Nancy L. Schwartz Lecture

ECONOMIC SURVIVAL

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NANCY L. SCHWARTZ

A dedicated scholar and teacher, Nancy Schwartz was the Morrison Professor of Decision Sciences and the Kellogg School's first woman faculty member appointed to an endowed chair. She joined Kellogg in 1970, chaired the Department of Managerial Economics and Decision Sciences, and served as director of the school's doctoral program until her death in 1981. Unwavering in her dedication to academic excellence, she published more than 40 papers and co-authored two books. At the time of her death she was associate editor of *Econometrica*, on the board of editors of the *American Economic Review*, and on the governing councils of the American Economic Association and the Institute of Management Sciences.

The Nancy L. Schwartz Memorial Lecture series was established by her family, colleagues, and friends in tribute to her memory. The lectures present issues of fundamental importance in current economic theory.

ECONOMIC SURVIVAL

1. *Introduction*[1]

Standard textbooks on microeconomic theory typically ascribe to consumers the goal of maximizing "utility," and to firms the goal of maximizing "profit" or the "value of the firm." Explicit consideration of the survival and failure of firms has scarcely been recognized by general equilibrium theory, in spite of the sophisticated development of the subject in the past forty years. The recent reawakening of interest in the evolution of economic behavior, especially among game theorists, implicitly brings with it a concern for the goal of "survival," but thus far most game-theoretic models of evolution do not bear much resemblance to even stylized pictures of economic institutions.[2]

Nevertheless, failure is a common occurrence in business. For example, during the 15-year period from 1967-1982, almost half of U.S. manufacturing firms exited from their industry each year. Even if we eliminate from each industry the group of smallest firms, producing one percent of the industry output, the annual exit rate was still about 37 percent. During the same period, more than 60 percent of such firms exited within their first five years in the industry, and almost 80 percent in their first ten years. (See Dunne, et al (1988), especially pp. 503-510.) Nor is the concept of "failure" a simple one - it has many degrees and manifestations. I shall return to this point briefly at the end of the paper.

Of course, the concept of "utility" is so broad that it easily encompasses the goal of survival. For example, we could ascribe to an economic agent a "utility" of one unit per period as long as he or she survives, and zero after that. In this case, maximizing total utility would be equivalent to maximizing the time to failure. However, this is not the kind of utility function that is usually ascribed to consumers. Indeed, I shall argue here that the explicit consideration of the goal of survival often leads to predictions of behavior that differ radically from those implied by the typical models of expected utility maximization

In recent decades there has been great progress in the ability of economic theory to deal with issues of uncertainty, and the connections between survival and uncertainty are particularly interesting. On the one hand, there seems to have been little disagreement among economic theorists that, in a world of certainty and complete markets it makes sense to ascribe to firms the goal of profit maximization. On the other hand, in a world of uncertainty and incomplete markets, the very definition of "profit" becomes problematic.[3] Some authors have suggested that there is a close link between survival and the maximization of expected profit, or even that the latter is necessary for the former.[4] I shall sketch a theoretical model in which, to the contrary, most of the surviving firms will not be maximizing expected profits.

My plan is to discuss these issues in the framework of a sequence of theoretical models, all of which are in some sense elaborations and extensions of the classical "Gambler's Ruin Problem." Although I shall use hardly any formal mathematical notation, I must admit that the exposition will nevertheless be rather abstract, and the nontheorist will probably need some patience to get through it. I hope that the figures will provide some additional help for the geometrically minded.

I have made no attempt to provide a systematic bibliography on the subject of economic survival. Most of the exposition here is based on research that I have done jointly with Professors Mukul K. Majumdar and Prajit K. Dutta, and I would like to acknowledge as well their helpful comments on the present paper. More details about the sources of the results reported here, and other references, are given in the Bibliographic Notes at the end of the paper.

Here is an outline of the rest of the paper:

2. The Gambler's Ruin and Survival

3. The Indebted Investor Who Wants to Survive

4. The Profit-Maximizing Investor

5. Survival and Selection

6. Concluding Remarks

2. *The Gambler's Ruin and Survival*

As every student of probability and statistics should know, the modern theory of probability dates from 1654, when Antoine Gombaud, Chevalier de la Mère, posed some some questions on games of chance to Blaise Pascal (1623-1662).

> Pascal communicated his solutions to Pierre de Fermat [1601-1665] for approval, and a correspondence ensued. At that time scientific journals did not exist, so it was a widespread habit to communicate new results by letters to colleagues. (Hald, 1990, p. 42.)

The "Gambler's Ruin Problem," which is the forerunner of the theories of survival that I shall discuss here, was evidently taken up two years later. Continuing with the account by Anders Hald:

> The correspondence of Pascal and Fermat was resumed in 1656 when Pascal posed to Fermat a problem that today is known as The Problem of the Gambler's Ruin. Through [Pierre de] Carcavi the problem was passed on to [Christian] Huygens [1629-1695] who described it in his treatise De Rationciniis in Ludo Aleae [1657] as the fifth problem to be solved by the reader. Pascal, Fermat, and Huygens all solved the problem numerically without disclosing their methods. (Hald, 1990, p. 63.)

Here is the problem:

> Problem 5. A and B each having 12 counters play with three dice on the condition that if 11 points are thrown, A gives a counter to B and if 14 points are thrown, B gives a counter to A , and that he wins the play who first has all the counters. Here it is found that the number of chances of A to that of B is 244,140,625 to 282,429, 536,481. (Hald, 1990, p. 76.)

This problem represented a new challenge in probability theory, because the number of plays before one player wins all the counters can be unboundedly large. (In modern terminology, the underlying probability space is not finite.)

In a more general statement of the Gambler's Ruin Problem, players A and B start with some given numbers of counters, and given probabilities of winning on any one trial. James Bernoulli (1654-1705) was apparently the first mathematician to find the general formula for the probability that A wins all of the counters before B does.This formula appeared, without proof, in his posthumously published book, *Ars Conjectandi* (1713), but evidently he had discovered it much earlier. The first published proof was by Abraham de Moivre (1667-1754), and appeared in his paper, *De Mensura Sortis* (1712), and later in his book, *Doctrine of Chances* (1718). (For further details, see Hald (1990), pp. 202 ff.)

The problems I shall be discussing here correspond formally to the case in which player B has infinitely many counters; we may think of B as "Nature" or "the rest of the market." The probability that player A never loses all of his counters, i.e., that A "is never ruined," or that A "survives forever," is given by:

(2.1)

$$P(a) = \begin{cases} 1-r^{a} & \text{if } r<1 \\ 0, & \text{otherwise,} \end{cases}$$

where a denotes A's initial number of counters, and r denotes the odds in favor of B on any one trial. We see from the formula that if the individual trials are favorable to player A ($r < 1$), then the probability of eventual ruin, $1 - P(a)$, decreases geometrically from unity, when $a = 0$, towards zero as a increases without bound. On the other hand, if the individual trials are unfavorable to A ($r > 1$), or even exactly fair ($r = 1$), then A is sure to be ruined eventually.

The problems that follow will be different from, and more general than, the Gambler's Ruin problem covered by this formula in several ways. First, player A's stock of counters will (typically) be replaced by a stock of real money or other liquid assets.[5] Accordingly, I shall refer to A as an "economic agent," "investor," "entrepreneur," or "manager," and to his stock of counters as his current "fortune" or "cash reserve." Second, A may gain or lose more than one unit in any trial (period). Third, A may be able to – or be required to – withdraw money from his current stock, e.g., for consumption or to service a debt. Fourth, at each play, A may have the option of choosing - from a suitably restricted set - which game he wants to play. For example, at the beginning of every market day an investor may have the option of revising his portfolio at current market prices. Fifth, A is "ruined" (fails, goes bankrupt, is fired) at the first time - if ever - that his stock falls below some prescribed value, which I shall conventionally take to be zero.

Finally, most of the results I shall describe are based on a mathematical model in which "play" takes place continuously, rather than at discrete times. This model has been adopted purely for mathematical convenience, since it turns out that the relevant formulas are often simpler and crisper in a model with continuous time. In any case, we may think of the continuous-time model as an approximation to the discrete-time model when transactions are sufficiently frequent. Accordingly, I shall adopt the following general scheme, with further elaborations or variations as needed. Underlying each problem will be a stochastic process that – for the time being – we may think of as the agent's cumulative net earnings. Thus the increment in the earnings process over any interval of time equals the agent's net earnings during that time interval. I shall make two important assumptions about this earnings process:

1. The earnings process evolves continuously in time.

2. Conditioned on the agent's actions, the earnings in non-overlapping intervals are statistically independent.

These assumptions are not entirely innocuous, so they are worth examining for a moment. Essentially, they represent a situation in which the agent's cumulative environment consists of a sequence of small but frequent events, small in the sense that no one event has an overwhelming effect on the agent's cumulative earnings at that moment. Thus I am ruling out infrequent catastrophes such as major earthquakes, stock-market crashes, etc. One might say that I am going to discuss problems of survival in "normal times."

A strong mathematical consequence of the above assumptions is that, conditioned on the agent's actions, the agent's earnings in any time interval has a Gaussian or normal distribution.[6] Roughly speaking, in any very small interval of time, the agent's earnings in that interval will be normally distributed with mean and variance proportional to the length of the interval. If the agent has any influence over the earnings process, he effectively does so by choosing that mean and variance at each moment of time, subject, of course, to some restrictions. The typical evolution of such a cumulative earnings process is shown in Figure 1.

Following standard terminology, I shall call such a process a controlled additive diffusion.[7] Such processes have become standard in the

modern theory of finance, following their introduction at the turn of the century by Bachelier, and later developments introduced by Samuelson (1965) and Black and Scholes (1972, 1973). (For a more recent account of applications of continuous-time processes in finance, see Merton (1990).)

I shall now give a formula for the probability of survival in the special case of a diffusion in which the agent does not exercise any control over the game being played, and essentially plays the "same game repeatedly." This is the continuous-time analogue of the Gambler's Ruin Problem solved by James Bernoulli. By this I mean that, in any time interval of length h, the agent's earnings is a normally distributed random variable with mean mh and variance vh,where m and v are fixed parameters. Following standard terminology, I shall call m the drift and v the volatility of the earnings process (sometimes called the "yield" and "risk," respectively). The drift may be positive or negative, but the volatility is of course non-negative. In fact, unless I indicate otherwise, I shall assume that the volatility is strictly positive; otherwise there would be no uncertainty about the evolution of the process, which would not be very interesting.

Suppose that the agent starts with a stock of money equal to y, and fails (is ruined) at the first time, if ever, that his stock falls to zero. Such a failure is illustrated in Figure 2, at time T. It can be shown (see, e.g., Harrison, 1985, p. 43, Corollary) that the probability that the agent survives forever (is never ruined) is given by the formula:

(2.2)

$$P(y) = \begin{cases} 1 - \exp(-2my/v), & \text{if } m > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note the similarity between (2.2) and (2.1). Player A's initial stock of counters, a, has been replaced by the agent's initial stock of money, y, and the odds ratio, r, has been replaced by the expression exp(-2m/v). Again, the probability of failure, 1 - P(y), decreases exponentially to 0 as the initial stock, y, increases without bound. Figure 3 illustrates the formula for the survival probability, with the initial stock, y, on the horizontal axis, and the survival probability, P(y), on the vertical axis.

Even though the formula (2.2) is valid only for the special case in which the drift and volatility are constant, I have taken some pains to display it because it contains information that will be relevant to the more complicated problems that I shall discuss later. In particular, we see that the survival probability is higher the larger is the ratio (m/v) whatever the initial stock, y.

Apart from the gambling metaphor, the model I have just described might be appropriate to represent a business of a fixed size, with two kinds of assets: (1) fixed assets, which are illiquid, and necessary to operate the business, and (2) a cash reserve, or other liquid assets, used to pay bills and other current obligations. Net earnings in periods of equal length (e.g., a quarter) have the same mean and variance. Earnings are added to the cash reserve and/or distributed to the owner(s). However, in this model, earnings cannot be reinvested to increase the scale of the business. Net earnings in any period may be negative, so the the cash reserve may decrease even if there are no distributions. If the cash reserve ever falls to zero so that the bills cannot be paid, the business fails. Note that the cash reserve will typically include a line of credit, in which case the critical level that defines "failure" is really some negative number. The important point is that there is some such critical level. I shall call this the constant size model.

If we want to represent a situation in which earnings can be reinvested to increase the scale of operations, then we must change the model. For example, a gambler in a large casino can stake his entire current fortune on each play, at least up to some large limit. Similarly, an investor in a securities market can reinvest his earnings by buying more securities, and the prices at which he can buy securities will be

independent of the scale of his purchases, at least until his fortune gets very large indeed. In both cases (up to some large limit), the agent's current net earnings are proportional to his current fortune; the factor of proportionality is determined by the rate of return on his current gamble or investment. Another feature of the gambler or securities investor is that his assets are liquid, so that he can remain in business as long as his fortune is positive, or at least above some minimum level. (I cannot buy or sell one penny's worth of AT&T stock, and in any case I would have to pay some minimum commission.) Suppose, for example, that the investor never spends any of his money, but continuously reinvests all of his net earnings. Since returns are multiplicative, and the agent's assets are liquid, his fortune will grow - or decline exponentially at a rate equal to the rate of return on the current investment. This rate of return will fluctuate, in part because of random factors and in part because of the agent's investment strategy. We can model this situation by postulating that the logarithm of the cumulative net earnings is a controlled additive diffusion, as described above. At any moment of time, the drift of this diffusion represents the expected current rate of return, and the volatility is its variance. I shall call this the constant returns-to-scale model.

These two models – constant size, and constant returns-to-scale – are the basis of the more elaborate constructions I shall describe in what follows. They are, of course, very special cases. The typical firm can invest to increase its scale, and its assets can be more or less liquid. Also, it may be subject to varying returns to scale, depending on its scale and other factors. Nevertheless, these two special cases lead to rather different results, and provide some hints as to what we can expect as rigorous analysis succeeds in exploring the rest of the map of technological possibilities

In addition to exploring two models of technology, I shall also focus on two contrasting models of preference. In the first, it is assumed that the agent wants to maximize the probability that he survives forever; I shall call such an agent a survivalist. To make the model more interesting and realistic, I shall suppose that the agent is obliged to withdraw funds from his cash reserve, or other liquid assets, at a constant rate per unit time, e.g., in order to service a debt. (Another interpretation is that this constant rate of withdrawal is required for the agent to maintain a "satisfactory" rate of consumption.) I shall call this the model of the indebted survivalist. In this case the agent can influence the probability of survival by dynamically controlling the drift and volatility of the cumulative earnings process.

In the second model, it is assumed that the agent wants to maximize the expected total discounted withdrawals. Here the agent can dynamically control the withdrawal rate, as well as the drift and volatility of the earnings process. I shall call such an agent a profit maximizer.

Combining the two models of technology with the two models of preference leads to the accompanying 2x2 table. In the next two sections, the four blanks in the table will be filled in with descriptions of the respective optimal strategies of the agents, and their corresponding probabilities of survival.

	Indebted Survivalist	Profit Maximizer
Constant Size		
Constant Returns to Scale		

3. *The Indebted Survivalist*

Most firms obtain at least part of their initial capital by borrowing money. In this section I shall consider a model of an investor who wants to maximize the probability of survival,

but has borrowed money and is obligated to make payments at a fixed rate per unit of time. I shall describe the rather different implications of such an obligation in the constant-size and constant-rate-of-return models.

Starting with the constant-size model, I need to introduce some additional concepts. Recall that the cumulative earnings process is modeled as a controlled additive diffusion. The evolution of the cash reserve is governed by the following simple accounting relation: During any time period of length h,

(3.1)

end-of-period cash reserve = beginning-of-period cash reserve plus period net earnings less c times h,

where c is the constant rate, per unit time, of payout. Recall that the investor survives if the cash reserve never reaches zero.[8]

To help fix ideas, let us first suppose that the investor has no control over the earnings process, so that its drift, m, and its volatility, v, are constant in time. It is intuitively clear from the accounting relation (3.1) that the cash reserve is also an additive diffusion process, with the same volatility, but with drift (m-c). It follows from equation (2.2) that the probability that the investor survives is

(3.2)

$$P(y) = \begin{cases} 1 - \exp[-2(m - c)y/v], & \text{if } m > c, \\ 0, & \text{otherwise.} \end{cases}$$

where y is the investor's strictly positive initial cash reserve (at time 0), and P(y) is the probability that the investor survives forever

Now suppose that the investor can influence the earnings process by controlling the drift and volatility through time. In other words, at each time t the investor can choose the drift and volatility that will govern the earnings process at that time. (Note the mathematical

abstraction used here: in practice, an investor will be able to change the drift and volatility at discrete times, like every day, or every month.) Since the investor is not clairvoyant, he will at each time be able to choose the drift and volatility at best as a function of the history of the process up to that time. In fact, in the class of problems we are considering here, he need not take account of the entire history, but only his current cash reserve.[9] The decision rule that determines his choice of current drift and volatility for each current reserve will be called the investor's strategy. For each strategy, there will be a corresponding probability of survival. Since the drift and volatility may vary with time, one cannot expect that the formula for the probability of survival will be as simple as (3.2). Indeed, for many quite simple strategies it will not be possible to find a (closed-form) formula at all, although numerical approximations will always be possible.

However, the investor will not be free to choose any drift-volatility pair he likes. In any situation there will be some constraints on the pairs available to him. Suppose, for simplicity, that the set of such feasible pairs - I shall call it A - is the same at all times. Such a set is illustrated in Figure 4, where the volatility is plotted on the horizontal axis, and the drift on the vertical axis. Three things are important about the feasible set A that is illustrated in the figure. First, notice that the volatility, v, is strictly positive everywhere in A. This means that there will be some randomness in the earnings process, whatever the choice of the investor (no risk-free investment). Second, notice that there is some part of the feasible set A where the drift, m, is strictly positive. This means that the investor can guarantee that the expected value of his earnings in any period is strictly positive, even though he cannot guarantee that the actual realized earnings will be so. Third, notice that the feasible set A is bounded; this means that the investor cannot make the drift of the earnings process arbitrarily large. There is some limit to how fast he can expect to make money!

What (feasible) strategy will maximize the investor's probability of survival? In our present case, the answer turns out to be quite simple: the investor should always use the same drift and volatility, namely, the pair (m,v) that maximizes – in the feasible set A – the ratio of (m - c) to v. This is illustrated in Figure 5, where the optimal drift-volatility pair is denoted by (m_c,v_c). Of course, in order for the investor's survival probability to be strictly positive, there must be some feasible drift that is strictly larger than the payout rate, as is clear from the previous formula

How does the optimal control depend on the payout rate c ? First, when the payout rate is zero, the optimal control maximizes the ratio of the drift to the volatility; this is denoted by the point (m_0,v_0) in Figure 6. On the other hand, when the payout rate equals the maximum feasible drift, say m^*, the optimal drift is also the maximum feasible, and with the corresponding volatility, v^*, i.e., the optimal control is (m^*,v^*), also illustrated in the same figure. Of course, at this payout rate, the probability of survival is zero. Finally, it is easy to show that, between these two limits, the optimal controls increase monotonically with the payout rate. Note, however, that the net drift of the investor's cash reserve, m - c, decreases monotonically with the payout rate, c.

I turn now to the case of constant returns to scale, as described in Section 2. Recall that, in this case, if the investor were to start with a dollar, and take no payout or consumption from his fortune, then his fortune would grow exponentially at some time-varying, stochastic rate of return. Here his cumulative rate of return, not his actual fortune, is a controlled additive diffusion. If the investor were to start with a fortune different from one dollar, then his fortune at time t would be proportionally different. The investor influences the cumulative rate of return by controlling its drift and volatility. Again, the choice of a drift-volatility pair is constrained to be in some feasible set [10], say A, as illustrated in Figure 4.

Recall that, in the situation we are studying in this section, the investor must pay his creditors at a constant rate c. In any short period of time, the investor's "gross return" (before subtracting the payout) is proportional to his fortune. Hence, whatever the expected rate of return of his fortune, if his fortune is small enough, this expected gross return per unit time will be less than his payout rate, c, and he can expect his net fortune to become even smaller. On the other hand, if the current fortune is very large, then so will be the expected total return per unit of time (provided the drift of the rate of return is positive), and he can expect his net fortune to become even larger.

I shall now describe the investor's optimal policy for controlling the drift and volatility of the rate of return. First, and not surprising, whatever drift-volatility pair he chooses at a moment of time, the drift will be as large as possible, given the volatility. This is illustrated in Figure 7, where the upper boundary of the feasible set, A, is indicated by a heavy curve. As shown in the figure, the curve is smooth and strictly concave, first increasing and then decreasing, as the volatility increases from its lower limit, v', to its upper limit, v''. For an optimal policy, whatever the investor's choice of volatility, his corresponding optimal drift will be on the heavy curve.

We have thus reduced the problem of determining an optimal control policy to the problem of choosing the optimal volatility as a function of the current fortune, y. One can show that this optimal volatility, or risk, decreases as the fortune increases, i.e., the larger the current fortune, the smaller will be the risk chosen by the investor. This is illustrated in Figure 8 by the direction of the arrows on the curve that define the upper boundary of the feasible set. As the figure shows, when the investor's fortune is very large, his choice corresponds approximately to the optimal control in the constant-size model, namely (m_0, v_0). Keep in mind, however, that in the present case the investor is controlling the rate of return, not the total return.

Second, and somewhat surprising, when the investor's fortune is sufficiently small, he behaves as if he were a "risk-lover." To be precise, let v^* denote the value of v for which the corresponding drift attains its maximum feasible value, say m^*. When the fortune is sufficiently small, the optimal risk will exceed v^*, and the optimal control will be on the part of the curve to the right of the point (m^*,v^*). This means that, the investor chooses the maximum feasible risk corresponding to the optimal drift. This behavior contradicts, of course, the well-known "efficiency property" of standard portfolio analysis, which would require that the investor choose the minimum volatility corresponding to the drift, i.e., the minimum risk for the given "yield." In fact, in the limit, as the investor's fortune approaches zero, his optimal risk approaches v', the maximum risk that is feasible for him.

This apparent risk-loving behavior is related to the fact, mentioned above, that, roughly speaking, the drift in the investor's fortune is proportional to the fortune. One can show that there is a critical fortune such that, if the investor's fortune is below it, then whatever his policy, he can expect his fortune to decline. On the other hand, if his fortune is above the critical value, then, using the optimal policy, he can expect his fortune to increase. In fact, it turns out that the critical fortune is exactly equal to the fortune at which it is optimal to choose the drift m^* and the volatility v^*. Thus we see that the apparent "risk-loving" behavior of the investor when his fortune falls below the critical value has nothing to with his attitude towards risk. The interval between zero and the critical fortune is a kind of trap, from which the investor tries to escape by taking sufficiently high risks, and the smaller his current fortune the greater the risk he must take.

Another insight into the investor's apparent attitude toward risk is obtained by examining the function that gives the maximum probability of survival, starting from any current fortune. Since the investor is constantly changing the drift and volatility of the rate of return,

there is no simple formula for this probability. It turns out to be convenient to measure the "state of the system" by the logarithm of the fortune, rather than by the fortune itself; I shall call this the logfortune. Figure 9 illustrates the maximum probability of survival, P(z), given that the initial logfortune is z. As we would expect, P(z) increases with z, approaching 1 asymptotically as z increases without bound. What is perhaps less expected is that P is S-shaped. The point z^* at which the "S" changes direction is, in fact, the logarithm of the critical fortune, i.e., the critical logfortune. In mathematical language, P is convex on the interval from 0 to z^*, and concave after that.

Now imagine that the investor is at time 0, with an initial logfortune z. Suppose that he adopts a drift for the rate of return process during a short time interval from 0 to h. If the interval is sufficiently small, then the probability that the investor fails in the interval 0 to h will be negligible. Therefore, the investor will want to control the process so as to maximize the expected value of the probability P at the end of the interval.

Those of you who have some familiarity with the theory of economic choice under uncertainty can now appreciate the significance of the shape of the function P. Since P is convex below the critical logfortune, z^*, the short-run behavior of the investor will appear to exhibit a love of risk in that region, whereas when his current logfortune exceeds z^* then his short-run behavior will exhibit apparent risk-aversion.

How do the optimal control and probability of survival depend on the payout rate? First, one can show that the optimal control is determined by the ratio, c/y, of the payout rate to the current fortune. It follows from what we know about the dependence of the optimal control on the current fortune that the optimal volatility increases monotonically with the payout rate, and that the optimal drift first increases from m to m^*, and then decreases.

Furthermore, as the payout rate, c, decreases to zero, for any given fortune the probability of survival increases towards unity; in other words, the entire curve P(z) shifts upwards.

4. *The Profit-Maximizing Investor*

Up to this point I have focused on the implications of the hypothesis that the investor wants to maximize the probability of surviving forever. I shall now switch to the hypothesis that the investor wants to maximize profits, and we shall see that the implications are quite different.

We first have to fix on a definition of "profit." I shall define profit to be the expected total discounted income from the investment, where future income is discounted at some fixed, exogenously given, rate. Although this may seem straightforward enough for some listeners, some comments may be in order for others. By "income" from the investment I mean money withdrawn from the capital stock or cash reserve for the purpose of consumption, debt repayment, and/or other payments to (other) investors. Thus money that is reinvested does not count as current income to the investor. Income may reflect realized capital gains, but since the investor's horizon is infinite, there is no terminal value of the capital stock. If the income is used for consumption, the addition of discounted income from different periods of time corresponds to the hypothesis that the investor's preferences are intertemporally independent, and the constancy of the rate of discount reflects the stationarity of his preferences (a constant rate of impatience).[11] Finally, taking the expected value reflects the investor's neutrality towards risk. Thus this definition of profit is not innocuous, and it will have strong implications for the investor's behavior. On the other hand, this is a fairly standard definition of the "profit" of a firm.

As in the previous section, the investor's optimal behavior will depend on his investment

"technology." Again, I shall consider the two polar cases of constant size and constant returns-to-scale. However, this time I shall first discuss the constant-returns-to-scale model. The reason for this switch in order is that the CRTS model will require very little of our time. The combination of profit maximization and CRTS results in a poorly posed optimization problem: either the investor can make an infinite profit, or he will want to terminate his investment and withdraw all of his liquid capital at the very beginning. I believe that this phenomenon is well known in the case of certainty, and it can also be shown to exist in the uncertain world of our model, but I shall not discuss it further here.

So I turn now to the constant-size model. Recall that, in this model, the "productive" capital stock is fixed and illiquid. Its cost is sunk, and so – for the purpose of characterizing the investor's optimal policy — it will not be necessary to subtract it from the profit. At any time, the current net earnings can be divided between a part that is added to the cash reserve and a part that is withdrawn. The amount that is "added" to the cash reserve can even be negative, as long as the cash reserve is strictly positive. On the other hand, the amount withdrawn in any period must be positive or zero. Thus we have the following simple accounting relation in any period

(4.1)

end-of-period cash reserve = beginning-of-period cash reserve plus period net earning less amount withdrawn.

This process continues until the first time, if ever, that the cash reserve falls to zero (call this the failure time), after which the earnings, cash reserves, and amounts withdrawn are all zero, i.e., the enterprise ceases to exist. The profit from the enterprise is defined to be the expected total of the discounted withdrawals.[12]

In the model of this section, there may be more

than one investor in the background, although these will not be described explicitly. For this reason, I shall call the decision maker the entrepreneur. In addition to managing the firm, the entrepreneur may also have money invested in it. The entrepreneur's policy will have two parts: (1) a control policy, for controlling the drift and volatility of the earnings process, and (2) a withdrawal policy. In Section 3, we saw a particularly simple example of a withdrawal policy, namely, a constant rate of payout (as long as the enterprise is solvent). As we shall see, this is not a profit-maximizing policy. In fact, the profit-maximizing withdrawal policy is also simple, but quite different

To prepare you for the description of the optimal (i.e., profit-maximizing) policy, I shall first describe a special class of withdrawal policies, which I shall call overflow policies. Imagine that the cash reserve is stored in a tank, as illustrated in Figure 10. Incoming rain adds to the water level in the tank (positive net earnings), but evaporation decreases it (negative net earnings). If tank ever goes dry, the firm fails. Near the top of the tank is a hole that feeds into a pipe; the pipe, in turn, empties into a bucket. Whenever the water level reaches the hole, any excess water (net earnings) flows into the pipe, and is thus withdrawn into the bucket; this corresponds to a withdrawal of funds from the cash reserve. The capacity of the tank up to the level of the hole, say b, is a parameter of the overflow policy, which we might call the overflow level.

The next two figures illustrate how the cash reserve will fluctuate with an overflow policy. Figure 11 shows a typical evolution of the cash reserve with no withdrawals, i.e., the cumulative earnings. Figure 12 shows how the cash reserve would evolve with the same net earnings, but with an overflow withdrawal policy.

If the entrepreneur uses an overflow policy, then the cash reserve will fluctuate between zero and the overflow level, but will never exceed the latter. Money is withdrawn from

the cash reserve only when the reserve level reaches the overflow level, and then only if a further accumulation of positive net earnings would raise the cash reserve above it. Furthermore, it can be shown that, with an overflow policy, the cash reserve will eventually reach zero in finite time, and hence the firm will not survive forever.[13]

The first important result about the profit-maximizing policy is that the optimal withdrawal policy is an overflow policy, for a suitably chosen overflow level, b. This characterization has an important corollary, namely, a profit-maximizing firm will fail in finite time! Although a rigorous proof of this requires the use of advanced mathematical techniques, some heuristic remarks may make it plausible. Recall that the cash reserve is not directly productive, but it is indirectly productive in that it provides insurance against a run of bad luck that would lead to failure. The larger the cash reserve, the greater is the protection that it provides, and hence the greater is the expected value of future withdrawals. However, this indirect (insurance) productivity of the cash reserve is subject to decreasing returns. The larger the cash reserve, the smaller is the marginal benefit - in terms of expected future profit – from a further increase in the reserve, compared to the benefit of an immediate withdrawal. On the other hand, it can be shown that, in order to have a positive probability of surviving forever, the firm must accumulate a larger and larger cash reserve, without bound; but beyond a certain point such accumulation is no longer profitable.

I turn now to the other part of the entrepreneur's policy, namely, the control of the drift and volatility of earnings. We shall see that there is a marked contrast with the behavior of the indebted investor of the previous section. The optimal policies are similar in that entrepreneur always chooses the maximum possible drift for any given volatility; in other words, he always chooses a point on the upper boundary of the feasible set. But here the similarity ends. First, the optimal control lies between the pair

that maximizes the ratio of the drift to the volatility, i.e., the yield/risk ratio – the familiar point (m_0,v_0) – and the pair that maximizes the yield - the point (m^*,v^*) in the figure. Thus the optimal control is always "efficient" in the sense of standard portfolio theory. Second, the optimal volatility is an increasing function of the cash reserve, which is just the opposite of the survival-maximizing control policy of the indebted investor. Third, it follows that the optimal drift is also an increasing function of the cash reserve. This direction of monotonicity of the drift and volatility is illustrated by the arrow in the upper boundary of the feasible set in the figure.

5. *Survival and Selection*

Although economists may admit that,a priori, different firms may have different goals and behaviors, it is often argued that market forces will tend to weed out all but the firms that display a certain specific behavior. In particular, it has been argued that firms that maximize profits (and are the most "efficient") will have the greatest chances for survival, and hence in the long run most of the existing firms will be maximizing profits.[14] I shall call the proposition in this more specific form the Neoclassical Selection Hypothesis (NSH). Thus Milton Friedman has written in his Essays in Positive Economics:

> ... under a wide range of circumstances individual firms behave as if they were seeking rationally to maximize their expected returns ... and had full knowledge of the data required to succeed in this attempt ... unless the behavior of businessmen in some way or other approxi mated the behavior consistent with the maximization of returns, it seems unlikely that they would remain in business for long. (Friedman, 1953, pp. 21-22).

Although some authors have criticized the NSH – notably Sidney Winter (see, e.g., Winter, 1982) – I think that it is fair to say that the issue has not received a thorough and

systematic treatment. It is, of course, tautological that in the long run most of the existing firms will be those with the largest probability of survival, but the results of the previous two sections might cast some doubt on the validity of the more specific NSH. After all, we have seen that (1) firms that maximize (expected) profits are sure to fail in finite time, whereas there are policies that produce a positive profit and yet have a positive probability of surviving forever . If both kinds of firms are present initially, then after a long time most of the surviving firms will be of the latter kind, and hence not be profit maximizers.

But, one might say, won't the competition for investment funds force each firm to pay the highest possible return? If this were the case, all existing firms would be profit-maximizers, and all would fail in finite time, although some would last longer than others.

Nevertheless, Prajit Dutta and I have argued that the NSH can be wrong under quite plausible conditions. I can only very briefly summarize the argument here. Successive cohorts of investors and potential new firms enter the capital market every period. The investors want to maximize their expected discounted returns. The firms are diverse in technology and behavior. In particular, if we call the maximum expected rate of return that a firm can offer its potential rate of return, then firms are diverse in their potential rates of return. At the market equilibrium, all new firms that are actually financed offer the same rate of return to outside investors. Firms whose potential rates of return are less than the equilibrium rate will not be financed. Firms whose potential rate exactly equals the market rate can be financed, but must maximize profit to do so, and hence will fail in finite time. A "supramarginal" firm, whose potential rate exceeds the market rate, will have some freedom to pursue goals other than profit maximization, for example, the goal of survival, in which case it will have a positive probability of surviving forever. The result is that, if each cohort contains some supramarginal "survivalists," then

as time goes on, the relative frequency of profit-maximizing firms becomes negligible.

What is happening, of course, is that the supramarginal firms are capable of earning "rents," which they can use in pursuit of various goals, e.g., survival. In this sense the capital market is "imperfect," in contrast to a so-called "perfect" capital market in which there would be an infinite supply of firms that offer the highest possible rate of return. I don't know what kind of capital market Milton Friedman had in mind, but I have no doubt in my own mind that the "imperfectly competitive" case is the normal one. In fact, it is quite common to call a market such as the one for investment funds I have described here "competitive," since the firms, although finite in number, are price-takers in the market for investment funds.

6. *Concluding Remarks*

The models I have described here are admittedly special, and need to be generalized in various directions. First, we need richer models of the technology and of the capital market. Second, my picture of "failure" is too stark. There are various forms and gradations of bankruptcy. There are also other crises that the firm may confront, such as hostile takeovers. Many of these crises may result in a change of management, but not in the disappearance of the firm itself.

Third, these considerations lead naturally to another set of issues that concern the separation of ownership and management. The models I have discussed here are perhaps suitable descriptions of a firm with a single entrepreneur/manager who raises investment funds from outside lenders and/or investors who, however, have no control over the firm except to force it into bankruptcy when it runs out of cash. They are less suitable as descriptions of a publicly held firm with shareholders and a board of directors, and a management team reporting to them. In the latter case, we should

deal more directly with the agency problems that such a structure entails. Both casual observation and game-theoretic research suggest that the threat of dismissal may be an effective ingredient in a potentially long-lasting principal-agent relationship.[15]

Nevertheless, I hope that I have been able to communicate the idea that the survival motive has interesting implications for behavior under uncertainty, implications that sometimes differ radically from the implications of profit maximization. I also hope that I have raised doubts in your mind about whether the connection between profit-maximization and survival is as straightforward as it is assumed to be.

7. *Bibliographic Notes and References*

I have already noted that the early history of the analysis of the Gambler's Ruin problem is described fully by Hald (1990). Modern treatments for the case of discrete time can be found in many textbooks and treatises on random walks and Markov chains, e.g., Spitzer (1976). Likewise, the analysis of the ruin problem for the case of a controlled diffusion is well-known; I have relied here on Harrison (1985), who, however, prefers the term "controlled Brownian Motion."

Regarding the indebted survivalist, the optimal policy for the constant-size technology is easy to derive, although I cannot find a convenient published reference. My exposition of the results for the case of constant returns to scale is based on Majumdar and Radner (1991). Further results about survival under production uncertainty under various conditions have been derived by Majumdar and Radner (1992) and Mitra and Roy (1993).

The fact that the problem of profit maximization is not not well posed in the model of constant returns to scale has been pointed out by Radner and Shepp (1995). The characterization of the profit-maximizing policy in the constant-size model was first given by Dutta and

Radner in 1993. They dealt with the case in which the set A of feasible drift-volatility pairs is compact and strictly convex; the rate of withdrawal can be unbounded (as was implicitly assumed here), or bounded above by some exogenously given number. A more explicit characterization of the optimal policy can be derived if the set A is finite; in this case it is sufficient to consider the extreme points of the convex hull of A; see Radner and Shepp (1995). It is conjectured that the analysis can be extended to the case in which A is only assumed to be compact.

Section 6 is also based on Dutta and Radner (1995). That analysis was inspired, in part, by the now extensive theoretical literature on the evolution of strategies in games, and more particularly by Dutta and Sundaram (1992) and Blume and Easley (1992).

ENDNOTES

[1] I would like to thank P.K. Dutta, H.–L. Huynh, E. Kalai, P.B. Linhart, and M.K. Majumdar for comments on a previous version of this paper. This paper was prepared while I was still at AT&T Laboratories. However, the views expressed here are those of the author, and not necessarily those of AT&T Bell Laboratories.

[2] For exceptions, see the references in Dutta and Radner (1993).

[3] See, e.g., Radner (1981).

[4] See Section 5 below.

[5] In the context of the so-called principal-agent problem, the agent's "stock" may be in units of some measure of performance, such as internal accounting profits.

[6] I make no attempt to be rigorous here. See, e.g., Breiman (1968) and Harrison (1985) for details.

[7] Some authors use the term "controlled Brownian motion." See, e.g., Harrison (1985).

[8] Under the assumptions of our model, the cash reserve must be a continuous function of time, and so it cannot become negative without actually taking on the value of zero.

[9] Technically, we are dealing with a stationary Markovian dynamic programming problem.

[10] To be precise, for the rest of this paper it will be assumed (unless stated otherwise) that the set A is closed, bounded and strictly convex, with smooth boundary, that volatility is everywhere positive in A, and that there is a point in A with positive drift.

[11] See Koopmans (1986).

[12] Note that it is implicitly assumed in (4.1) that the cash reserve earns zero interest. However, this assumption could be relaxed.

[13] In fact, it can be shown that the expected time to failure is finite.

[14] See, for example, Alchian (1950), Friedman (1953), and other references cited in Dutta and Radner (1993).

[15] See Dutta and Radner (1995) for a survey.

FIGURE CAPTIONS

Figure 1

A Cumulative Earnings Process

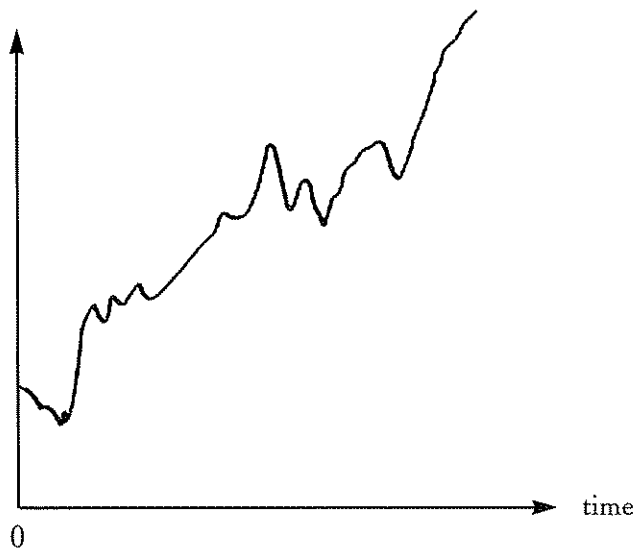


Figure 2

Failure

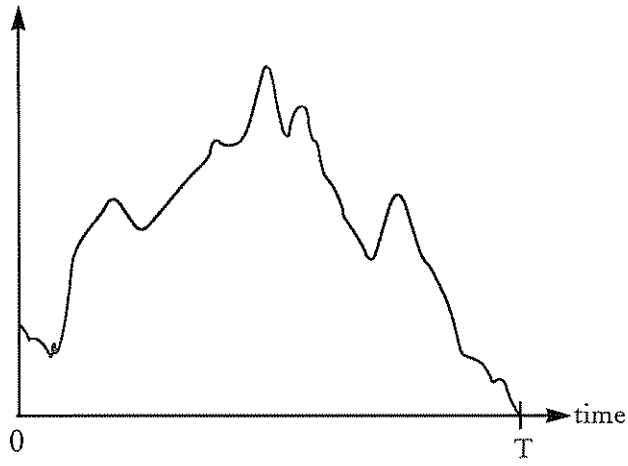


Figure 3

Probability of Survival

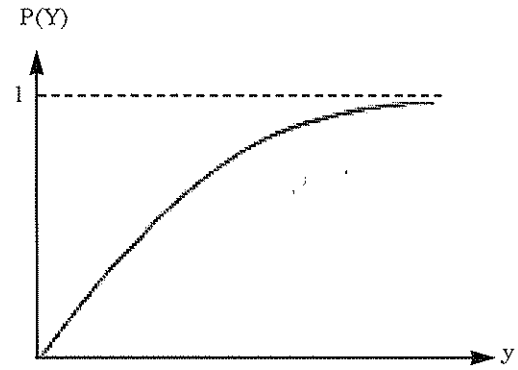


Figure 4

The Set of Feasible Drift-Volatility Pairs

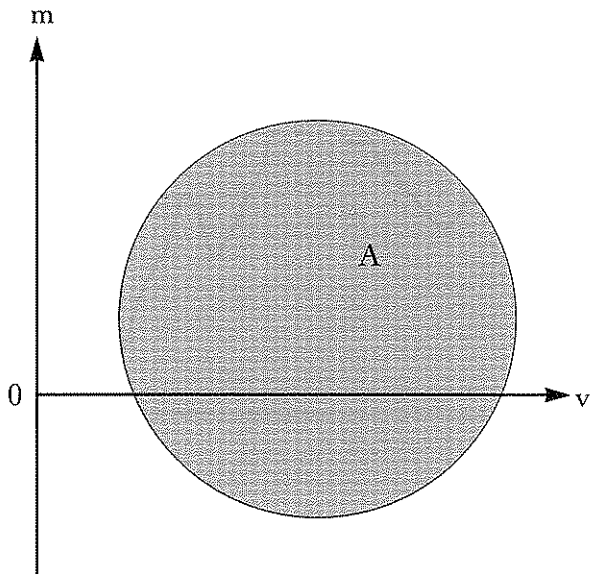


Figure 5

Constant Size:
Optimal Drift and Volatility

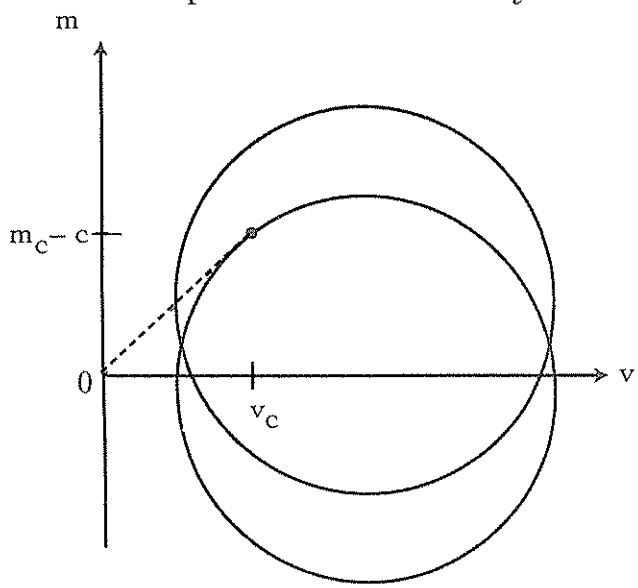


Figure 6

Constant Size:
Optimal Drift and Volatility for c=0
and c=m*

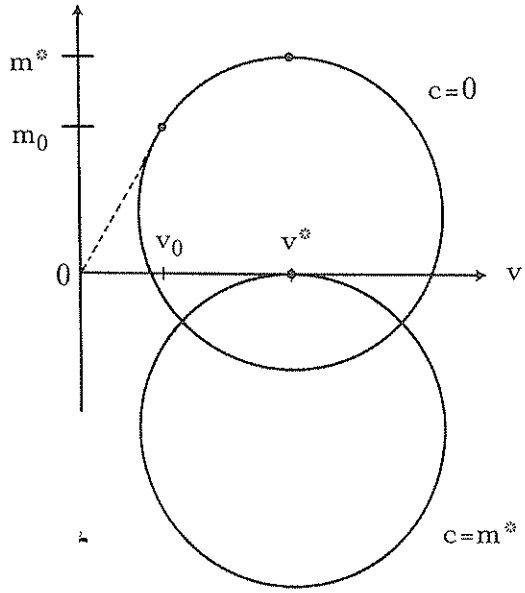


Figure 7
CRTS: Locus of Optimal Drift-Volatility Pairs

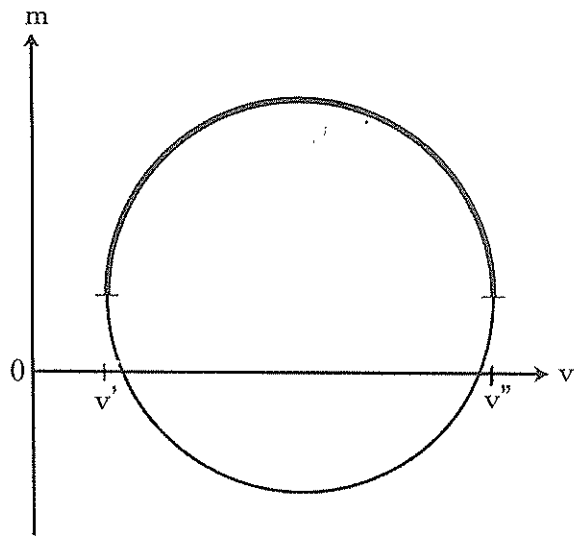


Figure 8
CTRS: Optimal Control Policy

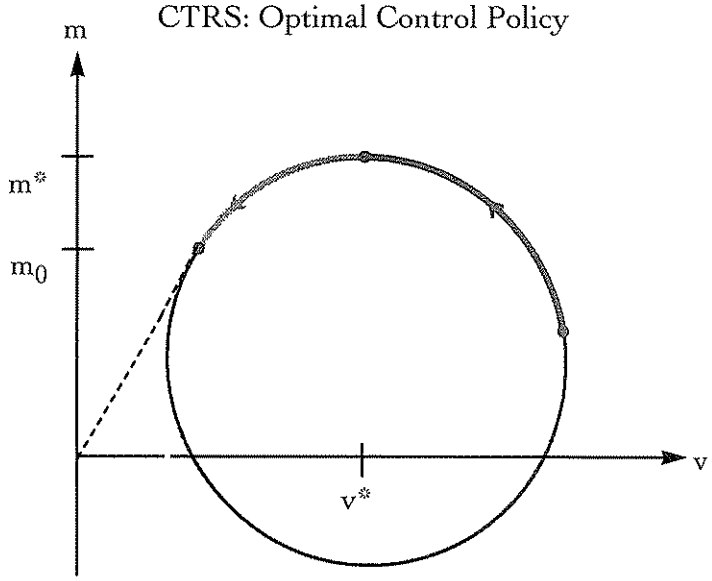


Figure 9
CRTS: Optimal Probability of Survival

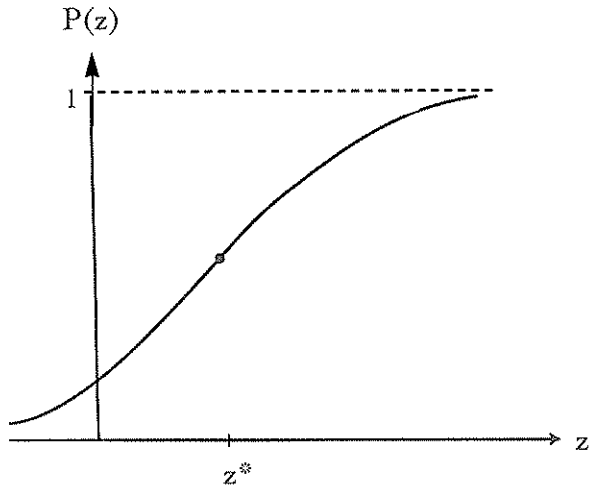


Figure 10
Overflow Withdrawal Policy

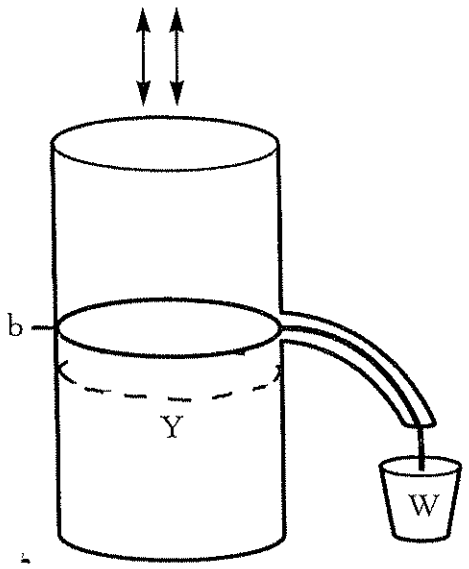


Figure 11

Cash Reserve:
No Withdrawals

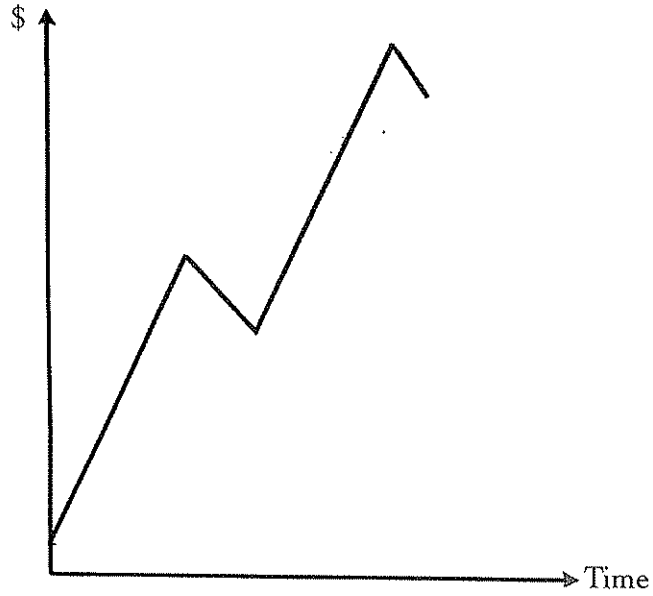


Figure 12

Cash Reserve:
Overflow Withdrawal Policy

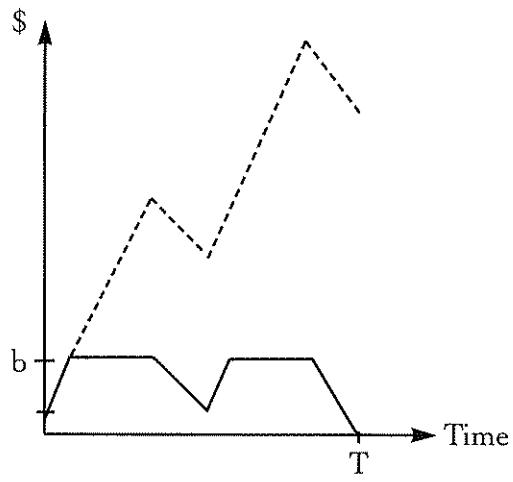
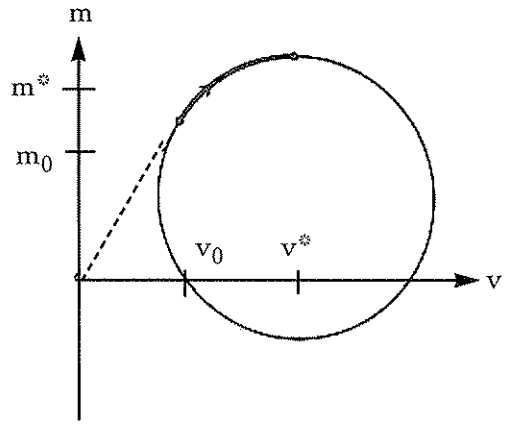


Figure 13

Profit-Maximizing Control Policy



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