Computationally Efficient Methods for Two Multivariate Fractionally Integrated Models

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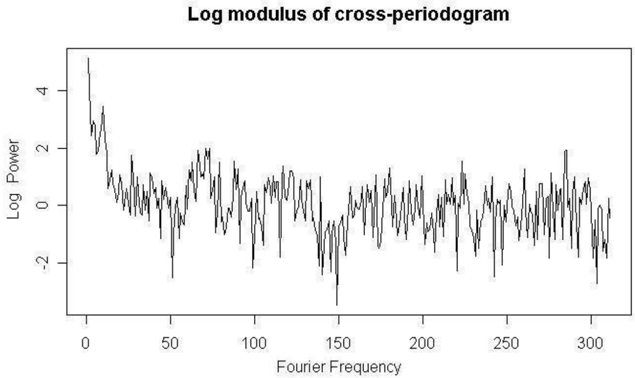
What is a cross-spectral density?

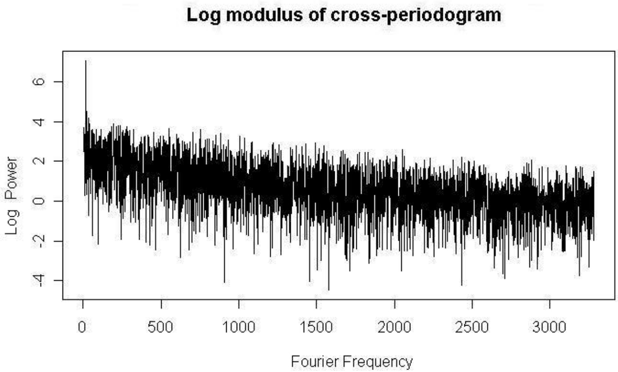
- For a univariate time series, the spectral density measures the contribution of a particular frequency to movements of the time series.
- If the spectral density is large near 0, the time series is more persistent.
- For multivariate time series, the cross-spectral density measures the relationship between two time series at a particular frequency.

Cross-spectral densities need not be finite at 0.



Monthly CPI goods and services inflation, 1956-2008.

Cross-spectral densities need not be finite at 0.



Daily wind speeds, in Valentia and Rosslare, Ireland, 1961-1978.

The most commonly used multivariate model is a vector autoregression.

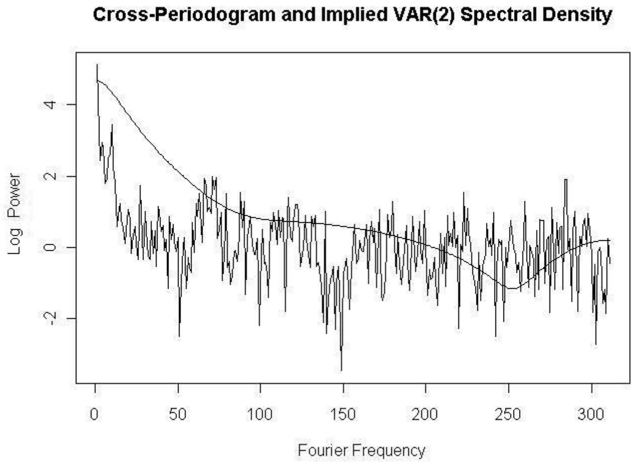
$$
\begin{aligned}
A(L)X_t &= \epsilon_t \\
\epsilon_t &\sim Normal(0, \Sigma)
\end{aligned}
$$

All the roots of $|A(L)|$ must be outside the unit circle for the model to be stationary.
The spectral density of a VAR is

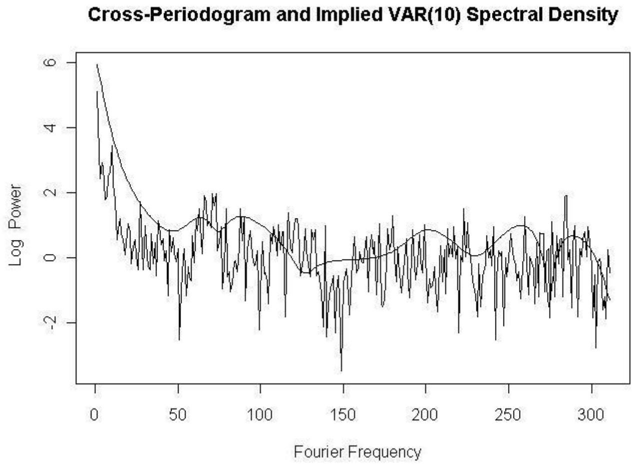
$$f_{VAR}(\lambda) = \frac{1}{2\pi} A(e^{-i\lambda})^{-1} \Sigma \left(A(e^{-i\lambda})^{-1} \right)^*$$

If the VAR is stationary, the spectral density is finite.

Vector autoregressions cannot fit peaks at 0.



Models that can match the peak require many lags.



Univariate long memory models have spectral densities that tend to infinity at the 0 frequency.

- Long memory models are defined in part by their differencing parameter, d.
- If $d = 0$, the models do not have long memory. Autoregressive models are one type of model with $d = 0$.
- If $0 < |d| < \frac{1}{2}$, then the model is stationary, and $f(\lambda) \sim C|1 - e^{-i\lambda}|^{-2d}$ as $\lambda \to 0^+$.

The simplest example of a univariate long memory model is fractionally integrated white noise.

$$
\begin{aligned}
(1-L)^d y_t &= \epsilon_t \\
\epsilon_t &\sim Normal(0, \sigma^2) \\
(1-L)^d &= \sum_{j=0}^{\infty} (-1)^j \binom{d}{j} L^j \\
\binom{d}{j} &= \frac{d(d-1)\cdots(d-j+1)}{j!}
\end{aligned}
$$

where $0 < |d| < \frac{1}{2}$. The spectral density is:

$$\frac{\sigma^2}{2\pi} |1 - e^{-i\lambda}|^{-2d}$$

We can combine fractionally integrated white noise and ARMA models to create ARFIMA models.

We can have an ARMA model driven by fractionally integrated white noise:

$$a(L)x_t = b(L)[(1-L)^{-d}\epsilon_t]$$

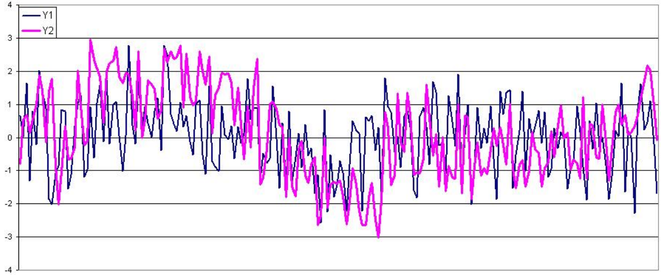
Equivalently, we can have an ARMA model which is then fractionally integrated:

$$x_t = (1-L)^{-d}\left(\frac{b(L)}{a(L)}\epsilon_t\right)$$

Now, we define multivariate fractionally integrated white noise.

$$\begin{aligned} D(L)Y_t &= \epsilon_t \\ \epsilon_t &\sim Normal(0, \Sigma) \\ D(L) &= \begin{pmatrix} (1-L)^{d_1} & 0 & \dots & 0 \\ 0 & (1-L)^{d_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (1-L)^{d_K} \end{pmatrix} \end{aligned}$$

What fractionally integrated multivariate white noise looks like:



$$d = (0.1, 0.4), \Sigma = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$$

Now, there are two possible combinations of fractionally integrated white noise and VAR models.

We can have a VAR model driven by fractionally integrated white noise (VARFI):

$$A(L)X_t = D(L)^{-1}\epsilon_t$$

We can also have a VAR model which is then fractionally integrated (FIVAR):

$$X_t = D(L)^{-1}\left(A(L)^{-1}\epsilon_t\right)$$

These two models are not equivalent.
(The difference between the two types of models was first noted by Lobato, 1997.)

These models have different properties.

- FIVAR: $A(L)D(L)X_t = \epsilon_t$
  - The k^{th} time series is integrated of order d_k.
  - There is no cointegration.
- VARFI: $D(L)A(L)X_t = \epsilon_t$
  - It is possible that each time series is integrated of order $\max(\vec{d})$.
  - It is often possible to find a linear combination of present and past values of the elements of X_t which is integrated of order less than $\max(\vec{d})$.

In our paper, we present efficient algorithms to:

- Compute the autocovariance and cross-covariance sequences.
- Simulate from multivariate long memory models.
- Estimate multivariate long memory models with Gaussian maximum likelihood:
  - Compute a quadratic form.
  - Compute a determinant.

Algorithms existed to do these computations, but...

- Our algorithms to compute covariances are faster by a factor of at least 100 for FIVAR processes and well over 1000 for VARFI processes.
- Our algorithms are conjectured to run in less than the quadratic time required for all the other algorithms.
- Our algorithms for simulation and for computing the quadratic form apply to general multivariate time series processes, not just FIVAR and VARFI models.

Faster computing time means that running Monte Carlo simulations and analyzing larger datasets becomes feasible.

Inflation is made up of multiple components.

- Goods and services are two broad categories that make up overall inflation.
- Since they measure the price changes of different products, they need not move together.
- The inflation rate in one category might affect the inflation rate in the other.
- There might be a long-term relationship between the two components of inflation.

Both types of inflation seem to be persistent but mean-reverting.

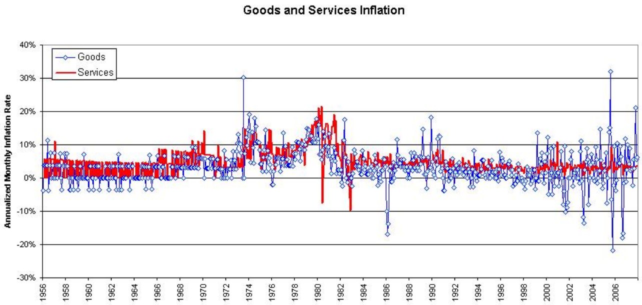
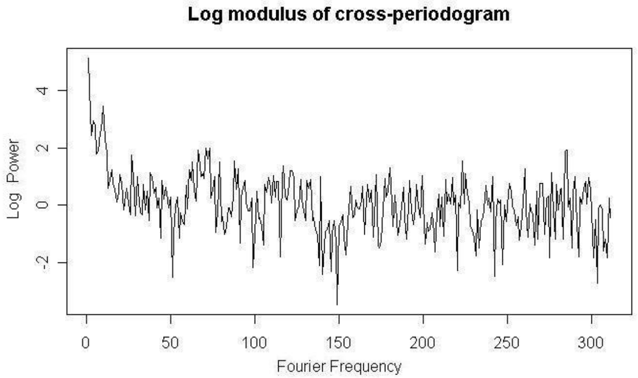


Figure: Goods and services monthly inflation at annualized rates, February 1956-January 2008.

Since both series seem to have long memory, a FIVAR or VARFI model might be useful.



This dataset is large enough that efficient computational algorithms are essential.

- Computing the likelihood one time using previously existing algorithms required over 60 seconds on a home computer.
- Computing the likelihood using our algorithms required about 2 seconds.
- Finding the maximum of the likelihood function requires many computations.

Comparing the log likelihoods at the maximum likelihood estimates, a VARFI model is preferred.

$$\begin{aligned} goods_t &= 0.3027 goods_{t-1} + 0.4245 services_{t-1} + u_{1t} \\ services_t &= -0.0237 goods_{t-1} - 0.3085 services_{t-1} + u_{2t} \\ \begin{pmatrix} u_{1t} \\ (1-L)^{0.484} u_{2t} \end{pmatrix} &\sim WhiteNoise\left(0, \begin{pmatrix} 20.2342 & 0.4605 \\ 0.4605 & 7.0783 \end{pmatrix}\right) \end{aligned}$$

Rearranging the terms of the first equation, we find a relationship like cointegration.

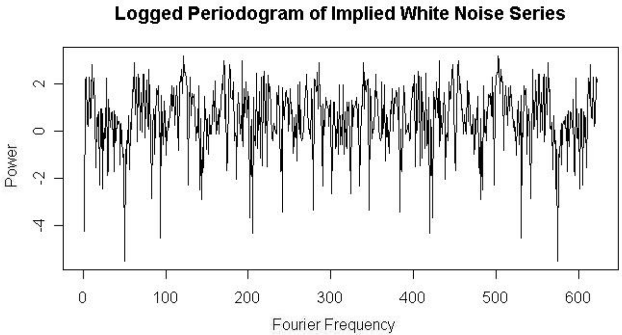
$$\begin{aligned}
goods_t &= 0.3027 goods_{t-1} + 0.4245 services_{t-1} + u_{1t} \\
services_t &= -0.0237 goods_{t-1} - 0.3085 services_{t-1} + u_{2t} \\
\begin{pmatrix} u_{1t} \\ (1-L)^{0.484} u_{2t} \end{pmatrix} &\sim WhiteNoise\left(0, \begin{pmatrix} 20.2342 & 0.4605 \\ 0.4605 & 7.0783 \end{pmatrix}\right)
\end{aligned}$$

Then,

$$u_{1t} = goods_t - 0.3027 goods_{t-1} - 0.4245 services_{t-1}$$

is estimated to be white noise.

The periodogram shows that it is quite close to white noise.



We have discussed two multivariate generalizations of the univariate ARFIMA model.

- FIVAR and VARFI models can both be used to model long memory time series, but they have different implications.
- We have discussed efficient algorithms for calculating their autocovariances, simulating from the models, and computing the Gaussian likelihoods.
- We have fit the models to inflation data and found a relationship like cointegration between the two series.

If you want to use these models, code is available online.

`http://pages.stern.nyu.edu/~rsela/VARFI/code.html`

There is much future work to be done.

- We have described a property akin to cointegration, but one could also use our algorithms to estimate models with traditional cointegration.
- There have been many applications of univariate long memory models to data – there are also many potential applications for FIVAR and VARFI models.

References

Lobato, Ignacio N. (1997), "Consistency of the averaged cross-periodogram in long memory time series," *Journal of Time Series Analysis* 18(2):137-155.

Sowell, Fallaw (1989a), "A decomposition of block Toeplitz matrices with applications to vector time series."

Sowell, Fallaw (1989b), "Maximum likelihood estimation of fractionally integrated time series models."