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Linear models of economic survival under production uncertainty*

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Summary. In this paper we consider the situation of an investor facing uncertainty, and whose objective is to survive. First we characterize the probability of survival of a passive agent who does not attempt to influence the evolution of the environment. Secondly, we look at an active agent who chooses investment opportunities affecting his fortune and who attempts to maximize the probability of survival. It is shown that, in some cases, the optimal investment policy will exhibit a risk-loving behavior whenever his fortune is below a critical level.

I. Introduction

Consider the situation of an investor facing uncertainty, and whose sole object is to survive. The basic constraint is that the investor must pay out – or alternatively consume – a constant sum per period of time. Any surplus capital stock is allocated among investment opportunities each of which yields a stochastic rate of return. If the investor's capital reaches zero at some time then failure (or, ruin) occurs at that time. We say that the investor *survives* if failure never occurs, and we suppose that the investor is interested in maximizing the probability of survival.

In the present paper we address several questions that arise concerning the situation we have just described. In particular, in the context of a continuous-time (diffusion) model, we are able to characterize the optimal policy of the investor in a relatively simple fashion. A striking feature of this optimal policy is that when the investor's capital is below a critical level he uses investments that are "inefficient" in the mean-variance sense, namely, there are other investments that have the same mean but a lower variance. Another interpretation is

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that the investor exhibits "risk-loving" or "risk-averse" behavior according as his capital is below or above some critical level.

It has been recognized that modelling economic decisions contingent on the survival of economic agents raises awkward problems, particularly in applying any "equilibrium" or "steady state" concepts to problems of intertemporal allocation. "Most authors have ignored the analytical difficulty of formulating a model that ensures the possibility of survival, blithely admitting any non-negative rates of consumption as sustainable," (Koopmans [10, p. 59]), and have not addressed the question as to how a temporary equilibrium or a steady state can be maintained if some agents face extinction in the very first period.

While it is easy to indicate the obvious limitations of a dynamic model that fails to come to grips with survival, it is difficult to make a move towards any "general" theory: the economic and ethical implications of "extinction" or "failure" appear to be quite specific to the role of the agent in the economy. It is hard to contemplate a framework that can adequately deal simultaneously with, for example, consumers facing extinction during a catastrophic famine, firms facing extinction in a Schumpeterian world, and banks facing "runs" in a period of liquidity or confidence crisis. In this paper, we do not pretend to provide a model that can deal with all such questions. Our point of departure is provided by simple dynamic models that explicitly treat an economic agent who is trying to meet a "performance" or "consumption target", and we attempt to identify some problems that arise naturally when such an agent has to act in an uncertain world. We focus on models in which the stochastic return or production function is *linear*, and consider two situations: first, we look at an agent who is "passive" and does not attempt to influence the stochastic law of evolution of the environment affecting the consequences of his investment decisions. Relevant questions in this case include a precise characterization of the probability of survival and the parameters (initial fortune, productivity coefficients, consumption target) that influence this probability. Secondly, we look at an "active" agent who can choose investment opportunities that affect his fortune, and who attempts to maximize the probability of survival. Our task is to characterize his optimal investment policy, and to contrast the implications of maximizing the chance of survival with those of expected utility maximization.

In application of the formal model, one can reinterpret the "consumption target" of the agent as a fixed periodic payment that an indebted person or country must make in order to avoid default. Another interpretation occurs naturally in certain models of a repeated principal-agent game (Radner [18]).

The paper is organized as follows: In Section II we introduce a discrete-time model motivating the analysis of a passive agent. An agent starts with an initial fortune $y > 0$ from which a positive constant $c > 0$ is subtracted. One can interpret c as the minimal amount of consumption needed to participate in economic activities. If $y - c$ is non-positive, the agent is "ruined"; if $y - c$ is positive, it is used as an input to generate the fortune in the next period according to the function $y_1 = (\exp R_1)(y - c)$, where R_1 is a random variable. The agent is, once again, required to consume c in the next period: if $y_1 - c$ is non-positive, he is ruined; if $y_1 - c$ is positive, the fortune becomes $y_2 = (\exp R_2)(y_1 - c)$ and the story is repeated. The agent is ruined in the first (random) period T (if any) at which his fortune is no more than c. If T is infinite, the agent *survives* (forever).

We begin by assuming that the sequence (R_t) is independent and identically distributed. Let $S_0 \equiv 0, S_t \equiv \sum_{\tau=1}^{t} R_\tau$, and $M_t \equiv \sum_{k=1}^{t} \exp(-S_k)$. If $P(y, c)$ is the probability of survival with initial fortune y, and target consumption c, then, as we shall show,

$$P(y, c) \equiv \text{Prob}\,[M \equiv \lim M_t \leq y/c] \ . \tag{1.1}$$

Thus, the probability of survival depends on y and c through their ratio. This formula is quite useful in identifying some cases in which ruin is inevitable, and others in which survival is possible with positive probability. Sections III and IV extend this model in two different directions. First, in Sect. III, we relax the assumption that (R_t) is a sequence of independent, identically distributed random variables. Instead, th environment is represented by a stochastic process (R_t) with the property that the absolute value of the conditional exception of R_t given the past is uniformly bounded. We show that if all the conditional expectations are non-positive, then $M \equiv \lim M_t$ is infinite almost surely; hence, by appealing to (1.1) which continues to hold, we conclude the inevitability of ruin. On the other hand, if these conditional expectations are bounded below by a positive constant, then M is almost surely finite, and the probability of survival approaches 1 as (y/c) increases without bound. In Sect. IV, more explicit calculation of survival probabilities is facilitated by considering a diffusion model, which in a precise sense is a limit of discrete time linear models. Thus, let h denote the length of time between dates, starting at date 0, and $t = nh$ where n is a positive integer. The agent's capital at date $t+h$ is

$$Y(t+h) = \exp(R_{(n+1)h})[Y(t) - ch] \tag{1.2}$$

Here, in (1.2), we take (R_{nh}) to be a seqence of independent, identically distributed Gaussian random variables each with mean mh and variance vh, and interpret c as a constant rate of consumption over the time interval $[t, t+h]$. Now, as h decreases to zero, the process (1.2) can be approximated by a diffusion process with a drift ("infinitesimal mean") $\mu(y)$ and variance ("diffusion coefficient") $\sigma^2(y)$ defined by

$$\mu(y) \equiv (m + v/2)y - c, \sigma^2(y) \equiv vy^2 \ . \tag{1.3}$$

For this diffusion, the probability $U(y)$ of survival, given $Y(0) = y$, can be computed as

$$U(y) \equiv K \int_{b/y}^{\infty} x^{a-1} e^{-x} dx \tag{1.4}$$

where K is an appropriate constant, and the parameters a and b are defined as $a \equiv 2m/v, b \equiv 2c/v$. Section IV contains an informal account of the basic steps involved in such a computation and provide some additional results on the rate of change of survival probabilities with respect to variations in the initial fortune (see (4.22) and (4.23)).

In Sect. V we develop a model of an active agent who is maximizing the probability of survival by choosing, at each instant of time, an appropriate investment opportunity, which is represented by a diffusion process, thus "controlling" the evolution of his fortune. We suppose that, depending on the fortune y at any instant of time, the agent can choose a pair (m, v), which defines the

infinitesimal mean and variance of a diffusion process according to the formula (1.3). The 'act' represented by (m, v) must be chosen from a set A (in $Rx(0, \infty)$) satisfying appropriate compactness, convexity, and smoothness properties (see Assumption 5.1). For any $v > 0$, let $f(v) = \text{Max}\,\{m : (m, v) \in A\}$. We assume that f is twice differentiable, strictly concave and f' satisfies some "boundary" conditions (sec 5.4). We derive an optimal policy as follows: let $z \equiv \log y$; and $v = \beta(z)$ be the unique solution of

$$f(v) - vf'(v) = ce^{-z} \ . \tag{1.5}$$

The pair $(m, v) \equiv (f(\beta(z)), \beta(z))$gives us the optimal choice at z. This function $\beta(z)$ is decreasing in z, and below a critical level ($y^* = c/m$), the agent displays a 'risk loving' behavior. We also indicate how the convexity and smoothness assumptions can be relaxed. Our exposition in Sect. IV and V is somewhat informal and heuristic. It does not presuppose expertise in diffusion processes. In the appendix we outline a formal rigorous approach to the survival problem of Sect. IV treated as a controlled diffusion. In Sect. VI we provide some (admittedly incomplete) bibliographical notes on several areas that are obviously related to the basic issues.

II. Survival under production uncertainty: a discrete-time model

Consider an economic agent with an initial fortune of $y > 0$. He is required to consume $c > 0$; if $y - c$ is non-positive, he is "ruined". If, however, $y - c$ is positive, his fortune in the text period is given by

$$y_1 = (\exp R_1)(y_1 - c) \ , \tag{2.1}$$

where R_1 is a random variable. If $y_1 - c$ is non-positive, he is ruined (in period one); otherwise his fortune in the subsequent period is

$$y_2 = (\exp R_2)(y_1 - c) \tag{2.2}$$

and the story is repeated. In general, let (R_t) be a sequence of independent, identically distributed random variables. Let T (a random variable) be the first period (if any) such that $y_T - c$ is non-positive; if T is finite we say that the agent is *ruined in period T*. If T is infinite, we say that the agent *survives* (forever). Let $P(y, c)$ be the probability of survival with initial fortune $y > 0$ and consumption target $c > 0$. Write

$$\begin{aligned} S_0 &\equiv 0 \\ S_t &\equiv R_1 + \ldots + R_t \\ M_t &= \exp(-S_0) + \quad + \exp(-S_t) \end{aligned} \tag{2.3}$$

Observe that M_t is an increasing sequence. Hence, M_t either converges to a finite limit or diverges to plus infinity. We can, therefore, define

$$M \equiv \lim_{t\uparrow\infty} M_t \tag{2.4}$$

with the understanding that the random variable M can be (plus) infinity. Of particular importance is the following:

Lemma 2.1.

$$P(y,c)\equiv \text{Prob}\,[M\leqq y/c] \tag{2.5}$$

Proof. The beginning of the period stocks, Y_t, evolve according to the equations

$$\begin{aligned} &Y_0=y\\ &Y_t=\exp(R_t)(Y_{t-1}-c),\, t\geqq 1 \end{aligned} \tag{2.6}$$

The solution to (2.6) is

$$Y_t=\exp(S_t)(y-cM_{t-1})\ . \tag{2.7}$$

Hence, $Y_t > c$ if and only if

$$y-cM_{t-1} > c(\exp(-S_t))$$

$$\text{or,}\quad y > c(M_{t-1}+\exp(S_t))=cM_t \tag{2.8}$$

The assertion (2.5) follows. Q.E.D.

Note that the survival probability depends on the *ratio* y/c. This lemma can be used to identify some situations where $P(y,c)=0$ for *all* y and c. If $\rho\equiv ER_t\leqq 0$, one can show that $P(y,c)=0$. On the other hand, if $\rho>0$, $P(y,c)$ approaches one as the *ratio* y/c approaches infinity. (See Sect. III). However, explicit calculation of $P(y,c)$ may be difficult, since the right side of (2.5) involves the limit variable M.

III. A general linear model

We now dispense with the assumption that the environment is represented by a sequence (R_t) of independent, identically distributed random variables. We deal with a much broader class of environments and establish two general results on the possibility of survival and inevitability of ruin.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $(\mathcal{F}_t)$ be a monotone non-decreasing sequence of sigma fields $(\mathcal{F}_t\subset\mathcal{F})$. Let (R_t) be any adapted sequence of real valued random variables. Assume that there is a positive α such that

$$\sup_t |R_t|\leqq\alpha \quad \text{a.s.} \tag{3.1}$$

Define

$$\begin{aligned} &\rho_1\equiv E(R_1)\\ &\rho_t\equiv E(R_t|\mathcal{F}_{t-1})\ ,\quad t\geqq 2 \end{aligned} \tag{3.2}$$

Observe, first, that Lemma 2.1 continues to hold in the general framework. Hence, in order to throw light on survival possibilities, we have to characterize the behavior of the limiting random variable M appearing on the right side of (2.5). The following proposition deals with two interesting cases.

Proposition 3.1.

(1) *If* $\rho_t\leqq 0$ *a.s. for all* t, *then* $M=\infty$ *a.s.* (3.3)

(2) *If for some* $\rho>0$, $\rho_t\geqq\rho$ *a.s. for all* t, *then* $M<\infty$ *a.s.* (3.4)

Before proving the Proposition, we note that to obtain a.s. finiteness of M [cf. (3.4)], it is not sufficient to require that $\rho_t>0$ a.s. for every t. To see this, consider the case in which R_t is not stochastic,

$$R_t=\rho_t>0\ ,\quad \text{and}$$

$$\sum_{t=1}^{\infty}\rho_t\equiv s<\infty\ ;$$

then $\lim S_t=s$, and $M=\infty$.

Proof of Proposition 3.1.

For any positive integer N let $T(N)$ be the first time t (if any) such that $S_t\leqq -N$. Let $S_t(N)$ denote the "stopped process", i.e.,

$$S_t(N)=\begin{cases} S_t\ , & t\leqq T(N)\ ,\\ S_{T(N)}\ , & t\geqq T(N)\ . \end{cases} \tag{3.5}$$

Since $\{S_t\}$ is a supermartingale, so is $\{S_t(N)\}$ (Chung, [2, Corollary to Theorem 9.3.4, p. 325]). Since the random variables (R_t) are uniformly bounded (cf. (3.1)),

$$S_t(N)\geqq -N-\alpha\ ;$$

hence $S_t(N)$ converges a.s. to a finite limit, say $\bar{S}(N)$; see Chung [2, Corollary to Theorem 9.4.4, p. 335].

To derive (3.3), we distinguish two cases:

Case 1. $\lim S_t=-\infty$. Then $M=\infty$.

Case 2. $\lim S_t>-\infty$. Then for all sufficiently large N, $S_k(N)=S_k$ for all k, so $\lim S_k$ exists and is finite, and $M=\infty$.

Now we derive (3.4). It follows from Levy's Strong Law of Large Numbers for dependent random variables (see Freedman [5, p. 912]), that

$$\lim_{t\to\infty}\frac{S_t}{t}\geqq\rho,\quad \text{a.s.}$$

or,

$$\overline{\lim_{t\to\infty}}\,(e^{-S_t})^{1/t}\leqq e^{-\rho},\quad \text{a.s.}$$

Hence, applying the Cauchy Criterion, we conclude that M is finite a.s. Q.E.D.

IV. A diffusion process with positive probability of survival

We start with a discrete-time model and then describe the corresponding diffusion model that arises in the limite as the time between successive returns and payouts approaches zero, i.e., as the investment returns and the payouts (consumption) occur in continuous time. We provide an informal derivation of the basic differential equation characterizing the survival probabilities of the limiting diffusion (see (4.13)) and indicate how an explicit formula for calculating the probability of survival can be derived (see (4.28)).

In the discrete-time model, let h denote the length of time between dates, starting at date 0. Let t be an integral multiple of h, say nh, and let $Y(t)$ denote the agent's capital at date t; then the agents capital date $(t+h)$ is

$$Y(t+h) = \exp(R_{(n+1)h})[Y(t) - ch] , \quad (4.1)$$

where c is a positive constant, and $\{R_{nh}\}$ is a sequence of independent and identically distributed Gaussian random variables, each with mean mh and variance vh.

The sequence $\{Y(nh)\}$ is a Markov chain, and we are interested in the probability that $Y(nh)$ never becomes zero or negative, given $Y(0) = y > 0$. We are able to provide an explicit expression for this probability in the limiting case in which h tends to be zero, and the process $\{Y(nh)\}$ tends to a diffusion process (see, e.g., Strook and Varadhan [25, Theorem 11.2 on p. 272]). To this end it will be more convenient to deal with the process

$$Z(t) \equiv \ln Y(t) . \quad (4.2)$$

More precisely, we define T to be the first t such that $Y(t) \leq 0$, and

$$Z(t) = \begin{cases} \ln Y(t) , & t < T , \\ -\infty , & t \geq T . \end{cases} \quad (4.2')$$

If the agent survives, then $T = +\infty$ and $Z(t)$ is always finite.

Since the process is stationary Markovian, it suffices to consider the conditional distribution of $Z(h)$ given $Z(0) = z \equiv \ln y$. (Note that, since $y > 0, z$ is well defined.)

For a given y, the smaller h the smaller is the probability that $Y(h) \leq 0$, i.e., that $Z(h) = -\infty$. In the following heuristic argument we shall suppose that h is so small relative to y that the probability that $Y(h) \leq 0$ is "negligible".

From (4.1) and (4.2),

$$\begin{aligned} Z(h) &= R + \ln(y - ch) \\ &= R + \ln(e^z - ch) \\ &= R + z + \ln(1 - hce^{-z}) . \end{aligned}$$

Hence,

$$Z(h) - z = R + \ln(1 - hC) \quad (4.3)$$

$$\text{where } C \equiv ce^{-z} ; \quad (4.4)$$

observe that C depends on z.

It will be useful to have expressions for the conditional mean and variance of $[Z(h) - z]$, given $Z(0) = z$. (In what follows, all expectations are to be understood as conditional on $Z(0) = z$). Expanding $\ln(1 - hC)$ in powers of h, we get from (4.3),

$$Z(h) - z = R - hC - \frac{h^2 C^2}{2} + 0(h^3) .$$

Hence

$$E[Z(h) - z] = (m - C)h + 0(h^2) , \quad (4.5)$$

$$\operatorname{Var}[Z(h) - z] = vh , \quad (4.6)$$

$$E[Z(h) - z]^2 = vh + 0(h^2) , \quad (4.7)$$

$$E[Z(h) - z]^3 = (E[Z(h) - z])^3 + 3E[Z(h) - z]\operatorname{Var}[Z(h) - z] = 0(h^2) . \quad (4.8)$$

Define, for $y \geq 0$,

$$U(y) \equiv \operatorname{Prob}(\text{Survival} \mid Y(0) = y) ,$$

$$V(z) \equiv U(e^z) ,$$

where $U(0) = V(-\infty) = 0$. Because the process is stationary Markovian, by invoking "the law of total probability" (see, e.g. Karlin and Taylor [9, pp. 169–173]), we get:

$$V(z) = EV[Z(h)] , \quad (4.9)$$

where, as before, the expectation is conditional on $Z(0) = z > 0$. Supposing V is sufficienctly smooth, we expand $V[Z(h)]$ in a Taylor's series:

$$\begin{aligned} V[Z(h)] = {} & V(z) + V'(z)[Z(h) - z] + \tfrac{1}{2} V''(z)[Z(h) - z]^2 \\ & + 0([Z(h) - z]^3) . \end{aligned} \quad (4.10)$$

Taking the expectation of both sides of (4.10), we have from (4.5)-(4.8),

$$EV[Z(h)] = V(z) + V'(z)(m - C)h + \tfrac{1}{2} V''(z)vh + 0(h^2) , \quad (4.11)$$

and so from (4.9),

$$V(z) = V(z) + V'(z)(m - C)h + \tfrac{1}{2} V''(z)vh + 0(h^2) ,$$

$$0 = V'(z)(m - C)h + \tfrac{1}{2} V''(z)vh + \tfrac{1}{2}0(h^2) . \quad (4.12)$$

Letting h tend to zero in (4.12) we get the differential equation

$$(m - ce^{-z})V'(z) + \frac{v}{2} V''(z) = 0 . \quad (4.13)$$

The general solution of this differential equation for V' is:

$$V'(z) = He^{-(az + be^{-z})} , \quad (4.14)$$

where

$$a \equiv \frac{2m}{v} , \quad b \equiv \frac{2c}{v} , \quad (4.15)$$

and H is an arbitrary positive constant whose particular value will be determined by the boundary conditions in the problem.

Before solving for V, we list some properties of V'. First, one can show that

$$V'(z) > 0 . \quad (4.16)$$

Second, *$V'(z)$ is decreasing if and only if*

$$k(z) \equiv az + be^{-z} \quad (4.17)$$

is increasing, or

$$k'(z) = a - be^{-z} > 0 \ , \tag{4.18}$$

which from (4.15) is equivalent to

$$z > \ln\left(\frac{c}{m}\right) \ , \tag{4.19}$$

or,

$$y > \frac{c}{m} \ . \tag{4.20}$$

Third, it is easy to verify that

$$\lim_{z \to -\infty} V'(z) = \lim_{z \to +\infty} V'(z) = 0 \ . \tag{4.21}$$

In summary, $V'(z)$ is strictly positive, and increases monotonically from 0 as z increases from $-\infty$ to $\ln(c/m)$, and then decreases monotonically towards zero as z increases beyond $\ln(c/m)$. It follows that

$$V''(z) \lesseqgtr 0 \quad \text{as} \quad z \lesseqgtr \ln\left(\frac{c}{m}\right) \ . \tag{4.22}$$

We turn now to the study of the function U. Recall that

$$U(y) = V(\ln y) \ ;$$

hence,

$$U'(y) = \left(\frac{1}{y}\right) V'(\ln y) \ ,$$

so that, from (4.15),

$$\begin{aligned} U'(y) &= \left(\frac{H}{y}\right) \exp\{-a(\ln y) - be^{-\ln y}\} \\ &= Hy^{-(a+1)} e^{-(b/y)} \ . \end{aligned} \tag{4.23}$$

Clearly $U(0) = 0$, so

$$U(y) = \int_0^y U'(s)ds \ . \tag{4.24}$$

Making the change of variable $x = b/s$ in (4.24) one can verify that

$$U(y) = Hb^{-a} \int_{b/y}^{\infty} x^{a-1} e^{-x} dx \ . \tag{4.25}$$

By an independent argument, one can show that

$$\lim_{y \to \infty} U(y) = 1 \ . \tag{4.26}$$

This is the boundary condition that will determine H. Since

$$\int_0^{\infty} x^{a-1} e^{-x} dx = \Gamma(a)$$

(recall that $a > 0$), it follows from (4.25) and (4.26) that

$$H = \frac{b^a}{\Gamma(a)} \ , \tag{4.27}$$

and hence that

$$U(y) = \frac{1}{\Gamma(a)} \int_{b/y}^{\infty} x^{a-1} e^{-x} dx \ . \tag{4.28}$$

Equation (4.28) can also be written as

$$U(y) = 1 - \left[\frac{1}{\Gamma(a)}\right] \int_0^{b/y} x^{a-1} e^{-x} dx \ . \tag{4.29}$$

The integral in the right-hand side of (4.29) is of course the incomplete gamma function. Thus (4.29) – or (4.28) – completely solves the problem of determining the probability of survival, starting from an initial capital y.

V. The optimal investment policy when the probability of survival is positive: a model of an active agent

We now derive turn to an "active" agent who maximizes the probability of survival. Consider the agent who can choose, at each instant of time, a pair (m, v) depending on the fortune y. This choice ("act") results in an investment opportunity described by a diffusion process with drift $\mu(y)$ and variance $\sigma^2(y)$ given by

$$\begin{aligned} \mu(y) &= (m + v/2)y - c \\ \sigma^2(y) &= vy^2 \end{aligned} \tag{5.1}$$

The agent's objective is to choose (m, v) so as to *maximize the probability of survival* (or minimize the probability that his fortune y hits 0, an absorbing state). We derive an optimal policy and characterize its qualitative properties.

First, we introduce some assumptions on A, the set that specifies all feasible acts.

Assumption 5.1. A is compact.

$$\text{(i)} \ \ 0 < v' \equiv \min\{v : (m, v) \in A\} < v'' \equiv \max\{v : (m, v) \in A\} \ . \tag{5.2}$$

For any v in $[v', v'']$ define the real valued function f as:

$$f(v) = \max\{m : (m, v) \in A\} \ . \tag{5.3}$$

Assumption 5.2. The function f is positive, twice continuously differentiable and strictly concave on (v', v''). Moreover

$$\lim_{v \downarrow v'} f'(v) = \infty \ , \quad \lim_{v \uparrow v''} f'(v) = -\infty \tag{5.4}$$

In our description of optimal choice, two values of v are of particular importance. Denote by v^* the unique point where f attains its maximum on $[v', v'']$. It follows that $f'(v^*) = 0$. Let v^0 be the unique point such that $(f(v^0), v^0)$ maximizes

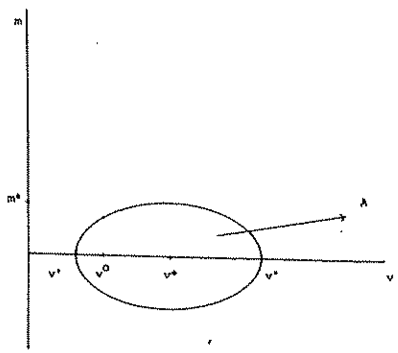

Fig. 1

$f(v)/v$ on A. It follows that $f'(v^0)=v^0 f(v^0)$. The points $v' < v^0 < v^* < v''$ are indicated in Fig. 1.

In order to describe the nature of optimal policy, we use the following Lemma. As in Sect. III, we let $z \equiv \ln y, y>0$.

Lemma 5.1. *For any* $z=\ln y, (y>0)$, *the equation*

$$f(v)-vf'(v)=ce^{-z} \tag{5.5}$$

has a unique solution $v=\beta(z)$. *The function* β *is strictly decreasing and*

$$\begin{aligned} &\lim_{z\downarrow -\infty} \beta(z)=v'' \ . \\ &\lim_{z\uparrow +\infty} \beta(z)=v^0 \ . \end{aligned} \tag{5.6}$$

Proof. Write $g(v)\equiv f(v)-vf'(v)$. Note that $g'(v)=-vf''(v)>0$ on (v', v''). Also, using the boundary conditions (5.4),

$$\begin{aligned} &\lim_{v\downarrow v'} g(v) = -\infty \\ &\lim_{v\uparrow v''} g(v) = +\infty \\ &g(v^0)=0 \end{aligned} \tag{5.7}$$

Since ce^{-z} is strictly decreasing from $+\infty$ to 0 as z increases from $-\infty$ to $+\infty$, the solution $v=\beta(z)$ of (5.5) exists, is unique, and clearly satisfies the asserted properties. Q.E.D.

It is intuitive that, given a particular "variance" v, the probability of survival is enhanced by choosing the maximum "infinitesimal mean" from the constraining set A. This is indeed the case, although the formal verification involves some tedious computation (see Lemma A.1 in the appendix). Hence, the solution to the problem of maximizing the probability of survival is given by the following:

Proposition 5.1. *For any* $y(t)>0$, *let* $Z(t)\equiv \ln Y(t)$. *If the current state is* $Z(t)=z$, *the optimal act is*

$$\begin{aligned} v&=\beta(z) \\ m&=f[\beta(z)] \ . \end{aligned} \tag{5.8}$$

These properties of the optimal policy have an interesting interpretation. Recall that f reaches a maximum at v^*, which is between v^0 and v''. The part of the graph of f between v^* and v'' is "inefficient" in the usual treatment of mean-variance portfolio analysis, since from any of these points one can reduce the variance without decreasing the mean. Nevertheless, when the agent's capital is sufficiently low, the optimal choice of (m, v) will be in the "inefficient" part of the boundary of A. Rather than call such choices "inefficient", we shall say that for the corresponding values of z the agent exhibits "risk-loving" behavior.

The critical value of z, call it z^*, below which the agent exhibits "risk-loving" behavior is easily calculated. Observe that v^* and z^* must satisfy (5.5); also, since f attains its maximum at $v^*, f'(v^*)=0$. Hence

$$f(v^*)=ce^{-z^*} \ .$$

Let

$$m^* \equiv f(v^*), y^*=e^{z^*} \ ;$$

then

$$y^*=\frac{c}{m^*} \ . \tag{5.9}$$

Compare this with (4.22).

We now establish Proposition 5.1 somewhat heuristically. Let $V(z)$ denote the *maximum* probability of survival starting from the state $Z(0)=z$, i.e., if the optimal policy is used. Imagine that the agent uses an investment (m, v) *throughout* the interval $[0, h]$, where h is "small", and then follows the optimal policy thereafter. The resulting probability of survival is approximately $EV[Z(h)]$. According to the Optimality Principle of Dynamic Programming, the optimal choice of (m, v) at time zero should (approximately) maximize $EV[Z(h)]$. Following the development in (4.10) and (4.11), if h is small then the optimal (m, v) will approximately maximize the sum of the terms of order h in the right-hand side of (4.11), i.e., the optimal (m, v) will maximize

$$V'(z)(m-ce^{-z})+\frac{1}{2}V''(z)v \ . \tag{5.10}$$

One can show independently that V is strictly increasing, so

$$V'(z)>0 \ . \tag{5.11}$$

Hence, for fixed z and v, (5.10) is maximized in m by taking $m=f(v)$. Hence, given z, v should be chosen to maximize

$$V'(z)[f(v)-ce^{-z}]+\frac{v}{2}V''(z) \ ,$$

or equivalently,

$$v=\beta(z) \text{ maximizes } V'(z)f(v)+\frac{v}{2}V''(z) \ . \tag{5.12}$$

Given the properties of f, the solution to this maximization problem is characterized by the first-order condition:

$$V'(z)f'(v) + \frac{1}{2}V''(z) = 0 \ . \qquad (5.13)$$

With the optimal choice of (m, v),

$$V(z) = EV[Z(h)] \ .$$

(This is the other part of the Optimality Principle of Dynamic Programming). As in Sect. IV, this implies that (in the limit as h tends to zero), V satisfies the differential equation (4.13), with (m, v) the optimal choice corresponding to z. Putting (4.13) and (5.13) together we obtain the equation

$$V'(z)[f(v) - ce^{-z} - vf'(v)] = 0 \ .$$

Since $V'(z) > 0$, this is equivalent to $f(v) - ce^{-z} - vf'(v) = 0$, or

$$g(v) = ce^{-z} \qquad (5.14)$$

where

$$g(v) \equiv f(v) - vf'(v) \ . \qquad (5.15)$$

The appendix contains a more formal derivation of the existence of optimal policy function and its characterization in terms of the optimality equation that was used above.

One can easily extend the foregoing analysis to the case in which the set A is an arbitrary compact set. First, suppose that A is also convex; then the function f defined by (5.3) will be concave, but not necessarily strictly concave, nor everywhere differentiable. Nevertheless, one can still find a function β that is nonincreasing and satisfies (5.6) and (5.12); it need not, however, be continuous. For example, β may pick out the extreme points on the upper boundary of A. Finally, if A is not convex (but is compact), replace A by its convex hull and use the preceding analysis.

VI. Bibliographical notes

While assessing the significance of the Walrasian equilibrium analysis in models with "dated" goods, Koopmans [10] pointed out the difficulties in ensuring the existence of an equilibrium where consumers can survive. Robinson [21] maintained that "an equilibrium position which contains...consumption of exhaustible resources or starvation of some group is in course of upsetting itself from within, and chance events may upset it from without". Survival of consumers (and firms) clearly raises problems in a mechanical application of any static equilibrium or steady state concepts to a dynamic framework. For a recent commentary on these problems, see Newman [17]. Of some interest is the concluding part of Ramsey [19].

The literature on exhaustible resources also contains discussions of the survival problem for an economy relying on an exhaustible resource as an essential input or consumption good. Of particular interest is the paper by Solow [24] on

the possibility of maintaining a positive steady state consumption. A detailed list of references on related topics can be found in Dasgupta and Heal [3].

There is by now a voluminous literature on poverty and malnutrition in which the idea of "subsistence" or "minimal" level or consumption/nutrition has figured prominently (since Adam Smith and Marx!). Precise descriptions of "the minimum necessities for the maintenance of physical efficiency" and attempts to obtain quantitative estimates have been somewhat controversial. For a review of the literature and an exhaustive list of references see Sen [22]. (The difficulty of using the Walrasian equilibrium apparatus to analyze problems of starvation and famines was also noted by Sen). The effects of malnutrition on productivity and life expectancy have been a central theme in development economics. In our terminology, the choice of "c" may itself affect the distribution of (R_t) or set an upper bound on the random variable T. For further references, see, for example, Gersovitz [6].

Turning to investment and finance, diffusion models of optimal investment and portfolio management have been developed following the lead of Samuelson [22] and Merton [14]. A useful survey is in Merton [15]. The optimality criterion usually studied is the maximization of the integral of discounted expected utility stream. A comprehensive discussion, paying particular attention to possibilities of hitting lower bounds on consumption, is found in Lehoczky et al. [11]. For discussions of risk-loving or "go for broke" policies, see also Gordon [7].

The case $c = 0$ is closely related to the literature on optimal gambling (see, e.g., Billingsley [1]). To complement our analysis, one may wish to consider an agent who faces a zero survival probability and tries to maximize the discounted expected time to ruin. The paper by Heath, Orey, Pestien and Sudderth [8] is of interest in this context. Individuals or economies facing "hopeless odds" may rely on short run borrowing to "tide over a crisis" (see Ray [20]) or engage in research to create a more favorable environment.

Finally, some results on survival probabilities with strictly concave return functions are contained in Mirman and Spulber [16] who deal with environments represented by a sequence of independent, identically distributed random variables, and in Majumdar and Radner [12] who allow for a number of stochastic laws to govern the evolution of the environment.

Appendix

In what follows we maintain Assumption 5.1 on the set A, and Assumption 5.2 on the function f defined in (5.3).

We consider diffusions with the state space $[0, \infty)$ where '0' is taken as an absorbing state. For any $y > 0$, the drift and diffusion coefficients are given by:

$$\mu(y) = [m(y) + 1/2v(y)]y - c \qquad (0.1)$$
$$\sigma^2(y) = v(y)y^2$$

where

$$(m(y), v(y)) \in A \ .$$

Let $\Psi(y)$ denote the probability that a diffusion with coefficients $\mu(\cdot), \sigma^2(\cdot)$ reaches '0' before reaching d, starting at $y > 0$ where $d > 0$ is fixed. Of independent interest is the following intuitive result:

Lemma A.1. *Consider two diffusions on* $[0,\infty)$, *with absorption at* 0 *having a common diffusion coefficient but distinct drifts satisfying* $\mu^{(1)}(y) \leqq \mu^{(2)}(y)$ *for every* $y>0$. *Then* $\Psi^{(2)}(y) \leqq \Psi^{(1)}(y)$ *for every* $y>0$.

Proof. It is known that: (see, e.g. Karlin and Taylor [9, pp. 192–5]).

$$\psi^{(i)}(y)=\int_y^d \exp\left\{-\int_0^u \frac{2\mu^{(i)}(x)}{\sigma^2(x)}dx\right\}du \Big/ \int_0^d \exp\left\{-\int_0^u \frac{2\mu^{(i)}(x)}{\sigma^2(x)}dx\right\}du \tag{0.2}$$

Write

$$\eta(y)=\mu^{(2)}(y)-\mu^{(1)}(y), \mu^{\varepsilon}(y)=\mu^{(1)}(y)+\varepsilon\eta(y)$$

and let $F(\varepsilon; y)$ denote the probability of reaching 0 before d starting at y, for a diffusion with drift coefficient $\mu^{\varepsilon}(\cdot)$ and diffusion coefficient $\sigma^2(\cdot)$. It is straightforward to check that

$$F(\varepsilon;y)=\Psi^{(1)}(y)(1-\varepsilon\gamma(y))+0(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0 , \tag{0.3}$$

where

$$\gamma(y)=\int_y^d \exp\left\{-\int_0^u \frac{2\mu^{(1)}(x)}{\sigma^2(x)}dx\right\}\left(\int_0^u \frac{2\eta(x)}{\sigma^2(x)}\right)du \Big/ \int_y^d \exp\left\{-\int_0^u \frac{2\mu^{(1)}(x)}{\sigma^2(x)}dx\right\}du$$

Since $\eta(y) \geqq 0$ for all y, it follows that

$$\gamma(y)>0 \tag{0.4}$$

unless $\mu^{(1)}(\cdot) \equiv \mu^{(2)}(\cdot)$. Hence, barring the case of identical drifts, one has

$$\frac{d}{d\varepsilon}F(\varepsilon;y)|_{\varepsilon=0}=-\psi^{(1)}(y)\gamma(y)<0 \tag{0.5}$$

Therefore, $F(\varepsilon; y)$ is strictly decreasing in ε in a neighborhood of $\varepsilon=0$, say on $[0,\varepsilon_0)$. Since one can continue beyond ε_0, by replacing $\mu^{(1)}(\cdot)$ by $\mu^{\varepsilon_0}(\cdot)$, the supermum over all such ε_0 is ∞. In particular, one can take $\varepsilon_0=1$. Q.E.D.

Suppose now that the agent wants to maximize the probability of reaching a fixed capital $d>0$ before 0; or, equivalently, to minimize the probability $\Psi(y)$ of reaching 0 before d starting at $y>0$. Using Lemma A.1, an optimal choice of $(m(y), v(y))$ should be the form $(f(v(y)), v(y))$. Such a stationary policy actually can be shown to be optimal among all non-anticipative strategies (see e.g., Fleming and Rishel [4, Chap. VI]). Hence, the optimization problem is reduced to that of an optimal choice of $v(y)$. To characterize the optimal choice of $v(y)$ in terms of a "dynamic programming equation" formalizing the principle of optimality, define for each constant $v \in [v', v'']$ the infintesimal generator A_v by

$$(A_v g)(y)=(vy^2/2)g''(y)+\{(f(v)+v/2)y-c\}g'(y) \tag{0.6}$$

For a given measurable function $v(\cdot)$ on $(0,\infty)$ into $[v', v'']$ define the infinitesimal generator $A_v(\cdot)$ by

$$(A_{v(\cdot)}g)(y)=[v(y)y^2/2]g''(y)+[(f(v(y))+v(y)/2))y-c]g'(y) \tag{0.7}$$

Let $\psi_{v(\cdot)}(y)$ denote the probability that a diffusion having generator $A_{v(\cdot)}$, starting at $y\varepsilon[0,d]$, reaches 0 before d. For simplicity consider $v(\cdot)$ to be differentiable, although the arguments are valid for all measurable $v(\cdot)$ (see, e.g., Mandl [13]). Write

$$\bar{\psi}(y)=\inf_{v(\cdot)}\psi_{v(\cdot)}(y) . \tag{0.8}$$

Proposition A.1. *The dynamic programming equation*

$$\min_{[v',v'']}\ldots \quad \textit{for} \quad 0<y<d ; \quad \lim_{y\downarrow 0}\psi(y)=1 , \quad \lim_{y\uparrow d}\psi(y)=0 . \tag{0.9}$$

has a solution ψ *which is the minimal probability of ruin* $\bar{\psi}$. *For each* $y \in (0,d)$ *there exists a unique* $\bar{v}(y) \in [v', v'']$ *such that the minimum is attained at* $v=\bar{v}(y)$. *The function* $\bar{v}$ *is strictly decreasing on* $(0,\infty)$ *and is optimal.*

Proof. Let $\psi(y)$ be a twice continuously differentiable function satisfying (0.9), and $\bar{v}(\cdot)$ a (measurable) function such that the minimum in (0.9) is attained at $v=\bar{v}(y)$. Then

$$A_{\bar{v}(\cdot)}\psi(y)=0 \quad \text{for} \quad 0<y<d ; \quad \lim_{y\downarrow 0}\psi(y)=1 , \quad \lim_{y\uparrow d}\psi(y)=0 \tag{0.10}$$

Then (see (0.7)),

$$y^2\psi''(y)=\frac{2}{\bar{v}(y)}\left\{(f(\bar{v}(y))+\frac{1}{2}\bar{v}(y))y-c\right\}\psi'(y) . \tag{0.11}$$

Using this in (0.9), one gets

$$\min_{v \in [v',v'']}[-v\{(f(\bar{v}(y))+\tfrac{1}{2}\bar{v}(y))y-c\}/\bar{v}(y)$$
$$+(f(v)+\tfrac{1}{2}v)y-c]\psi'(y)=0 . \tag{0.12}$$

It follows from (0.10), since its solution is unique, that $\psi(y)$ is the probability of reaching zero before d, starting at y, for the diffusion generated by $A_{\bar{v}(\cdot)}$. Also, from (0.2), $\psi'(y)<0$. Therefore, to determine $\bar{v}(y)$ we find the critical point(s) of the expression within the square brackets in (0.12) by setting its derivative equal to zero. This leads to solving for each $y>0$, as v satisfying:

$$f(v)-vf'(v)=\frac{c}{y} \tag{0.13}$$

The arguments in Lemma 5.1 can be readily adapted to prove the existence of a unique $\bar{v}(y)$ with the monotonicty property asserted.

It remains to show that ψ is minimal. For this consider an arbitrary (measurable) choice $v(\cdot)$. It follows from (0.9) that

$$(A_{v(\cdot)}\psi)(y) \geqq 0 \quad \text{for all} \quad y \in (0,d) . \tag{0.14}$$

Since $(A_{v(\cdot)}\psi_{v(\cdot)})(y)=0$ and ψ and $\psi_{v(\cdot)}$ satisfy the same boundary conditions, letting $u(y) = \psi(y) - \psi_{v(\cdot)}(y)$, one gets

$$\begin{aligned} &(A_{v(\cdot)}u)(y) \geqq 0 \quad \text{for all} \quad y \in (0, d) \ , \\ &\lim_{y \downarrow 0} u(y) = 0 \ , \quad \lim_{y \uparrow d} u(y) = 0 \ . \end{aligned} \tag{0.15}$$

We would like to show that $u(y) \leqq 0$ for all y, i.e.,

$$\psi(y) \leqq \psi_{v(\cdot)} \quad \text{for all} \quad y \in [0, d] \ . \tag{0.16}$$

To this effect first assume $(Au)(y) > 0$ for $y \in (0, d)$ where A is any second order differential operator (see (0.1) and [9, p. 199]). If possible, let may $u(y)$ be attained at $y_0 \in (0, d)$. Then

$$u''(y_0) \leqq 0 \ , \quad u'(y_0) = 0 \ .$$

Hence

$$(Au)(y_0) - \frac{1}{2}\sigma^2(y_0)u''(y_0) \leqq 0 \ ,$$

a contradiction.

Now assume $(Au)(y) \geqq 0$ for $y \in (0, d)$. Consider the function $u_\varepsilon(y) = u(y) + \varepsilon Q(z)$, where Q is a twice differentiable function on $(0, d)$, continuous on $[0, d]$, satisfying $(AQ)(y) = 1$ for $y \in (0, d)$. For example, one can take Q as

$$Q(y) = \int_c^y \left(\int_v^x \frac{2}{\sigma^2(v)} \left\{ \exp\left(- \int_v^x \frac{2\mu(z)}{\sigma^2(z)} dz \right) dv \right\} dx \right)$$

Then for every $\varepsilon > 0$, $Au_\varepsilon(y) > 0$ for $y \in (0, d)$. Hence the maximum of u_ε cannot be attained in $(0, d)$, i.e., $u(y) + \varepsilon Q(y) < \max\{u(0) + \varepsilon Q(0), u(d) + \varepsilon Q(d)\}$ for all $z \in (0, d)$. Letting $\varepsilon \downarrow 0$, one gets $u(y) \leqq \max\{u(0), u(d)\}$.

It is important to note that the optimal choice represented by

$$\bar{\mu}(y) = (f(\bar{v}(y)) + \frac{1}{2}\bar{v}(y)) - 0 \ , \quad \text{and } \bar{v}(y) \ , \tag{0.16}$$

does not depend on d. Therefore, with this choice one also minimizes the probability ρ_{y0} of ever reaching 0, *starting from* $y > 0$. For this simply let $d \uparrow \infty$ in the inequality

$$\psi_{\bar{v}(\cdot)}(y) \leqq \psi_{v(\cdot)}(y) \ . \tag{0.17}$$

References

1. Billingsley P (1979) Probability and measure. Wiley, New York
2. Chung KL (1974) A course in probability theory. New York, Academic Press
3. Dasgupta P, Heal G (1979) Economic theory and exhaustive resources. Cambridge, Nibert/Cambridge University Press
4. Fleming, WH, Rishel RW (1975) Deterministic and stochastic optimal control. Springer, Berlin Heidelberg New York
5. Freedman D (1973) Another note on the Borel-Cantelli lemma and the strong law. Ann Probab 1: 910–25
6. Gersovitz M (1983) Savings and nutrition at low incomes. J Political Econ 91: 841–855
7. Gordon MJ (1986) The portfolio policy that maximizes the probability of long run survial. Discussion Paper, Faculty Management Studies, Toronto, University of Toronto
8. Heath D, Orey S, Pestian V, Sudderth W (1987) Maximizing or minimizing the expected time to reach zero. Siam J Control Optimization 25: 195–205

9. Karlin S, Taylor H (1981) A second course in stochastic processes. New York, Academic Press
10. Koopmans TC (1957) Three essays on the state of economic science. New York, McGraw Hill
11. Lehoczky J, Sethi S, Shreve S (1983) Optimal consumption and investment policies allowing consumption constraints and bankruptcy. Math Operations Res 8: 613–636
12. Majumdar M, Radner R (1988) Survival under production uncertainty. In: Equilibrium and dynamics. Brock W, Majumdar M (eds). London, Macmillian Press
13. Mandl P (1968) Analytical treatment of one dimensional markov processes. Berlin Heidelberg New York, Springer
14. Merton RC (1971) Optimum consumption and portfolio rules in a continuous time model. J Econ Theory 3: 373–413
15. Merton RC (1982) On the microeconomic theory of investment under uncertainty. In: Handbook of Mathematical Economics, vol. 2. Arrow K, Intrilagator M (eds). Amsterdam North Holland, pp. 601–666
16. Mirman LJ, Spulber D (1984) Uncertainty and markets for renewable resources. J Econ Dyn Control 8: 239–264
17. Newman P (1989) Consumption sets. In: The new Palgrave: general equilibrium. Eatwell J, Milgate M, Newman P (eds). London, Macmillan Press, pp. 108–110
18. Radner R (1986) Repeated moral hazard with low discount rates. In: Uncertainty, information, and communication, essays in honor of Kenneth J. Arrow. Heller WP, Starr RM, Starrett D (eds) Vol. III. Cambridge, Cambridge University Press, pp. 25–64
19. Ramsey F (1928) A mathematical theory of savings. Econom J 38: 543–49
20. Ray D (1984) Intertemporal borrowing to sustain exogenous consumption standards under uncertainty. J Econ Theory 33: 72–87
21. Robinson J (1962) Essays in the theory of economic growth. London, Macmillian Press
22. Samuelson PA (1969) Lifetime portfolio selection by dynamic stochastic programming. Rev Econ Stat 51: 239-246
23. Sen AK (1981) Ingredients of famine analysis: availability and entitlements. Quart J Econ 95: 433–464
24. Solow RM (1974) Integerrational equity and exhaustible resources, Rev Econ Stud. Symposium on Exhaustible Resources
25. Strook DW, Varadhan SRS (1979) Multi-dimensional diffusion processes. Berlin Heidelberg New York, Springer