Notes on Economic Welfare [Preliminary and Incomplete Draft]

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1 Introduction

[To be written.]

2 Naive Surplus Theory

2.1 Consumers who buy 0 or 1 unit

I introduce the concept of surplus in one of the simplest models of demand. [This is the model underlying (Bakos and Brynjolfsson, 1999).] Consider a very large population of K consumers facing a market for a single good. Each consumer will buy either one or zero units of this good (per period), depending on the price. Each consumer has a *willingness-to-pay (WTP)*, or "reservation price" for one unit, and will buy one unit if and only if his or her WTP exceeds the price. Let F_K denote the *cumulative distribution function* (cdf) of the WTP in the population, i.e.,

$$F_K(p) \text{ is the fraction of the population whose WTP } \leq p. \tag{1}$$

The *demand function* for the good is therefore

$$D_K(p) = 1 - F_K(p), \tag{2}$$

i.e., if the price is p, the demand for the good equals the fraction of the population whose WTP $> p$. I shall assume that the cdf F_K has the following properties:

$$F_K(0) = 0, \quad F_K(p) \text{ is nondecreasing}, \quad F_K(v_{\max}) = 1 \text{ for some } v_{\max} > 0. \tag{3}$$

Thus the demand function has the usual property that it is everywhere decreasing (or constant). In addition,

$$D_K(0) = 1, \qquad D_K(v_{\max}) = 0. \tag{4}$$

Since there are only finitely many consumers, the cdf F_K is a "step function," i.e., it is constant except at a finite number of points. The set of these points is exactly the set of different values of the WTP in the population of consumers. At such a point w, the cdf F_K jumps up by an amount equal to the fraction of consumers with a WTP equal to w. The demand function jumps down correspondingly at the same set of point.

It is sometimes convenient to approximate the above model by a "smooth" one. Imagine that the population of K consumers is a random sample from a population for which the cdf of the WTP is F. Assume that

$$F \text{ has a derivative, denoted by } f, \tag{5}$$

$$f(w) \quad > \quad 0 \text{ for } 0 < w < v_{\max}. \tag{6}$$

The function f is called the *density function* corresponding to F. Thus, for small Δ, the fraction of the population whose WTP is between w and $w + \Delta$ is approximately $f(w)\Delta$. As is well known, as K increases without bound, the sequence of cdfs, F_K, approaches the cdf F as a limit (see a standard text on probability theory or statistics). We therefore adopt, as an approximation, the (fictional) model of the consumers as a *continuum*, with the convention that *the total mass of consumers is unity.* Corresponding to (4), in this model, the demand function is given by $D(p) = 1 - F(p)$. With our assumption about F, the demand function D is strictly decreasing for $0 < w < v_{\max}$.

If a person with WTP w consumes one unit of the good, we shall say that his or her (incremental) *gross value* derived from this consumption is w. Hence, speaking heuristically, for small Δ, the consumers whose WTP is between w and $w + \Delta$ derive a total gross value that is approximately $wf(w)\Delta$. Hence the total gross value derived by all consumers whose WTP is greater than p is

$$\tau(p) = \int_p^{v_{\max}} wf(w)dw. \tag{7}$$

Correspondingly, if a consumer with WTP w buys one unit of the good at price p, we shall say that his or her *surplus* from the transaction is $(w-p)$. Hence, speaking heuristically, for small Δ, the consumers whose WTP is between w and $w+\Delta$ have a surplus that is approximately $(w-p)f(w)\Delta$, provided that $w > p$. Hence, if all consumers face the same price, p, then the total *consumer surplus at that price* is

$$\gamma(p) = \int_p^{v_{\max}} (w - p)f(w)dw. \tag{8}$$

From this last equation, and the definition of D, is straightforward to verify that

$$\gamma'(p) = -D(p). \tag{9}$$

2.2 Consumers who buy multiple units

If a person buys more than one unit of a good, it is intuitively plausible that the WTP for successive additional units will be decreasing as the total number consumed increases. (For the time being, assume that the quantity purchased must be a nonnegative integer.) Pursuing this line of thinking, given the price of a good, assume that a person will want to increase his or her consumption as long as the incremental willingness to pay for the next additional unit (strictly) exceeds the price. (Of course, if the WTP for the first unit is less than the price, then the person will consume zero units.) Let w_{ix} denote person i's incremental WTP for the "last" unit of the good if his or her total consumption of the good is x. Extending the idea of value from the previous subsection, we suppose that the total gross value to i of consuming q units is

$$W_i(q) = \sum_{x=1}^{q} w_{ix}. \tag{10}$$

Now suppose that consumers can buy fractional units of the good, i.e., the quantity bought can be any nonnegative real number. Adapting the notation of the previous paragraph to this case, let $W_i(q)$ denote the consumer's WTP for a quantity q. I make the following assumptions about the function W_i:

The derivative $W_i^{'}(x) = w_i(x)$ exists and is continuous for $0 < x < \mu_i$,

$$\begin{aligned} w_i(x) &> 0 \text{ and is strictly decreasing for } 0 \le x < \mu_i, & (11)\\ w_i(x) &= 0 \text{ for } x \ge \mu_i. & (12) \end{aligned}$$

[In particular, if $\mu_i = 0$, then $w_i(x) = 0$ for all $x \ge 0$.] Thus, corresponding to equation (10) we have

$$W_i(q) = \int_0^q w_i(x)dx. \tag{13}$$

With these assumptions,

$$\begin{aligned} W_i(q) &> 0 \text{ and is strictly concave and increasing for } 0 < q < \mu_i, & (14)\\ W_i(q) &= W_i(\mu_i) \text{ for } q \ge \mu_i. & (15) \end{aligned}$$

The value $w_i(x)$ is sometimes called the consumer's *marginal willingness-to-pay at quantity* x. Since the marginal WTP is zero for $q \ge \mu_i$, the quantity μ_i is called the consumer's *point of satiation* for the good. To simplify the exposition, I make the further assumption that

$$w_i \text{ is continuous at } x = \mu_i. \tag{16}$$

(In what follows, the reader should think about the possible implications of a discontinuity of the marginal WTP at the point of satiation.)

Now assume that the the consumer's "net utility" if he consumes a quantity q and pays a (per-unit) price p is

$$U_i(q,p) = W_i(q) - pq. \tag{17}$$

Given the price, the consumer demands a quantity, $D_i(p)$, that maximizes his net utility. For a fixed price p, the function $U_i(q,p)$ is strictly concave on the interval $(0, \mu_i)$, and linear and strictly decreasing for $q \geq \mu_i$. (The reader should draw a figure.) The partial derivative of $U_i(q,p)$ with respect to q is

$$\frac{\partial U_i(q,p)}{\partial q} = W_i'(q) - p = w_i(q) - p. \tag{18}$$

Hence

$$w_i(0) < p \text{ implies } D_i(p) = 0, \tag{19}$$

whereas

$$w_i(0) > p \text{ implies that } D_i(p) \text{ is the solution of } w_i[D(p)] = p. \tag{20}$$

The last two conditions determine the consumer's demand function, D_i. (What can happen if the marginal WTP is discontinuous at the point of satiation?) Define

$$v_i = \text{ the smallest price } p \text{ at which } D_i(p) = 0. \tag{21}$$

We are now in a position to characterize the consumer's *surplus*, $S_i(p)$, at price p. The surplus is defined as

$$S_i(p) = W_i[D_i(p)] - pD_i(p) \tag{22}$$

Case 1. $w_i(0) > p$. Make the change of variable,

$$x = D_i(y),$$

so that

$$\begin{aligned} dx &= D_i'(y)dy, \\ x &= 0 \Rightarrow y = v_i, \\ x &= D_i(p) \Rightarrow y = p, \end{aligned}$$

By the optimality condition (20),

$$w_i(x) = w_i[D_i(y)] = y. \tag{23}$$

Hence

$$W_i[D_i(p)] = \int_0^{D_i(p)} w_i(x)dx = -\int_p^{v_i} yD_i'(y)dy.$$

Integrating by parts, we get

$$\int_p^{v_i} yD_i'(y)dy = v_iD_i(v_i) - pD_i(p) - \int_p^{v_i} D_i(y)dy.$$

Hence, recalling that $D_i(v_i) = 0$,

$$S_i(p) = \int_p^{v_i} D_i(y)dy. \tag{24}$$

This expression for the individual consumer's surplus can be interpreted geometrically as the area under his demand curve and above the price axis, from p to v_i.

Case 2. $w_i(0) \leq p$. In this case, the consumer's demand is zero for all prices $\geq p$, and so the surplus is zero, and the above characteriation of the surplus is also valid for this case.

The *total consumer surplus* is defined as the sum of the individual consumer surpluses:

$$S(p) = \sum_i S_i(p) = \sum_i \int_p^{v_i} D_i(y)dy. \tag{25}$$

Define

$$v = \max\{v_i\},$$

and note that, for all i,

$$\begin{aligned} p &> v_i \Rightarrow D_i(p) = 0, \text{ so that} \\ \int_p^{v_i} D_i(y)dy &= \int_p^{v} D_i(y)dy. \end{aligned}$$

Hence, from (25),

$$\begin{aligned} S(p) &= \sum_i \int_p^{v_i} D_i(y)dy & (26) \\ &= \int_p^{v} \left[\sum_i D_i(y)dy \right]. & (27) \end{aligned}$$

However, the total demand from all consumers at the price p is:

$$D(p) = \sum_i D_i(p). \tag{28}$$

Hence

$$S(p) = \int_p^{v} D(y)dy. \tag{29}$$

so we have, for the entire market, *the total consumer surplus is equal to the area under the market demand curve and above the price axis, from p to v.* (See Figure 1.)

Let P denote the inverse demand function, i.e., $P(x)$ is the solution of

$$D[P(x)] = x. \tag{30}$$

If we redraw Figure 1 with quantity on the horizontal axis and price on the vertical axis, we see that the consumer surplus is now the area between the graph of the inverse demand function and the horizontal line $y = p$, and between zero and the quantity demanded. (See Figure 2.) Define

$$\theta(q) = \int_0^q P(x)dx, \tag{31}$$

$$\Gamma(q) = S[P(q)]. \tag{32}$$

We may call $\theta(q)$ the *gross value of consuming the quantity q*. With this notation, if the total demand is q and the corresponding price is $P(q)$, then the consumer surplus is.

$$\Gamma(q) = \theta(q) - qP(q), \tag{33}$$

i.e., it is the difference between the gross value and the total cost to the consumers.

2.3 Efficient Output

With the machinery of the previous section, we can now address the question: what would a "socially efficient" level of output of the good be? Of course, the answer will depend on the cost of production. Suppose that the society as a whole has available a method of production such that the cost of producing an output q is

$$C^T(q) = c_0 + C(q). \tag{34}$$

Here c_0 is the *fixed cost*, independent of output, and $C(q)$ is the *variable cost*. By definition, $C(q) = 0$. I shall also assume that $C(q)$ is an increasing, convex, differentiable function. The *net value* to society of producing and consuming a quantity q is defined to be the difference between the gross value and the cost. The *total surplus* is defined to be the difference between the gross value and the variable cost.

$$\sigma(q) = \theta(q) - C(q). \tag{35}$$

A quantity is *socially efficient* (in the short run) if it maximizes the total surplus. From (31),

$$\theta'(q) = P(q), \tag{36}$$

which is decreasing; hence θ is a concave function. Furthermore, the variable cost function is convex, so $-C$ is concave. Hence σ is concave. Assume that the socially efficient output is strictly positive. The first-order-condition (FOC) for a maximum is

$$\sigma'(q) = \theta'(q) - C'(q) = 0,$$

or

$$P(q) = C'(q), \tag{37}$$

which gives rise to the recipe, "price equals marginal cost." [Of course, the recipe is valid only if the FOC does indeed determine the socially efficient output. The reader should consider other possibilities.] Let $\hat{q}$ the socially efficient output.

The total surplus can represented graphically as the area between the inverse demand curve and the marginal cost curve, and between zero and the socially efficient output. (See Figure 3.)

2.4 Monopoly is inefficient

Consider a market with consumers as in the previous sections, and a monopolist with a variable cost function C, as in the previous section. If the firm produces and sells a quantity q, its revenue will be $qP(q)$, and its variable cost will be $C(q)$. Define the *producer surplus* to be

$$\psi(q) = qP(q) - C(q). \tag{38}$$

(It's profit will equal its surplus minus its fixed cost.) Hence, by (33) and (35), the sum of the producer and consumer surpluses equals the total surplus.

The first-order condition determining the output at which the firm's profit or surplus is maximized is

$$\psi'(q) = qP'(q) + P(q) - C'(q) = 0. \tag{39}$$

Call the profit-maximizing output q^*, and assume that it is strictly positive. Hence

$$P(q^*) = C'(q^*) - q^*P'(q^*). \tag{40}$$

Since $q^*P'(q^*) < 0$, at the profit-maximizing quantity the price exceeds marginal cost. *Hence, compared to the socially efficient outcome, the monopolist's quantity is too low and the price is too high.* The difference between the maximum total surplus and the total surplus at the monopolist's quantity is called the *dead weight loss.* This is illustrated in Figure 4.

2.5 Competition is efficient

Another recipe common to our modern economic cookbook is that "competition is efficient." I shall now show that this recipe is correct in a particular elaboration of our model. (This is not meant to imply that the recipe is universally useful.) Suppose that there are a number of firms that produce the good in question, numbered $1, ..., J$. Roughly speaking, the market is said to be perfectly (or purely) competitive if each firm assumes that varying its output will not affect the market price. This assumption is approximately plausible if there are many firms, and no firm is "large" relative to the the other firms. More precisely, a *competitive equilibrium* of the market for this good is a vector, $(p, q_1, ..., q_J)$, such that

(1) for every firm j, its output q_j maximizes its profit, *given the price* p;

(2) demand equals supply, i.e.,

$$D(p) = \sum_{j=1}^{J} q_j. \tag{41}$$

Suppose that for every firm j, it's cost function, C_j^T, satisfies the assumption of the previous section (about the monopolist), and that the total demand function for the good also satisfies the previous assumptions. If firm j produces an output q_j, and the price is p, then its profit is

$$\pi_j(q_j) = pq_j - C_j^T(q_j) = pq_j - c_{0j} - C_j(q_j).$$

Assume that its optimal output is determined by the FOC,

$$\begin{aligned} \pi_j'(p) &= p - C_j'(q_j) = 0, \\ C_j'(q_j) &= p, \quad j = 1, ..., J. \end{aligned} \tag{42}$$

These last J equations, together with (41), form a system of $(J+1)$ simultaneous equations in $(J+1)$ unknowns, i.e., the outputs of the firms and the market price. In what follows, I shall assume that there is at least one solution to this system.

In order to apply the analysis of Section 2.3, we need to identify the "global cost function" that is implied by the individual cost functions of the J firms. Suppose that society wants to produce a quantity q of the good, using the production capabilities of the J firms. A reasonable notion of "efficiency" would require that it do so at minimum cost, or equivalently, at minimum variable cost. Accordingly, define

$$C(q) = \min \sum_{j=1}^{J} C_j(q_j) \text{ subject to } \sum_{j=1}^{J} q_j = q. \tag{43}$$

Call the resulting vector $(q_1, ..., q_J)$ an *efficient allocation of the total output*, q, among the J firms. Assume that in an efficient allocation every firm produces strictly positive output. It follows that at an efficient allocation all the firms must have the same marginal cost, say λ, i.e.,

$$C_j'(q_j) = \lambda, \quad j = 1, ..., J. \tag{44}$$

To see this, suppose, to the contrary, that there are 2 firms, say j and k, such that $C_j'(q_j) < C_k'(q_k)$. In this case, total cost can be reduced by reducing k's output by some small amount and increasing j's output by the same amount. Hence the original allocation was not efficient.

If we add the constraint,

$$\sum_{j=1}^{J} q_j = q, \tag{45}$$

then (given the total quantity q) we have a system of $(J+1)$ simultaneous equations in $(J+1)$ unknowns, i.e., the J outputs of the firms and λ. For the purpose of this analysis, denote the solutions by

$$\begin{aligned} q_j &= Q_j(q), \quad j = 1, ..., J, \\ \lambda &= \Lambda(q), \end{aligned}$$

to emphasize their dependence on the total output, q. Hence the global cost function is given by

$$C(q) = \sum_{j=1}^{J} C_j[Q_j(q)]. \tag{46}$$

From Section 2.3, the socially efficient output is characterized by "marginal cost equals price," or

$$C'(q) = P(q), \tag{47}$$

where, as before, P is the inverse demand function. Differentiating both sides of (46) with respect to q, we get

$$C'(q) = \sum_{j=1}^{J} C_j'[Q_j(q)]Q_j'(q). \tag{48}$$

From (44),

$$C_j'[Q_j(q)] = \Lambda(q), \quad j = 1, ..., J, \tag{49}$$

so from the previous equation,

$$C'(q) = \Lambda(q) \sum_{j=1}^{J} Q_j'(q).$$

Now observe that

$$\sum_{j=1}^{J} Q_j(q) = q,$$

so that

$$\sum_{j=1}^{J} Q_j'(q) = 1,$$

and hence, by (49),

$$C'(q) = \Lambda(q) = P(q). \tag{50}$$

Therefore, from (44) and (45),

$$C_j'(q_j) = P(q), \quad j = 1, ..., J, \tag{51}$$

$$\sum_{j=1}^{J} q_j = q \tag{52}$$

Compare these equations with the equations (42) and (41) for a competitive equilibrium:

$$\begin{aligned} C_j'(q_j) &= p, \quad j = 1, ..., J, \\ \sum_{j=1}^{J} q_j &= D(p). \end{aligned}$$

Since $q = D(p)$ and $p = P(q)$, the conditions for a competitive equilibrium are equivalent to those for a socially efficient output and price.

2.6 Example: An application to bundling information goods

This example is based on [Bakos and Brynjolfsson, 1999]. Consider a market for N "information goods." For each good, a consumer purchases either 1 or 0 units, as in Sec. 2.1. Let $V_n$denote a consumer's willingness-to-pay (WTP) for good n. *[Warning! The notation used in this subsection is not entirely consistent with that used in the preceding sections.]* Assume that the variables V_n are independently and identically distributed (IID) in the population of consumers. This population is "very large," so we represent it by a continuum with "unit mass," and the total consumption of a good as unity if every consumer buys that good. [Cf. the "smooth" model in the second part of Sec. 2.1.] Thus the total consumption of a good equals the fraction of consumers who buy it.

As a consequence of these assumptions, the fraction of consumers for which $V_1 \leq v_1, V_2 \leq v_2, ..., V_N \leq v_N$, is given by

$$\Pr\{V_n \leq v_n : n = 1, ..., N\} = \prod_{n=1}^{N} F(v_n), \tag{53}$$

where F is a cumulative distribution function (cdf). For convenience of exposition, assume that F is differentiable.

Suppose now that the variable cost of producing these goods is zero, at any level of output. (This may be a good approximation for many information goods.) From Sec. 2.3, for each good the socially efficient price and output are 0 and $D(0) = 1$, respectively. By (7) the corresponding maximum total surplus for the sale of a single good is

$$\tau(0) = \int_0^{v_{\max}} v f(v) dv = m, \tag{54}$$

which is the mean (mathematical expectation) of the distribution F. The maximum total surplus for all N goods together is thus Nm.

Suppose further that the N goods are provided by a monopolist, who will be denoted by the acronym "Mon." Since variable cost is zero, Mon's surplus is equal to his revenue. We shall consider the question: which is better for Mon, to sell the goods separately or in a single "bundle"? We first consider the case in which Mon sells the goods separately. Let p_1 and q_1 denote Mon's optimal price and output for a single good, respectively. Under plausible assumptions, about F, $p_1 > 0$ and $q_1 < 1$, and Mon's surplus will be strictly less than the maximum, i.e., in the notation of Sec. 2.4, $\psi(q_1) < m$, or equvalently,

$$\psi(q_1) = km, \qquad 0 < k < 1. \tag{55}$$

Thus, if Mon sells the N goods separately, his total surplus will be

$$N\psi(q_1) = Np_1 q_1 = Nkm. \tag{56}$$

Figure 1 illustrates the two surpluses for a single good with monopoly pricing. The maximum total surplus is the area under the demand curve between a zero price and a a price equal to $v_{\max}$, whereas Mon's surplus is the area of the rectangle.

An alternative marketing strategy for Mon is to sell all of the goods in a single bundle, at a single price, say ϕ, per bundle. The WTP of a (random) consumer is

$$S_N = V_1 + ... + V_N,$$

and the consumer will buy the bundle if $S_N > \phi$, or equivalently, if $\bar{W}_N > \bar{p}_N$, where

$$\bar{W}_N = \frac{S_N}{N}, \qquad \bar{p}_N = \frac{\phi}{N}. \tag{57}$$

Note that $\bar{W}_N$ is the average WTP *per good* in the bundle, and $\bar{p}_N$ is the corresponding average price per good (APPG). Let F denote the cdf of $\bar{W}_N$, and let $D_N(\bar{p}_N)$ be the demand for bundles if the APPG is $\bar{p}_N$; then

$$D_N(\bar{p}_N) = 1 - F_N(\bar{p}_N). \tag{58}$$

The mean (mathematical expectation) of $\bar{W}_N$ is the same as the mean of every V_n, namely m. The variance of $\bar{W}_N$ is s^2/N, where s^2 is the variance of every V_n. As we shall see, by the "Law of Large Numbers," the distribution of $\bar{W}_N$ is more concentrated about m than that of the each individual V_n. This is illustrated in Fgure 5. The corresponding total, producer, and consumer surpluses are illustrated in Figure 6, which suggests that, for large N, the average producer's surplus per good is close to the maximum total (social) surplus, and the consumer surplus is close to zero. I shall show below that this is the case. More precisely, I shall demonstrate the following proposition.

Proposition 1 *For any $\epsilon > 0$, there exists a number M_ϵ such that, for every $N \geq M_\epsilon$, if the monopolist sells the N goods as a single bundle, then the average producer surplus per good will be at least $m - \epsilon$.*

Before proving Proposition 1, I shall show that it implies a second proposition:

Proposition 2 *For large enough N, it is better for the monopolist to bundle N goods than to sell them separately.*

An easy corollary of this last proposition is:

Corollary 3 *For any bundle size, there is always a larger bundle size that is better for the monopolist.*

I first show that Proposition 1 implies Proposition 2. From Proposition 1, we see that $N \geq M_\epsilon$ implies that Mon can achieve a producer surplus of at least $N(m - \epsilon)$ by bundling the N goods. By (55), this is greater than the surplus attained by selling the goods separately if and only if $N(m - \epsilon) > Nkm$, or

$$(m - \epsilon) > km,$$

which is true for ϵ sufficiently small. This completes the proof of Proposition 2. To prove the Corollary, simply regard the bundle of N_1 goods as a single good, and then apply Proposition 2.

I now prove Proposition 1. As above, let s^2 denote the variance of the distribution of a single good, i.e.,

$$s^2 = \int_0^{v_{\max}} (v-m)^2 f(v)dv.$$

By Chebycheff's Inequality (see a standard probability or statistics textbook), for any $h > 0$,

$$\Pr\left\{|\bar{W}_N - m| > h\right\} \le \frac{s^2}{Nh^2}.$$

Hence, for any $h > 0$, there exists a number N_h such that, for all $N \ge N_h$,

$$\Pr\left\{|\bar{W}_N - m| > h\right\} \le h. \tag{59}$$

From this last statement,

$$\begin{aligned} F_N(w_i) &\le h, \qquad 0 \le w_i \le m-h, \\ F_N(w_i) &\ge 1-h, \qquad m+h \le w \le v_{\max}. \end{aligned}$$

Equivalently,

$$\begin{aligned} D_N(\bar{p}_N) &\ge 1-h, \qquad 0 \le \bar{p}_N \le m-h, \\ D_N(\bar{p}_N) &\le h, \qquad m+h \le \bar{p}_N \le v_{\max}. \end{aligned} \tag{60}$$

Now observe that the maximum WTP for a bundle is $Nv_{\max}$. For the purpose of this proof, let Δ denote the demand function for bundles of size N. By the definition of D_N,

$$D_N(\bar{p}_N) = \Delta(N\bar{p}_N). \tag{61}$$

Replacing D by Δ and $v_{\max}$by $Nv_{\max}$, the maximum total (social) surplus is from consuming bundles is

$$\int_0^{Nv_{\max}} \Delta(p)dp.$$

But the maximum total surplus is also the mean of the distribution of S_N, which is Nm, so

$$Nm = \int_0^{Nv_{\max}} \Delta(p)dp.$$

In the last integral, make the change of variable $\bar{p}_N = p/N$, or $p = N\bar{p}_N$; then the integral becomes

$$\int_0^{v_{\max}} \Delta(N\bar{p}_N)Nd\bar{p}_N,$$

which is equal to

$$N\int_0^{v_{\max}} D_N(\bar{p}_N)d\bar{p}_N.$$

Hence,

$$m = \int_0^{v_{\max}} D_N(\bar{p}_N) d\bar{p}_N,$$

To prove the proposition, it is not necessary to find the optimal monopoly price for bundles. It is sufficient to find a pricing method such that Mon's average surplus per good is within ϵ of m for sufficiently large bundle size. Figure 6 and (58) suggest that we take the average price per good to be close to, but less than, the mean WTP per good. Accordingly, let the APPG be

$$\bar{p}_N = m - h. \tag{62}$$

The bundle size, N, and the number h will be chosen appropriately. Mon's average surplus per good at this price is $(m-h)D_N(m-h)$. Hence we want to make the difference,

$$\delta = \int_0^{v_{\max}} D_N(\bar{p}_N) d\bar{p}_N - (m-h)D_N(m-h), \tag{63}$$

small. Write the above integral as a sum of three integrals:

$$\int_0^{v_{\max}} D_N(y)dy = \int_0^{m-h} D_N(y)dy + \int_{m-h}^{m+h} D_N(y)dy + \int_{m+h}^{v_{\max}} D_N(y)dy.$$

Also, write

$$(m-h)D_N(m-h) = \int_0^{m-h} D_N(m-h)]dy.$$

Hence, from (58),

$$\begin{aligned} \int_0^{m-h} D_N(y)dy - (m-h)D_N(m-h) &= \int_0^{m-h} [D_N(y) - D_N(m-h)]dy \\ &\leq (m-h)h; \end{aligned}$$

$$\int_{m+h}^{v_{\max}} D_N(y)dy \leq (v_{nax} - m - h)h.$$

In addition, since $0 \leq D_N(y) \leq 1$,

$$\int_{m-h}^{m+h} D_N(y)dy \leq 2h.$$

Combining the last three inequalities, we get

$$\delta \leq (m-h)h + 2h + (v_{\max} - m - h)h = (v_{\max} + 2 - 2h)h.$$

Now take h small enough so that $\delta < \epsilon$, and take N_h large enough so that (58) is satisfied, and the proposition is proved.

The statement containing the inequality (59) is a form of the *Weak Law of Large Numbers*. Proposition 1 can be generalized to include cases in which the variables V_n are neither independent nor identically distributed, provided that the Weak Law of Large Numbers is still valid. For example, it is sufficient that they be independent, their variances be bounded between two positive numbers, say $s^2 < t^2$, and their means, m_n, have a well-defined long-run average,

$$m = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} m_n.$$

[Note: Proposition 1 of Bakos and Brynjolfsson, 1999, is incorrect as stated, in that Assumption 2 is too weak. Correspodingly, Proposition 1A is also incorrect as stated.]

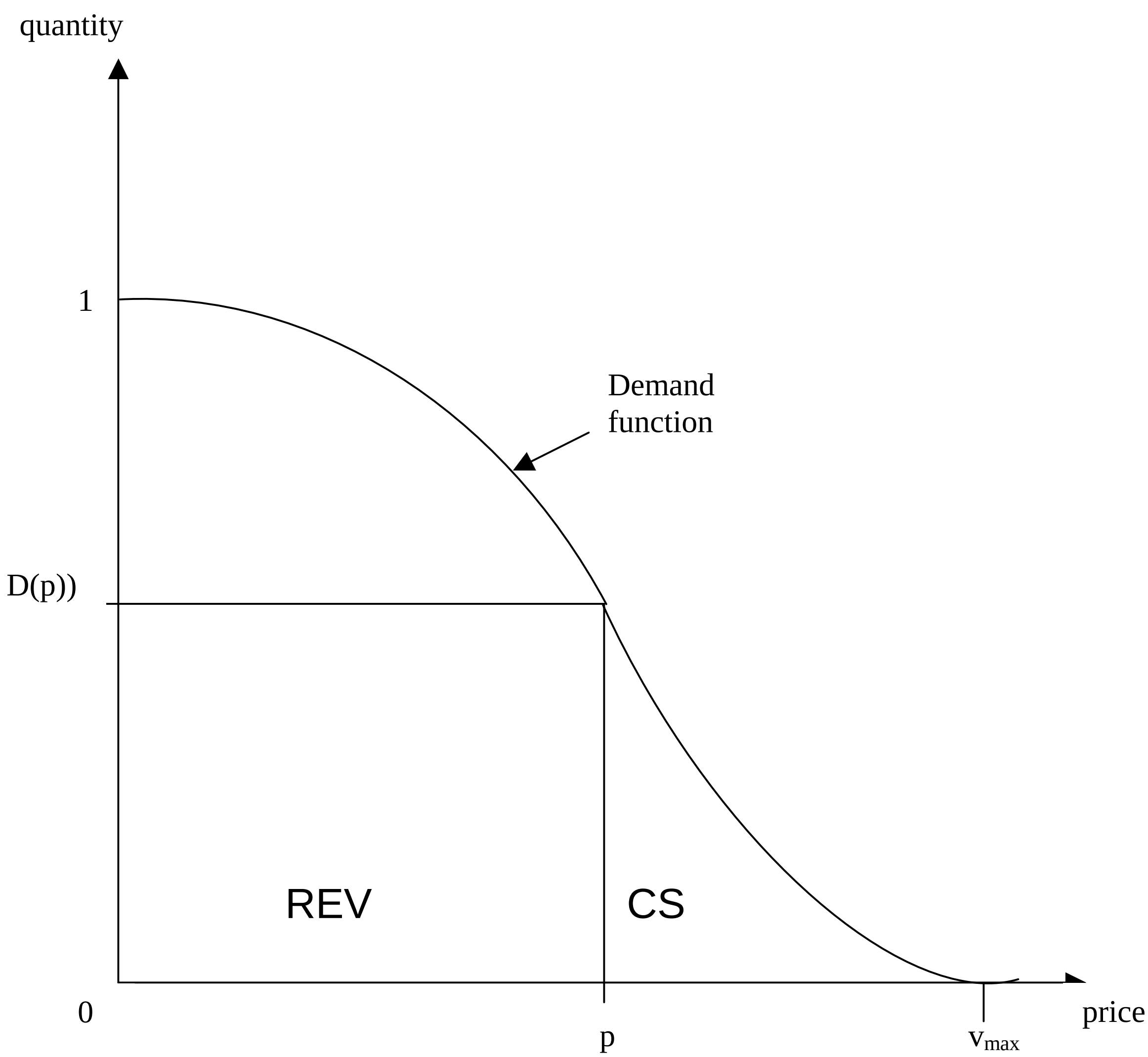

Figure 1

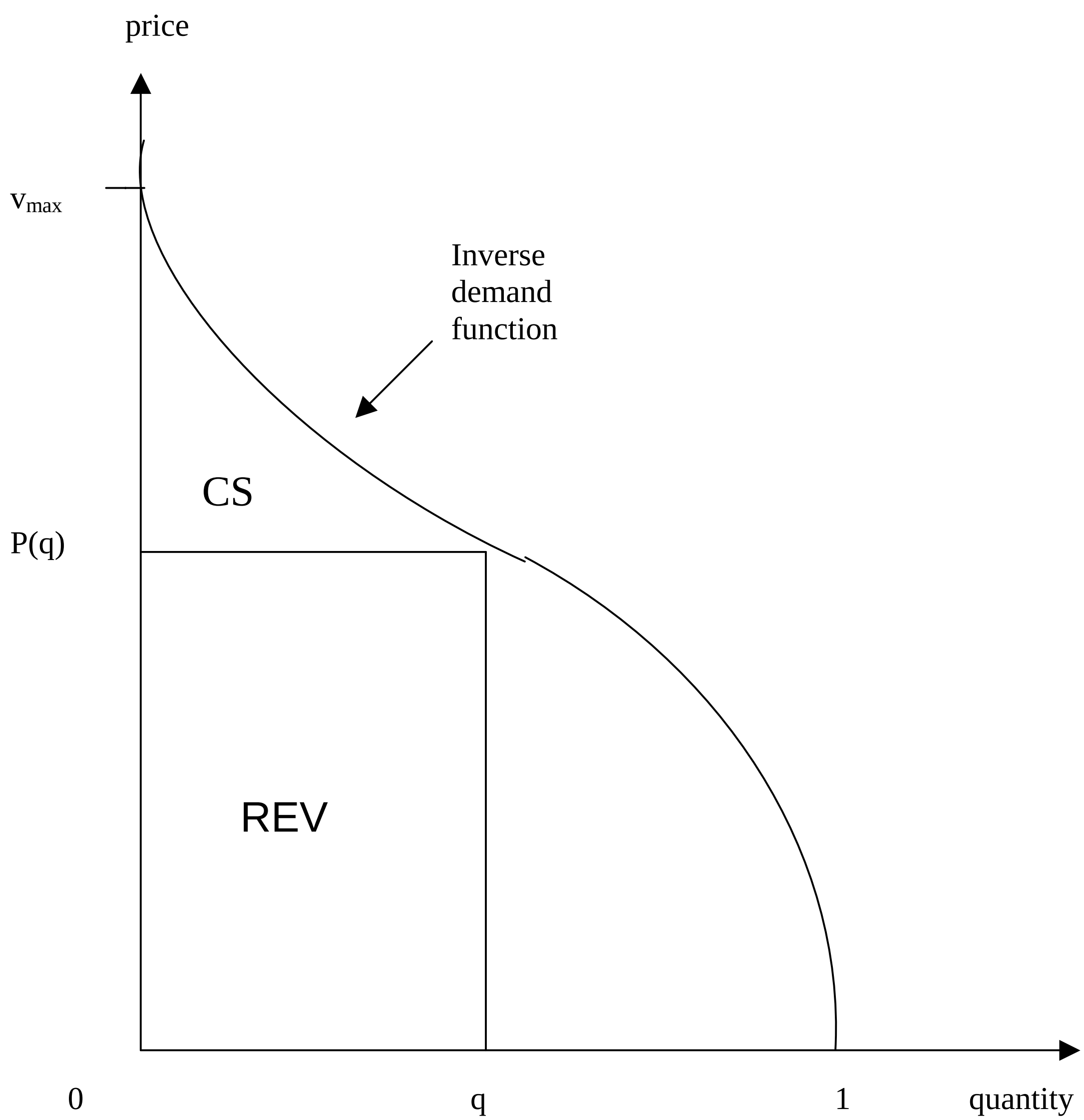

Figure 2

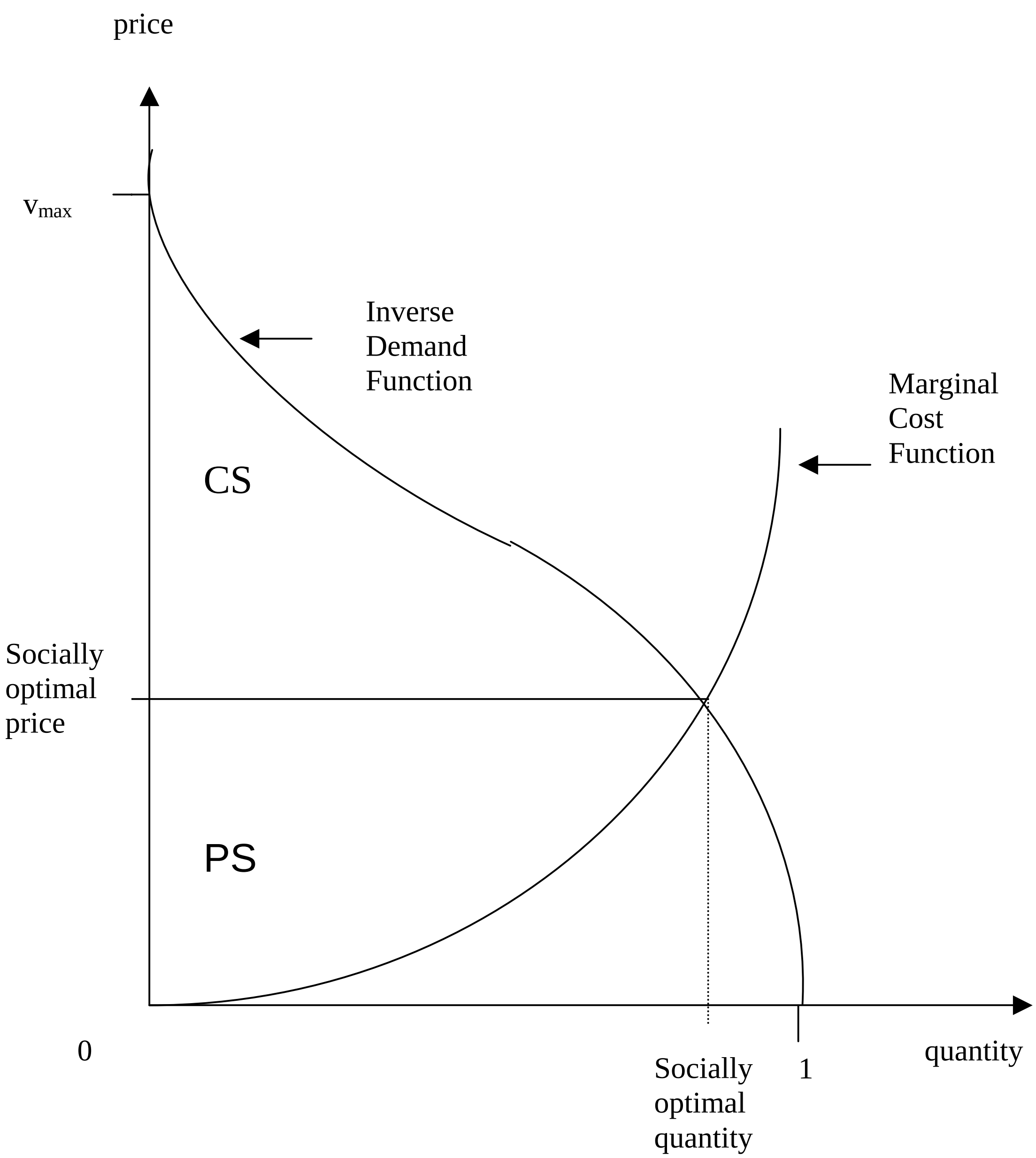

Figure 3. Socially optimal quantity and price

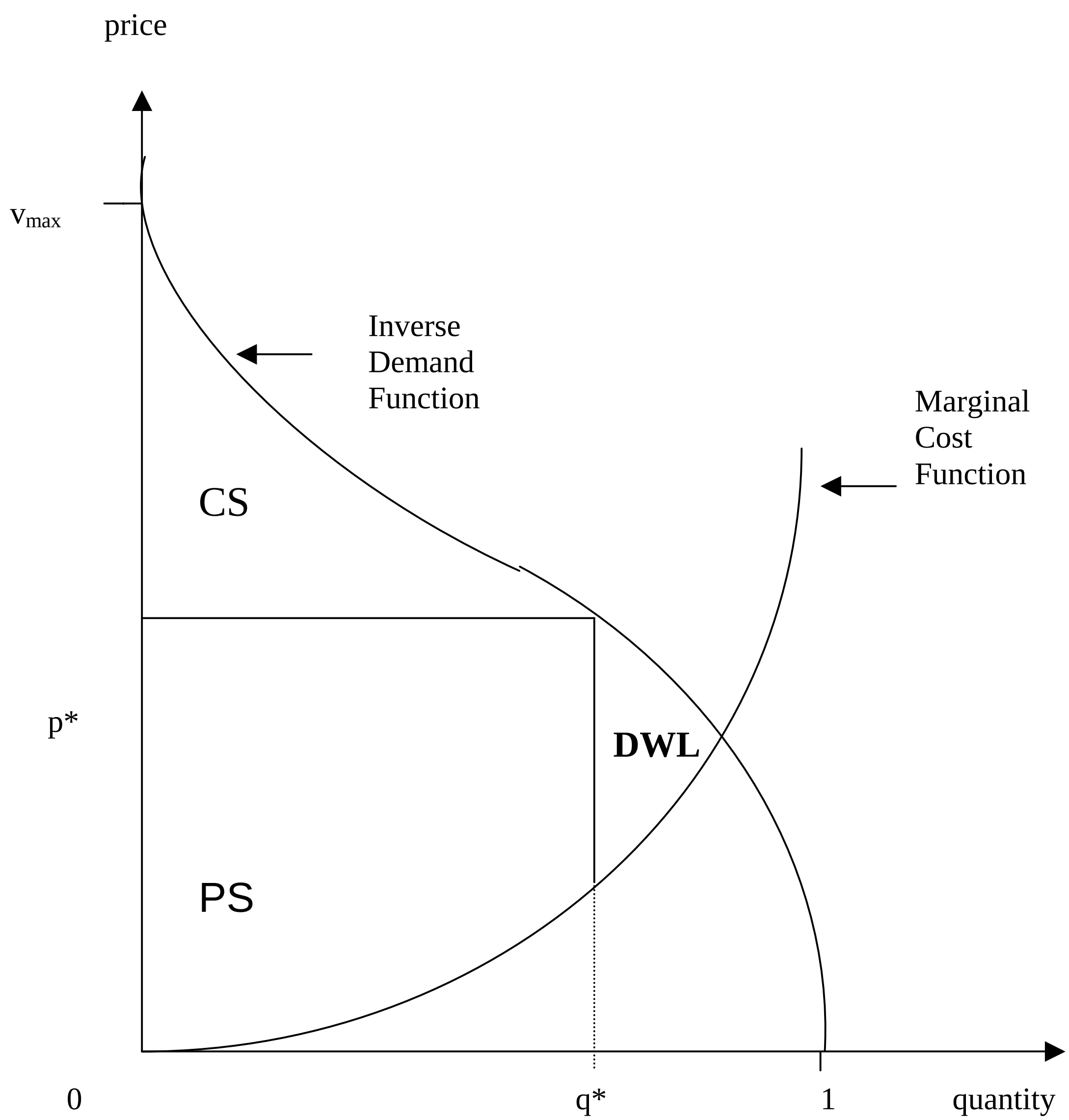

Figure 4. Monopolist's quantity and price

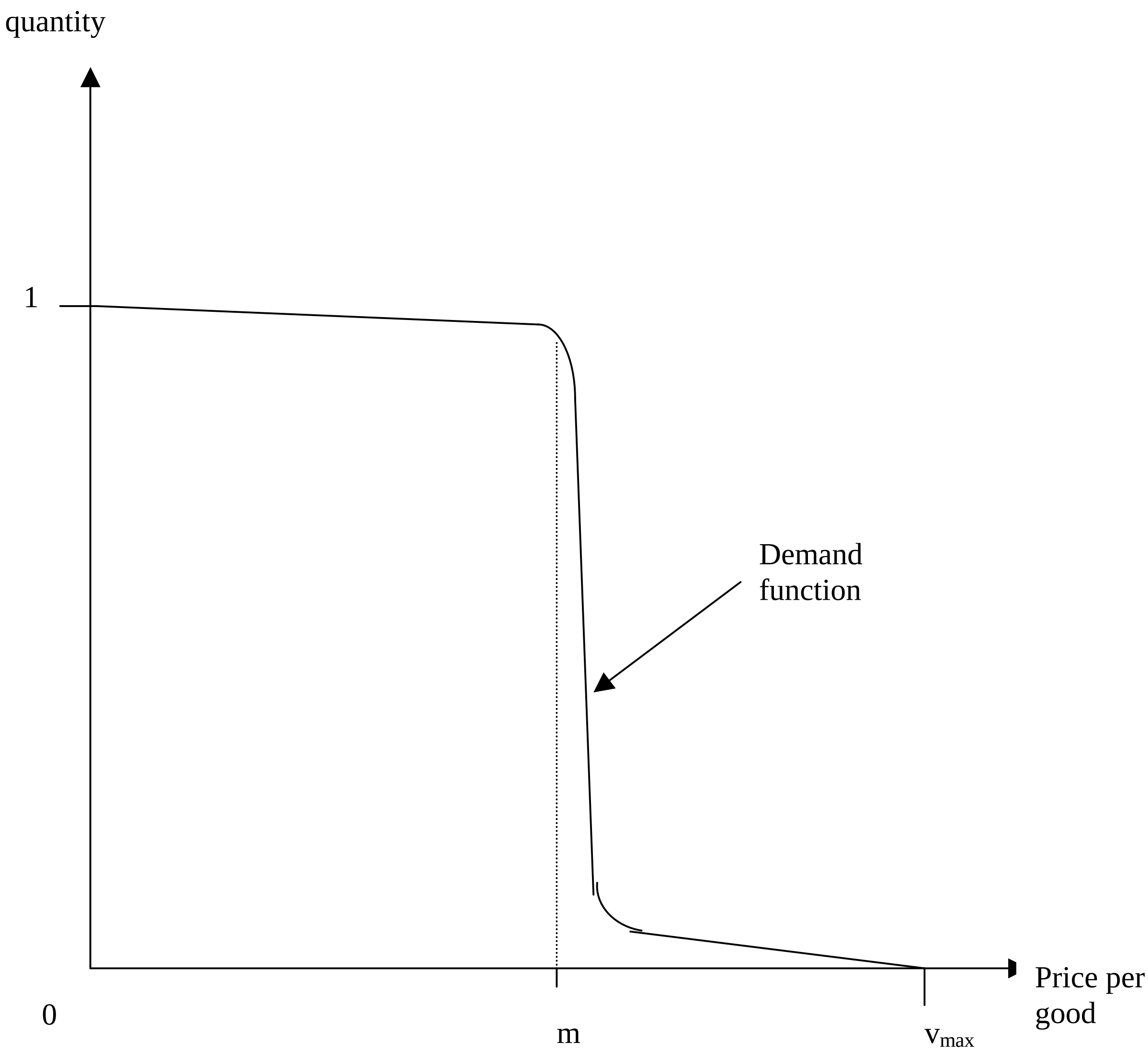

Figure 5. Demand function for a large bundle.

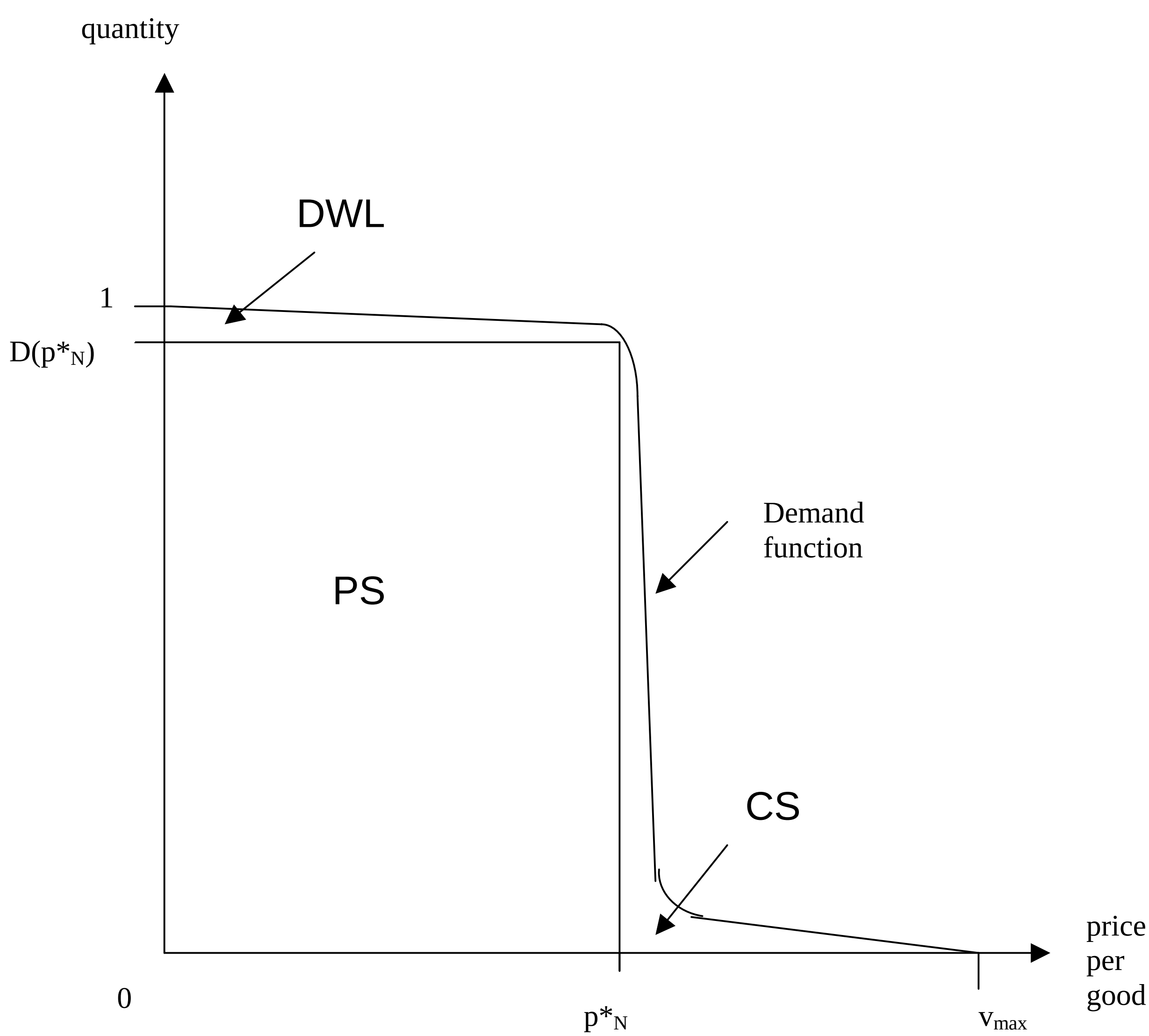

Figure 6. Monopoly pricing for a large bundle.