point x*, would produce more of the ith good than the two countries produce together at the equilibrium point x. The lemma does *not* assert that more would be produced if one country were actually the sole producer. For if that were to happen, we would have a normalized national income different from $Z(x)$, with different wages and therefore possibly a different outcome. It also important to realize that the lemma does not assert that for any $0 \leqslant x_{i,j} \leqslant 1$ the inequality holds, but only for those $x_{i,1}$ *that are part of an equilibrium* x. Without that restriction the result is not true.

Proof. At the equilibrium x, we have for each i prices and wages such that $p_1 f_{i,1} = w_i l_{i,1}$ and $p_i f_{i,2} = w_i l_{i,2}$. If we form the ratio of these two expressions and use the relations $f_{i,1}(l_{i,1}) = q_{i,1}(x_{i,1}, Z(x))$ and $l_{i,1} = x_{i,1}(L_1/Z_1)(d_{i,1}Z_1 + d_{i,2}Z_2)$, together with similar relations for $f_{i,2}$ and $l_{i,2}$, we obtain $q_{i,1}(x_{i,1}, Z(x))/q_{i,2}(x_{i,2}, Z(x)) = x_{i,1}/x_{i,2}$ or equivalently $q_{i,1}(x_{i,1}, Z(x))/x_{i,1} = q_{i,2}(x_{i,2}, Z(x))/x_{i,2} = C$. Since $x_{i,1} + x_{i,2} = 1$, $q_{i,1}(x_{i,1}, Z(x)) + q_{i,2}(x_{i,2}, Z(x)) = C$.

Since the q's are the quantities produced and the $x_{i,j}$ are proportional to the amounts of labor at the fixed $Z = Z(x)$, the production economies of scale condition asserts that $q_{i,1}(x_{i,1}, Z(x))/x_{i,1}$ grows with $x_{i,1}$ so

$$q_{i,1}(1, Z(x))/1 \geqslant q_{i,1}(x_{i,1}, Z(x))/x_{i,1} = C = q_{i,1}(x_{i,1}, Z(x)) + q_{i,2}(x_{i,2}, Z(x)).$$

The reasoning for $q_{i,2}$ is the same, so this proves the lemma.

We now assume, as usual, that $Z_1(x)$ is above the classical level. Let x' be the integer eqilibrium point obtained from x by rounding down all the non-integer $x_{i,2}$ to 0. Since all the coefficients in the inequality are non-negative and x satisfies (3.4b), x' satisfies i-(3.4b) and therefore is a feasible solution to (3.9). We compare $u_1(x', Z(x))$ with the utility of x.

Since the maxium value of the objective function in (3.9) is $B_l(Z(x))$ we already have $B_l(Z(x)) \geqslant u_1(x', Z(x))$. If we can show that $u_1(x', Z(x))$ is $\geqslant u_1(x, Z(x))$, we would have $B_l(Z(x)) \geqslant u_1(x', Z(x)) \geqslant u_1(x, Z(x))$ which would prove the theorem. To compare the values of $u_1(x, Z(x))$ and $u_1(x', Z(x))$ we look at the individual terms in the two u_1 expressions. The terms are of the form $d_{i,1} \ln F_i(Z(x))\, Q_i(z, Z(x))$ where z is x in $u_1(x, Z(x))$ and z is x' in $u_1(x', Z(x))$. Whenever the components of x and x' are different, because of the rounding, the x' components are always 0 while the x components come from an intermediate equilibrium point. So the conditions of Lemma 5.1 are fulfilled and we always get an equal or larger Q_i from the rounded term. This shows that $u_1(x', Z(x))$ is $\geqslant u_1(x, Z(x))$, which establishes the theorem.

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