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Analysis of linear trade models and relation to scale economies

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ABSTRACT **We discuss linear Ricardo models with a range of parameters. We show that the exact boundary of the region of equilibria of these models is obtained by solving a simple integer programming problem. We show that there is also an exact correspondence between many of the equilibria resulting from families of linear models and the multiple equilibria of economies of scale models.**

In Gomory and Baumol (1), we discussed the equilibria that arise from the classical linear Ricardian model of international trade when the productivity parameters $e_{i,j}$ of the model are allowed to vary, limited only by a maximal productivity constraint, $e_{i,j} \le e_{i,j}^{\max}$. We plotted each equilibrium as a point in a (Z_1, U_1) diagram, where Z_1 is country 1's share of total world income, $Z_1 = Y_1/(Y_1 + Y_2)$, and U_1 is the utility of country 1 at that equilibrium. We had similar diagrams in which we plotted (Z_2, U_2), where $Z_2 = 1 - Z_1$ is country 2's share and U_2 is country 2's utility.

We showed that in each case the resulting region of equilibria (R_1 for country 1 and R_2 for country 2) could be bounded above by a curve $B_j(Z_1)$ obtained by solving a very simple linear programming problem for each value of Z_1. This upper bounding curve has a characteristic hill shape that persists over a wide range of models. Furthermore, as the number of industries in the model increases, we showed that the actual upper boundary of the region rapidly approaches this boundary curve.

The economic significance of these results comes from the characteristic hill shape of the region of equilibria. The hill shape implies that there is inherent conflict in international trade, that the best equilibria for one country are poor ones for the other, and that a country is better off with a partly developed trading partner than with a fully developed one. The fundamental mechanism at work is complementary to but different from the mechanisms employed in the analyses of international trade that also have shown the possibility of conflict in Hicks (2), Dornbush *et al.* (3), and Krugman (4). An excellent summary of the relevant history appears in Grossman and Helpman (5).

In this note we complete one component of this analysis by showing that the upper boundary of the region is given exactly by solving a closely related integer programming problem. The relation between the linear programming problem and the integer programming problem is that they are two different relaxations of the economies of scale problem introduced in Gomory (6). We discuss the close connection between the linear family and economies of scale models below.

This result enables us to examine models with a small number of products; models in which there is a considerable gap between the boundary given by the linear programming approximation and the actual boundary of the region of equilibria. This includes, for example, the famous model of trade in textiles and wine given by David Ricardo. These small models can and do turn out to have special characteristics, due to their small size, that disappear in all but the most contrived large models. These characteristics cause small models not to exhibit the inherent conflict that is present in almost all large models.

The Exact Boundary Theorem

As in Gomory (6) and Gomory and Baumol (7), we use $x_{i,j}$ for the market share of country j in the ith industry. We again use Z_C for the classical income-share value, which here is simply the income share of the countries when $e_{i,j} = e_{i,j}^{\max}$. We use the same linearized utility

$$Lu_1(x,Z,\varepsilon) = \sum_i \{x_{i,1}[d_{i,1}\ln F_{i,1}(Z)q_{i,1}(1,Z,\varepsilon)] + x_{i,2}[d_{i,1}\ln F_{i,1}(Z)q_{i,2}(1,Z,\varepsilon)]\}, \qquad [1]$$

which is the same as the Cobb–Douglas utility at every equilibrium, to measure utility. In Eq. **1**, ε is the vector $\{e_{i,j}\}$ of productivities per unit of (labor) input and $F_{i,j}(Z)$ is country j's consumption share of the ith good derived from a Cobb–Douglas utility, so $F_{i,j}(Z) = d_{i,j}Z_1/(d_{i,2}Z_1 + d_{i,2}Z_2)$. $q_{i,j}$ in Eq. **1** is the quantity of good i produced in country j when it is the sole producer of i, so that

$$q_{i,j}(l,Z,\varepsilon) = e_{i,j}l_{i,j} = (e_{i,j}/w_j)(d_{i,1}Z_1 + d_{i,2}Z_2) = (e_{i,j}L_j/Z_j)(d_{i,1}Z_1 + d_{i,2}Z_1). \qquad [2]$$

In Eq. **2**, w_j is the wage level in country j and L_j is country j's labor-force size. With this notation we assert the following theorem:

EXACT BOUNDARY THEOREM. *The upper boundary $B_{1,1}(Z)$ of the region of equilibria R_1 for $Z_1 \le Z_C$ is the solution of the maximization problem in integer x:*

$$B_{I,1}(Z_1) = Max_x\ Lu(x,Z_1,\varepsilon^{\max})$$

$$\sum_i x_{i,1}(d_{i,1}Z_1 + d_{i,2}Z_2) \le Z_1, \qquad x_{i,1} + x_{i,2} = 1, \qquad [3a]$$

while for $Z_1 \ge Z_C$ it is the solution of

$$B_{I,1}(Z_1) = Max_x\ Lu(x,Z_1,\varepsilon^{\max})$$

$$\sum_i x_{i,1}(d_{i,1}Z_1 + d_{i,2}Z_2) \ge Z_1, \qquad x_{i,1} + x_{i,2} = 1. \qquad [3b]$$

Note that the ε that appears in the linearized utility in the theorem is $\varepsilon^{\max} = \{e_{i,j}^{\max}\}$, the vector of maximal productivities.

Proof: Part 1. $B_{I,1}(Z_1)$ lies above any equilibrium. Consider any equilibrium (x, Z_1, ε) with market shares x, productivities ε, and income share $Z_1 \le Z_C$, so that Eq. **3a** applies. The market shares x satisfy the inequality in Eq. **3a** as an equality. From x construct an integer x' by setting $x'_{i,1} = 1$ and $x'_{i,2} = 0$ in the industries where country 1 is the sole producer and $x'_{i,1}$

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$= 0, x'_{i,2} = 1$ in all other industries. This new integer x' has less producing industries in country 1 ($x'_{i,1} \leq x_{i,1}$). So x' satisfies the inequality in Eq. **3a** and is, therefore, a feasible integer solution to Eq. **3a**. The objective function value for x', $Lu(x', Z_1, \varepsilon^{\max})$, is also greater than or equal to the utility $Lu(x, Z_1, \varepsilon)$ of the original equilibrium. This can be seen by a term-by-term (industry-by-industry) comparison of $Lu(x', Z_1, \varepsilon^{\max})$ with $Lu(x, Z_1, \varepsilon)$. If the ith industry had only one producer in x, for example $x_{i,1} = 1$, it still has only one producer in x', so the only change is the replacement of $e_{i,1}$ by $e_{i,j}^{\max}$, which only increases utility. If, however, the ith industry was shared in x, it now has $x'_{i,2} = 1$ in x'. But for shared industries both countries produce at the same cost so $e_{i,1}/w_1 = e_{i,2}/w_2$ and we can see from Eq. **2** that $q_{i,1} = q_{i,2}$. Therefore, the shift in x does not by itself affect the linearized utility. We then argue, as we just have done for the specialized industries, that the coefficient of $x'_{i,2}$ is greater than the coefficient of $x_{i,2}$ and this only increases utility. So $Lu(x', Z_1, \varepsilon^{\max}) \geq Lu(x, Z_1, \varepsilon)$. Because for any equilibrium (x, Z_1, ε) we can construct the corresponding integer x', $B_{I,1}(Z_1)$, the maximum over integer solutions must be as large as the utility of any equilibrium with share Z_1.

Proof: Part 2. There is an equilibrium with the same linearized utility as $B(Z)$. Let x be the maximizing solution to the integer programming problem for that same Z_1. Since $Z_1 < Z_C$, we know from Gomory (6) or Gomory and Baumol (1) that with market shares x satisfying the inequality in Eq. **3a**, there is an industry in which country 1 can be the cheaper producer, but in which it is not producing. In symbols, there is an i for which $e_{i,1}^{\max}/w_1 > e_{i,2}^{\max}/w_2$, but $x_{i,1} = 0$. If we increase this $x_{i,1}$, the objective function increases. If we increase $x_{i,1}$ to 1, we have a new integer x with a larger value of the objective function. Since x was already the maximizing integer solution satisfying Eq. **3a**, we must conclude that increasing $x_{i,1}$ to 1 must have violated the inequality in Eq. **3a**. Therefore, there is a value of $x_{i,1} < 1$ that produces equality in Eq. **3a**. We adopt this value of $x_{i,1}$ thus forming a new vector x' with one noninteger component. We next choose a new smaller value for $e_{i,1}$ that makes both countries equally cheap producers in industry i; that is, we choose $e_{i,1}$ so that $e_{i,1}/w_1 = e_{i,2}^{\max}/w_2$. With this new $e_{i,1}$, no change in utility results in going from x to x'. But x' now has share Z_1 and satisfies the conditions for an equilibrium. Thus we have produced an equilibrium with share Z_1 and with a linearized utility value as great as the maximizing integer solution. Parts 1 and 2 together prove the theorem for equilibria with $Z_1 \leq Z_C$; the proof for $Z_1 \geq Z_C$ is almost identical.

Using the Exact Boundary Theorem

We can solve Eqs. **3a** and **3b** by using standard integer programming techniques. Our model is in fact the simplest of all integer programming problems, the knapsack problem. We have solved a series of small problems by using dynamic programming to obtain the data in the figures below. Although our techniques allow us to catalogue all small models, we will not do that, but rather we will watch the evolution of one small model as the number of industries increases.

Fig. 1 shows the results for Ricardo's classical textile-wine example. There are only two industries. Country 1 (England) excels in one of them (textiles), so that $e_{1,1}^{\max} = 1$ whereas $e_{1,2}^{\max} = 0.55$, and the other country (Portugal) excels in the other industry (wine), so that $e_{2,2}^{\max} = 1$ and $e_{2,1}^{\max} = 0.45$. All equilibria with $e_{i,j} \leq e_{i,j}^{\max}$ are plotted. In Fig. 1*A* we have plotted world output as measured by country 1's utility function. In Fig. 1*B* we show country 1's utility, and in Fig. 1*C* we show country 2's utility measured in country 2 utility units. The boundary in Fig. 1*B* is simply the boundary in Fig. 1*A* multiplied by country 1's share of world income, Z_1. The boundary in Fig. 1*C* would be world output utility multiplied by country 2's share if the world output were measured by country 2's utility. There is a sharp peak in world output above the classical level Z_C. This is where

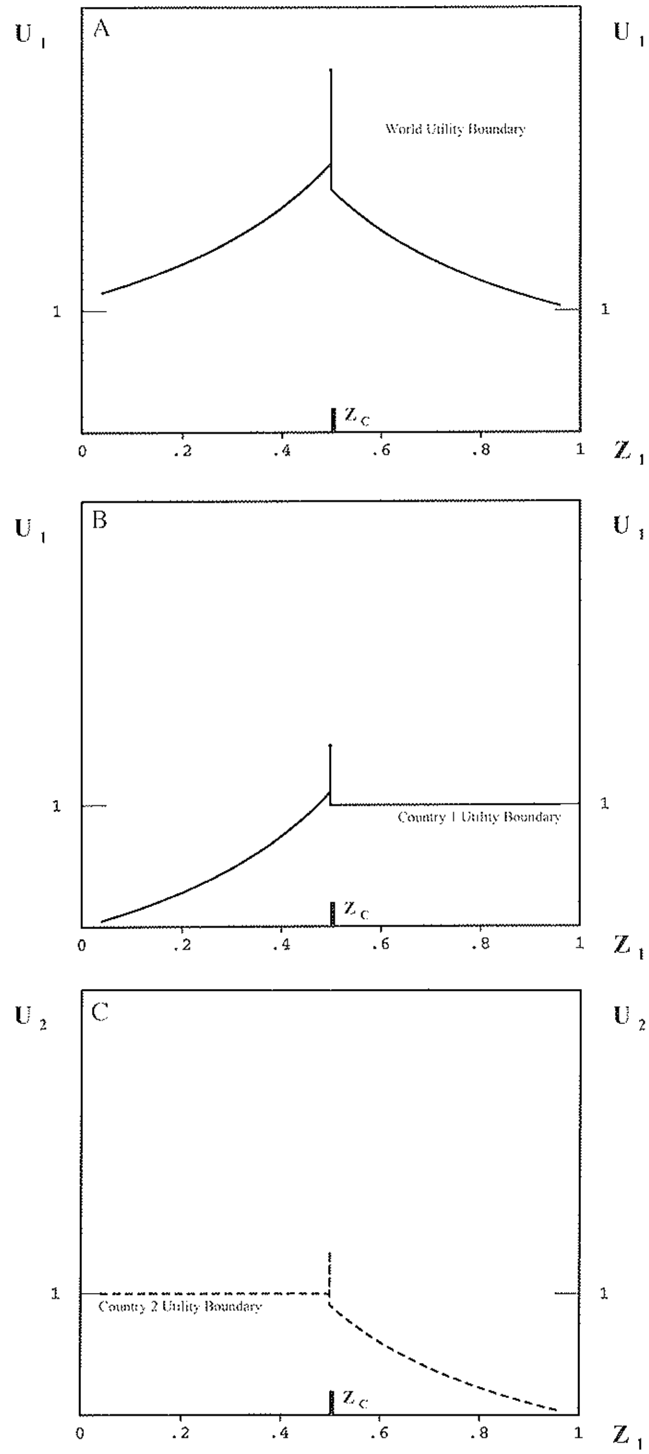

FIG. 1. (*A*) Two industries, world utility. (*B*) Two industries, country 1 utility. (*C*) Two industries, country 2 utility.

England specializes in textiles and Portugal specializes in wine, and both have attained maximal productivity. This peak is high enough that the best outcome for each country is attained there, as Fig. 1 *B* and *C* shows. So in the two-product case, the classical specialized outcome is the best possible result for both countries. But this is far from typical.

In the three-industry model shown in Fig. 2, country 1 is better in two industries with combined demand somewhat less than the demand for the one industry in which country 2 is best. We show world utility in Fig. 2*A*, together with country 1's utility, and the utility of country 2 in Fig. 2*B*. Although the

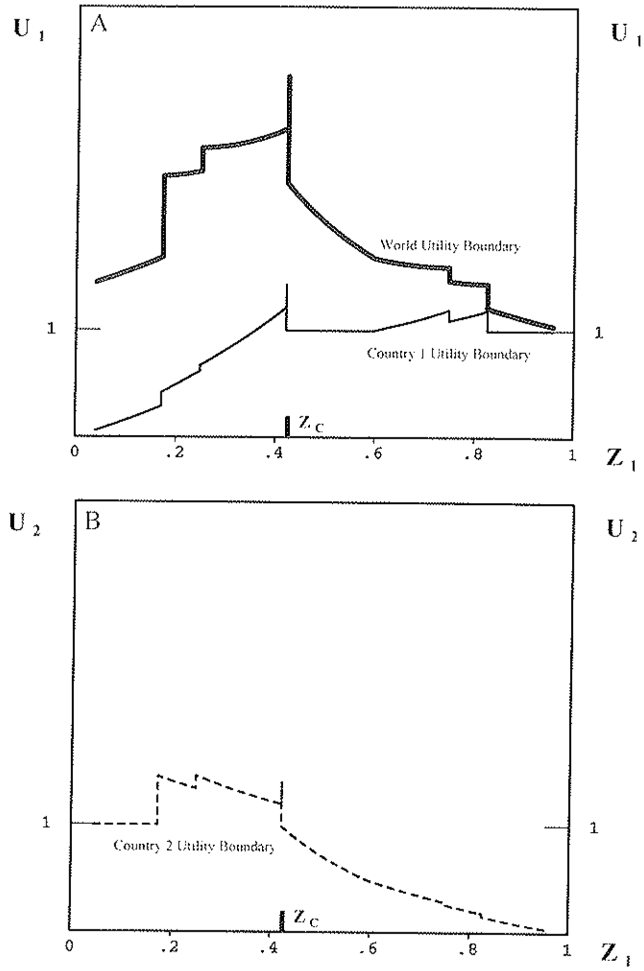

FIG. 2. (*A*) Three industries, world and country 1 utility. (*B*) Three industries, country 2 utility.

classical level is still good for both countries, there are points to the left in Fig. 2*B* that are about as good for country 2.

In Fig. 3, a four-industry model, country 1 is better in two industries and country 2 is better in two industries. We have combined all the utilities in a single figure. Country 1 utility is read from the right vertical axis and country 2 utility is read from the left. Already the best equilibrium is clearly different for the two countries.

In Fig. 4, a six-industry model, we have added the smooth

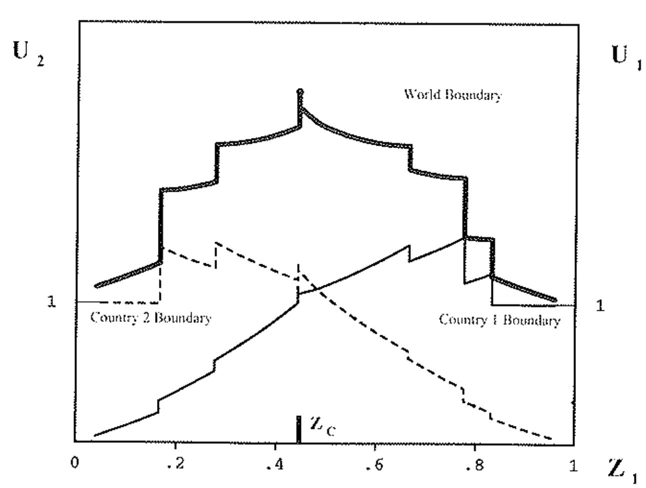

FIG. 3. Four industries, all utilities.

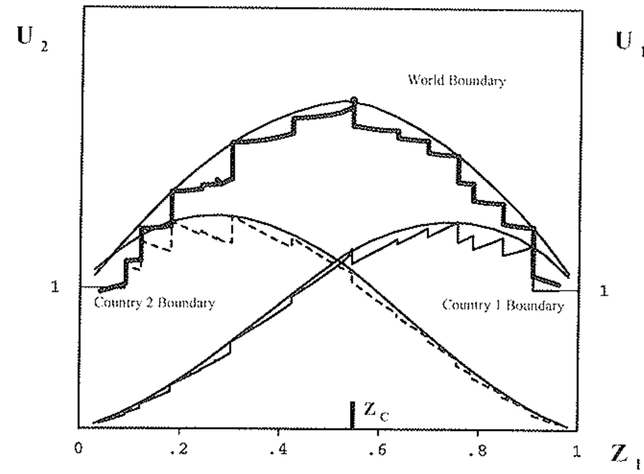

FIG. 4. Six industries, all utilities and linear programming boundaries.

linear (not integer) programming boundaries that we referred to in the introduction. Note that they are already close to the exact boundaries and that the equilibria best for the two countries are clearly different from one another. As in Gomory and Baumol (1), the equilibrium that is best for country 1 is rather poor for country 2 and vice versa. We have returned to the inherent conflict that in Gomory (6) characterizes models with large n. This is avoided only in the models with the very smallest number of products.

The Connection with Scale Economies Models

All of this, together with our earlier work, point to a close connection between families of linear models and models with economies of scale. We have explored that connection in Gomory and Baumol (7) and summarize it herein as follows:

The Correspondence Principle. We say that a scale-economies model $M(f_{i,j})$ corresponds to a linear family model if it has the same labor-force sizes L_1 and L_2 and the same country demand values $d_{i,j}$. However, instead of linear production functions $e_{i,j}l_{i,j}$, the model $M(f_{i,j})$ has production functions $f_{i,j}(l)$ with economies of scale, defined as nondecreasing average productivity, $f_{i,j}(l)/l$. We assume that there is a well-defined derivative $df_{i,j}(l)/dl$ at $l = 0$ and that $f_{i,j}(L_j)/L_j$, which is the largest productivity value that $f_{i,j}(l)/l$ can attain in the model, is $e_{i,j}^{\max}$.

THEOREM 5.1 (CORRESPONDENCE THEOREM). *From any specialized equilibrium* (x, Z_1) *of the scale-economies model, we can construct a corresponding equilibrium* (x, Z_1, ε) *of the linear family having the same* x *and* Z_1 *and an* ε *given by:* (i) *the* $e_{i,j}$ *for producers is average productivity at the economies equilibrium, so* $e_{i,j} = f_{i,j}(l_{i,j})/l_{i,j}$, *and* (ii) *the* $e_{i,j}$ *for nonproducers is marginal productivity at output zero, so that* $e_{i,j} = df_{i,j}(0)/dl_{i,j}$.

Many Corresponding Equilibria. If the economies model has many equilibria, each will clearly correspond to a different equilibrium (x, Z_1, ε) of the family of linear models. One economies model is, therefore, a way of looking at a large sample of the equilibria of a family of linear models. Fig. 5 shows the (linear programming) boundary for country 1's region of equilibria from a linear family model, together with the equilibrium points corresponding to one rather small economies model. The set of economies equilibria display the characteristic shape described in Gomory (6).

The location of the equilibria corresponding to $M(f_{i,j})$ in the region of equilibria of the linear family depends on the nature of the scale economies. If the production functions $f_{i,j}(l)$ have productivities $f_{i,j}(l)/l$ that go on increasing until $l = L_j$, the corresponding equilibria tend to be low in the region of