Internet Appendix for
"A Pyrrhic Victory?
Bank Bailouts and Sovereign Credit Risk"

This Internet Appendix serves as a companion to the paper "A Pyrrhic Victory? Bank Bailouts and Sovereign Credit Risk". It contains the proofs to the propositions and other results that were not included in the main text in order to conserve space.

IA.1 Derivations

IA.1.1 Proof of Lemma 1

Use (6) to substitute for w_s in the financial sector's first-order condition and then take the derivative with respect to the transfer T_0:.

$$\frac{d^2 f(K_0, s_0)}{ds_0^2}\frac{ds_0}{dT_0}p_{solv} + w_s\frac{dp_{solv}}{dT_0} - c''(s_0)\frac{ds_0}{dT_0} = 0$$
$$\frac{ds_0}{dT_0} = -w_s\frac{dp_{solv}}{dT_0} / \left(\frac{d^2 f(K_0, s_0)}{ds_0^2}p_{solv} - c''(s_0)\right) \tag{IA.1}$$

Since $dp_{solv}/dT_0 = p(\underline{A}_1)$, this term is positive so long as $\underline{A}_1$ is in the support of $\tilde{A}_1$ and the transfer increases the probability of solvency by decreasing the solvency threshold $\underline{A}_1$. Hence the numerator of the right hand side in the second line is negative. That the denominator is also negative follows from the concavity of f and the convexity of c. This establishes that the right side is positive and hence $ds_0/dT_0 > 0$.

IA.1.2 A Candidate for $V(K)$ based on $f(K, s)$

Consider the frictionless counterpart to our setting, with $p_{solv} = 1$. In a dynamic setting, the expression for V would reflect the value of future production of the non-financial sector as a function of its future capital, K. For simplicity, consider one extra period of output. The case of more than one future period should be similar as it is the sum of multiple one-period output. The output of the additional period is given by $\max_s f(K, s)$. It is natural then to let

$$V(K) = \max_s f(K, s) - w_s s$$

with w_s determined by the financial sector's first-order condition. With $f(K, s) = \alpha K^{1-\vartheta} s^{\vartheta}$, this implies that

$$V(K) = (1 - \vartheta)\alpha K^{1-\vartheta} s^{*\vartheta}$$

where s^* is the optimal choice of s.

Let $c(s) = \frac{1}{m} s^m$ for $m \geq 2$. Then the first-order condition of the financial sector implies that $w_s = s^{m-1}$ and the first-order condition of the non-financial sector implies that:

$$\vartheta \alpha K^{1-\vartheta} s^{\vartheta-1} = w_s = s^{m-1}$$

Solving for s^*, substituting into the expression above for $V(K)$, and simplifying gives:

$$s^* = (\vartheta\alpha)^{\frac{1}{m-\vartheta}} K^{\frac{1-\vartheta}{m-\vartheta}}$$

$$V(K) = (1 - \vartheta)\alpha^{\frac{m}{m-\vartheta}} K^{\gamma} \quad \text{where} \quad \gamma = \frac{(1-\vartheta)}{1 - \frac{\vartheta}{m}}$$

Hence, $V(K)$ has the power form that is used in the paper. Moreover,for $m \geq 2$ (which is assumed), $\gamma < 1$.

IA.1.3 Properties of Expected Tax Revenue: $\mathcal{T}$

For the assumed parametric forms, we obtained the following results:

$$\begin{aligned}
\mathcal{T} &= \theta_0 \gamma^{\frac{\gamma}{1-\gamma}}(1-\theta_0)^{\frac{\gamma}{1-\gamma}} \\
\frac{d\mathcal{T}}{d\theta_0} &= \gamma^{\frac{\gamma}{1-\gamma}}(1-\theta_0)^{\frac{\gamma}{1-\gamma}} - \theta_0 \frac{\gamma}{1-\gamma}\gamma^{\frac{\gamma}{1-\gamma}}(1-\theta_0)^{\frac{\gamma}{1-\gamma}-1} = \frac{\mathcal{T}}{\theta}\left(1 - \frac{\gamma}{1-\gamma}\frac{\theta_0}{1-\theta_0}\right) \\
\frac{d^2\mathcal{T}}{d\theta_0^2} &= -2\frac{\gamma}{1-\gamma}\gamma^{\frac{\gamma}{1-\gamma}}(1-\theta_0)^{\frac{\gamma}{1-\gamma}-1} + \frac{\theta_0}{1-\theta_0}\left(\frac{\gamma}{1-\gamma} - 1\right)\frac{\gamma}{1-\gamma}\gamma^{\frac{\gamma}{1-\gamma}}(1-\theta_0)^{\frac{\gamma}{1-\gamma}-1}
\end{aligned}$$

The second line shows that $d\mathcal{T}/d\theta_0 > 0$ on $[0, \theta_0^{max})$ and $d\mathcal{T}/d\theta_0 < 0$ on $(\theta_0^{max}, 1)$ where θ_0^{max} solves: $\frac{\gamma}{1-\gamma}\frac{\theta_0^{max}}{1-\theta_0^{max}} = 1$. It is zero at θ^{max} and at 1 (where $\mathcal{T} = 0$).

The third line implies that $d^2\mathcal{T}/d\theta_0^2 < 0$ on $[0, \theta_0^{max}]$ so $\mathcal{T}$ is *increasing* but *concave* on this region. To see this, note that the third line can be rewritten as:

$$\frac{d^2\mathcal{T}}{d\theta_0^2} = \left(-2 + \frac{\gamma}{1-\gamma}\frac{\theta_0}{1-\theta_0} - \frac{\theta_0}{1-\theta_0}\right)\frac{\gamma}{1-\gamma}\gamma^{\frac{\gamma}{1-\gamma}}(1-\theta_0)^{\frac{\gamma}{1-\gamma}-1}$$

We know that $-1 + \frac{\gamma}{1-\gamma}\frac{\theta_0}{1-\theta_0} < 0$ on $[0, \theta_0^{max}]$ and so, on this region, the leading term in parenthesis is negative. Since the remaining terms are positive, $d^2\mathcal{T}/d\theta_0^2 < 0$ in this region.

IA.1.4 The Government's First-Order Condition

From (3) we obtain the following first order condition of the government for the tax rate, θ_0:

$$\left[\frac{\partial f(K_0, s_0)}{\partial s_0} - c'(s_0)\right]\frac{ds_0}{dT_0}\frac{dT_0}{d\mathcal{T}}\frac{d\mathcal{T}}{d\theta_0} + [V'(K_1) - 1]\frac{dK_1}{d\theta_0} = 0 \tag{IA.2}$$

Note that the derivatives of s_0 and $\mathcal{T}$ here are total derivatives, since the government's choices are subject to the equilibrium choices of the financial and non-financial sectors.

As shown above, $d\mathcal{T}/d\theta_0$ is positive and decreasing (towards zero), but remains positive, on $[0, \theta_0^{max}]$. Therefore, the mapping from tax level (θ_0) to the marginal rate of transformation of taxes into tax revenue ($d\mathcal{T}/d\theta_0$), is invertible on this region. A high tax rate corresponds to a low marginal rate of transformation of taxes into tax revenue and vice versa. Note that the optimal tax rate must be in the region $[0, \theta_0^{max}]$, since any further increase in θ_0 beyond θ_0^{max} reduces tax revenue and investment. Hence, we can limit the consideration of the optimal tax rate to this region. Since $d\mathcal{T}/d\theta_0$ is positive and the mapping from θ_0 to $\mathcal{T}$ is invertible in this region, we can instead consider the government's first order condition with respect to $\mathcal{T}$, which turns out to be more intuitive for analyzing the government's problem. Dividing (IA.2) through by $d\mathcal{T}/d\theta_0$, and rewriting $(dK_1/d\theta_0)/(d\mathcal{T}/d\theta_0) = dK_1/d\mathcal{T}$ we obtain this alternative first-order condition:

$$\left[\frac{\partial f(K_0, s_0)}{\partial s_0} - c'(s_0)\right]\frac{ds_0}{dT_0} + [V'(K_1) - 1]\frac{dK_1}{d\mathcal{T}} = 0 \tag{IA.3}$$

where the term $dT_0/d\mathcal{T}$, which equals 1 under a no-default government policy, is omitted from the expression.

IA.1.5 Under-Investment Loss Due to Taxes

We want to obtain an expression for the second term in (8), the transfer version of the government's first-order condition:

$$\frac{[V'(K_1) - 1]\frac{dK_1}{d\theta_0}}{\frac{d\mathcal{T}}{d\theta_0}}$$

The first-order condition for investment of the non-financial sector, (7), and the parametric form for V imply that:

$$\begin{aligned} V'(K_1) - 1 &= \theta_0 V'(K_1) \\ &= \theta_0 \gamma K^{\gamma-1} \end{aligned}$$

Substituting in the parametric form also gives:

$$\frac{dK_1}{d\theta_0} = \frac{1}{1-\theta_0}\frac{1}{\gamma-1}K_1$$

Moreover, from (7) we can solve for the equilibrium K_1 as a function of θ_0:

$$K_1 = \gamma^{\frac{1}{1-\gamma}}(1-\theta_0)^{\frac{1}{1-\gamma}}$$

We can obtain the numerator to our fraction of interest by multiplying the expressions for the two terms together:

$$\begin{aligned}[V'(K_1)-1]\frac{dK_1}{d\theta_0} &= \frac{\theta_0\gamma}{(1-\theta_0)(\gamma-1)}K^{\gamma}\\ &= \frac{\theta_0}{1-\theta_0}\frac{\gamma}{\gamma-1}\gamma^{\frac{\gamma}{1-\gamma}}(1-\theta_0)^{\frac{\gamma}{1-\gamma}}\\ &= \frac{\mathcal{T}}{\theta_0}\frac{\theta_0}{1-\theta_0}\frac{\gamma}{\gamma-1}\end{aligned}$$

where the second line follows by substituting in the expression for K_0 and the third line follows by substituting in the expression for $\mathcal{T}$. Appendix IA.1.3 derives $d\mathcal{T}/d\theta_0$. Dividing the expression for the numerator by the expression for $d\mathcal{T}/d\theta_0$ shows that the marginal loss per transfer is given by:

$$\frac{d\mathcal{L}}{d\mathcal{T}} = \frac{[V'(K_1)-1]\frac{dK_1}{d\theta_0}}{\frac{d\mathcal{T}}{d\theta_0}} = \frac{-\frac{\theta_0}{1-\theta_0}\frac{\gamma}{1-\gamma}}{1-\frac{\theta_0}{1-\theta_0}\frac{\gamma}{1-\gamma}}$$

From this it is clear that $d\mathcal{L}/d\mathcal{T} \to -\infty$ as $\theta_0 \to \theta^{max}$ (since at θ^{max} the denominator is 0). Additionally, we have:

$$\frac{d^2\mathcal{L}}{d\mathcal{T}^2} = \frac{d^2\mathcal{L}}{d\theta_0 d\mathcal{T}}\frac{d\theta_0}{d\mathcal{T}} < 0 \qquad \text{for} \quad \theta_0 \in [0, \theta^{max}) \quad .$$

Hence, the marginal loss to the economy is increasing in magnitude (getting worse) as the tax rate increases up to θ^{max} and expected tax revenue rises to $\mathcal{T}^{max}$. In other words, marginal tax revenues becomes increasingly expensive to raise as the marginal loss to the economy from underinvestment rises in the tax rate/level of tax revenues.

IA.1.6 Proof of Proposition 1A

Substituting (6) into (5) and solving, we obtain the equilibrium level of s_0 (note that we refer to the *equilibrium* level of s_0 also as s_0, an abuse of notation intended to reduce clutter):

$$s_0 = \left(\frac{\vartheta\alpha}{\beta}\right)^{\frac{1}{m-\vartheta}} K_0^{\frac{1-\vartheta}{m-\vartheta}} p_{solv}^{\frac{1}{m-\vartheta}}$$

Now substitute this into the expression for $d\mathcal{G}/d\mathcal{T}$ to get:

$$\frac{d\mathcal{G}}{d\mathcal{T}} = \frac{\partial f(K_0, s_0)}{\partial s}(1 - p_{solv})\frac{ds_0}{dT_0} = \frac{1}{m-\vartheta}\left(\vartheta\alpha K_0^{1-\vartheta}\right)^{\frac{m}{m-\vartheta}}\beta^{\frac{-\vartheta}{m-\vartheta}}p_{solv}^{\frac{\vartheta}{m-\vartheta}-1}(1-p_{solv})\frac{dp_{solv}}{dT_0}$$

Taking derivative again with respect to $\mathcal{T}$ shows that:

$$\begin{aligned}\frac{d^2\mathcal{G}}{d\mathcal{T}^2} \propto \quad & \left(\frac{\vartheta}{m-\vartheta} - 1\right)p_{solv}^{\frac{\vartheta}{m-\vartheta}-2}(1-p_{solv})\frac{dp_{solv}}{dT_0} \\ & - p_{solv}^{\frac{\vartheta}{m-\vartheta}-1}\left(\frac{dp_{solv}}{dT_0}\right)^2 \quad + p_{solv}^{\frac{\vartheta}{m-\vartheta}-1}(1-p_{solv})\frac{d^2p_{solv}}{dT_0^2}\end{aligned}$$

where $dT_0/d\mathcal{T} = 1$ is omitted. Since the second term in the above expression is always negative, a sufficient condition to ensure that $d^2\mathcal{G}/d\mathcal{T}^2 < 0$ is to ensure that the first and third terms in the above expression are non-positive. The condition: $m - 2\vartheta \geq 0$ ensures that the first term is non-positive. The third term is negative if the slope of the probability density of $\tilde{A}_1$ at $\underline{A}_1$ is non-positive. Letting $\tilde{A}_1$ take a uniform distribution sets this term to zero.[1]

Since we have shown that both $\mathcal{G}$ and $\mathcal{L}$ are concave in $\mathcal{T}$, the government's problem is concave in $\mathcal{T}$. Furthermore, the optimum tax revenue, $\hat{\mathcal{T}}$, must correspond to a tax rate $\hat{\theta} < \theta^{max}$, because the first-order condition is *negative* at θ^{max}. To see that this is the case, note that $d\mathcal{L}/d\mathcal{T} \to \infty$ as $\theta \to \theta^{max}$ while $d\mathcal{G}/d\mathcal{T}$ is finite for $p_{solv} > 0$.

IA.1.6.1 Impact of L_1 and N_D on $\mathcal{T}$

Let $x = L_1$ or N_D. Rewriting (8) using the gain and loss notation as $d\mathcal{G}/d\mathcal{T} + d\mathcal{L}/d\mathcal{T} = 0$ and then taking the derivative with respect to x gives:

$$\frac{d^2\mathcal{G}}{dxd\mathcal{T}} + \frac{d^2\mathcal{L}}{dxd\mathcal{T}} = 0 \qquad \text{(IA.4)}$$

[1] Using an exponential distribution would also be sufficient. For the log-normal distribution, this term will be negative for a range of values below a cutoff.

Using the Implicit Function Theorem, the two terms on the right side evaluate to the following:

$$\frac{d^2\mathcal{G}}{dxd\mathcal{T}} = \frac{d}{dp_{solv}}\left(\frac{d\mathcal{G}}{d\mathcal{T}}\right)\left\{\frac{\partial p_{solv}}{\partial T_0}\left(\frac{\partial T_0}{\partial \mathcal{T}}\frac{d\mathcal{T}}{dx} + \frac{\partial T_0}{\partial x}\right) + \frac{\partial p_{solv}}{\partial x}\right\}$$
$$\frac{d^2\mathcal{L}}{dxd\mathcal{T}} = \frac{d^2\mathcal{L}}{d\mathcal{T}^2}\frac{d\mathcal{T}}{dx}$$

Substituting into (IA.4) and combining the terms multiplying $d\mathcal{T}/dx$ yields:

$$\frac{d\mathcal{T}}{dx}\left[\frac{d}{dp_{solv}}\left(\frac{d\mathcal{G}}{d\mathcal{T}}\right)\frac{\partial p_{solv}}{\partial T_0}\frac{\partial T_0}{\partial \mathcal{T}} + \frac{d^2\mathcal{L}}{d\mathcal{T}^2}\right] = -\frac{d}{dp_{solv}}\left(\frac{d\mathcal{G}}{d\mathcal{T}}\right)\left\{\frac{\partial p_{solv}}{\partial T_0}\frac{\partial T_0}{\partial x} + \frac{\partial p_{solv}}{\partial x}\right\} \quad \text{(IA.5)}$$

Note for the left-hand side term in parenthesis:

$$\frac{d}{dp_{solv}}\left(\frac{d\mathcal{G}}{d\mathcal{T}}\right)\frac{\partial p_{solv}}{\partial T_0}\frac{\partial T_0}{\partial \mathcal{T}} + \frac{d^2\mathcal{L}}{d\mathcal{T}^2} = \quad \frac{d^2\mathcal{G}}{d\mathcal{T}^2} + \frac{d^2\mathcal{L}}{d\mathcal{T}^2} \quad < 0$$

For $x = N_D$:

$$\frac{\partial p_{solv}}{\partial T_0}\frac{\partial T_0}{\partial x} + \frac{\partial p_{solv}}{\partial x} = \frac{\partial p_{solv}}{\partial T_0}(k_A - 1) < 0$$

since $\partial T_0/\partial N_D = -1$ and $\partial p_{solv}/\partial N_D = (\partial p_{solv}/\partial T_0)k_A$.

For $x = L_1$:

$$\frac{\partial p_{solv}}{\partial T_0}\frac{\partial T_0}{\partial x} = 0 \quad \text{and} \quad \frac{\partial p_{solv}}{\partial x} < 0$$

so for either value of x, the term in braces on the right side is negative. Finally, the intermediate steps in the proof of the concavity of G in $\mathcal{T}$ show that

$$\frac{d}{dp_{solv}}\left(\frac{d\mathcal{G}}{d\mathcal{T}}\right) < 0$$

Combining these results shows that $d\mathcal{T}/dx > 0$ for $x = L_1$ or N_D.

IA.1.6.2 Impact of N_D on T_0

To show how T_0 changes with N_D, begin by using the result above for $\mathcal{T}$. In particular, letting $x = N_D$ in (IA.5) and simplifying the right-side expression using $\frac{\partial p_{solv}}{\partial T_0}\frac{\partial T_0}{\partial x} + \frac{\partial p_{solv}}{\partial x} =$

$\frac{\partial p_{solv}}{\partial T_0}(k_A - 1)$ and $d^2\mathcal{G}/(dT_0 d\mathcal{T}) = d^2\mathcal{G}/d\mathcal{T}^2$ gives:

$$\frac{d\mathcal{T}}{dN_D}\left[\frac{d^2\mathcal{G}}{d\mathcal{T}^2} + \frac{d^2\mathcal{L}}{d\mathcal{T}^2}\right] = (1-k_A)\frac{d^2\mathcal{G}}{d\mathcal{T}^2}$$
$$\frac{d\mathcal{T}}{dN_D} = \frac{(1-k_A)\frac{d^2\mathcal{G}}{d\mathcal{T}^2}}{\frac{d^2\mathcal{G}}{d\mathcal{T}^2} + \frac{d^2\mathcal{L}}{d\mathcal{T}^2}} \quad \Rightarrow \quad 0 < \frac{d\mathcal{T}}{dN_D} < 1 - k_A$$

Since $T_0 = \mathcal{T} - N_D$,

$$\frac{dT_0}{dN_D} = \frac{d\mathcal{T}}{dN_D} - 1 \quad \Rightarrow \quad -1 < \frac{dT_0}{dN_d} < -k_A$$

Moreover, this shows that $T_0 + k_A N_D$, the *gross* transfer to the financial sector, is *decreasing* in N_D.

IA.1.7 Proof of Proposition 1B

The tradeoff involved in default is the loss of the deadweight cost D, versus the benefit of a larger transfer with reduced underinvestment made possible by diluting pre-existing debt. The net benefit of this tradeoff can be written as follows:

$$\int_{\hat{T_0}^{no\_def}}^{\hat{T_0}^{def} - kN_D} \frac{d\mathcal{G}}{dT_0} dT_0 + \int_{\hat{\mathcal{T}}^{no\_def}}^{\hat{\mathcal{T}}^{def}} \frac{d\mathcal{L}}{d\mathcal{T}} d\mathcal{T} - D \tag{IA.6}$$

where the first integral is the gain due to increasing the (gross) transfer, while the second integral is the reduction in underinvestment loss due to reducing tax revenue. Note that $d\mathcal{G}/dT_0$ here is evaluated at the no-default values. If (IA.6) is positive, it is optimal for the sovereign to choose default, while if it is negative then no-default is optimal.

To prove point (1), take the derivative of (IA.6) with respect to L_1 and simplify the resulting expression to obtain:

$$\int_{\hat{T_0}^{no\_def}}^{\hat{T_0}^{def} - k_A N_D} \frac{d}{dL_1}\left(\frac{d\mathcal{G}}{dT_0}\right) > 0$$

The intermediate steps in Appendix IA.1.4 show that the derivative in the integrand is positive. As shown in Appendix IA.1.6.2, the *gross* transfer is decreasing in N_D, so $T_0^{def} > k_A N_D + T_0^{no\_def}$ and hence the integral is positive.

To prove the statement for N_D, take the derivative of (IA.6) with respect to N_D. Simplifying the derivative at the upper integration boundary gives $-k_A d\mathcal{G}/dT_0\big|_{\hat{T_0}^{def}-k_A N_D}$ while from the lower boundary we get we get $d\mathcal{G}/dT_0\big|_{\hat{T_0}^{no\_def}}$. The remaining part of the derivative is:

$$\int_{\hat{T_0}^{no\_def}}^{\hat{T_0}^{def}-k_A N_D} \frac{d}{dN_D}\left(\frac{d\mathcal{G}}{dT_0}\right) = k_A \int_{\hat{T_0}^{no\_def}}^{\hat{T_0}^{def}-k_A N_D} \frac{d}{dT_0}\left(\frac{d\mathcal{G}}{dT_0}\right)$$
$$= k_A \left(\frac{d\mathcal{G}}{dT_0}\bigg|_{\hat{T_0}^{def}-K_A N_D} - \frac{d\mathcal{G}}{dT_0}\bigg|_{\hat{T_0}^{no\_def}}\right)$$

Combining the three parts of the derivatives gives: $(1-k_A)d\mathcal{G}/dT_0\big|_{\hat{T_0}^{no\_def}} > 0$. To show that the benefit of defaulting is convex in N_D, take a second derivative to obtain: $(1-k_A)d^2\mathcal{G}/dT_0^2\big|_{\hat{T_0}^{no\_def}} dT_0^{no\_def}/dN_D > 0$.

Finally, taking the derivative with respect to k_A, we obtain $-(d\mathcal{G}/dT_0)N_D < 0$ at the upper integration boundary and 0 at the lower boundary. In the interior we obtain

$$\int_{\hat{T_0}^{no\_def}}^{\hat{T_0}^{def}-k_A N_D} \frac{d}{dk_A}\left(\frac{d\mathcal{G}}{dT_0}\right) = N_D \int_{\hat{T_0}^{no\_def}}^{\hat{T_0}^{def}-k_A N_D} \frac{d}{dT_0}\left(\frac{d\mathcal{G}}{dT_0}\right) < 0$$

so the derivative is negative.

IA.1.8 Optimal Tax Revenue Under Uncertainty

Since $N_T = (\mathcal{T} - N_D/H)H$ and P_0 is given by (10) under uncertainty, T_0 can be written in terms of $\mathcal{T}$ and H as follows:

$$T_0 \quad = \quad N_T P_0 \quad = \quad (\mathcal{T} - \frac{N_D}{H}) E_0\left[\min\left(H, \tilde{R}_V\right)\right] . \tag{IA.7}$$

As earlier, the first order condition for the government's choice of $\mathcal{T}$ is given by:

$$\frac{d\mathcal{G}}{dT_0}\frac{dT_0}{d\mathcal{T}} + \frac{d\mathcal{L}}{d\mathcal{T}} = 0$$

Whereas under certainty $dT_0/d\mathcal{T}$=1, this is no longer the case. Taking the derivative of T_0 in (IA.7) with respect to $\mathcal{T}$ (while holding H constant) and then using (9) to substitute into the resulting expression gives $dT_0/d\mathcal{T} = P_0 H$. Therefore, the first-order condition for $\mathcal{T}$ is:

$$\frac{d\mathcal{G}}{\partial T_0} H P_0 + \frac{d\mathcal{L}}{d\mathcal{T}} = 0 \tag{IA.8}$$

with T_0 given in (IA.7). The loss due to underinvestment, $\mathcal{L}$, is the same as under certainty. Recall that it is concave, with the magnitude of the marginal loss, $d\mathcal{L}/d\mathcal{T}$, increasing in $\mathcal{T}$. Similarly, $d\mathcal{G}/dT_0$, the gain to the economy from the increased provision of financial services, remains the same with uncertainty and is decreasing in T_0. However, the rate at which T_0 increases in $\mathcal{T}$ is now HP_0 rather than 1. Note that this rate is a constant in $\mathcal{T}$, as P_0 is only a function of H, and is less than 1.[2] Finally, the second-order condition for $\mathcal{T}$ holds

$$\frac{d^2\mathcal{G}}{\partial T_0^2} (HP_0)^2 + \frac{d^2\mathcal{L}}{d\mathcal{T}^2} < 0$$

as $\mathcal{G}$ and $\mathcal{L}$ are concave and HP_0 is a function only of H.

IA.1.9 Optimal Probability of Default Under Uncertainty

Changing H affects two components of the government's objective. As can be seen from (IA.7), increasing H changes T_0. Unlike the case with $\mathcal{T}$, however, increasing H does not have any effect on investment. Instead, the cost associated with increasing H is that it increases the probability of default, and so also the expected deadweight cost. The first-order condition for H shows this tradeoff:

$$\frac{d\mathcal{G}}{dT_0} \frac{dT_0}{dH} - D \frac{dp_{def}}{dH} = 0 \tag{IA.9}$$

From (10), it is clear that $dp_{def}/dH > 0$ and we can think of choosing H exactly as choosing the probability of default. The effect on $T_0 = P_0 N_T$ is less immediately clear, since increasing H increases N_T, but decreases P_0. However, (IA.7) shows that $dT_0/dH > 0$. To see this we break up T_0 into two terms based on (IA.7) and consider their derivatives:

$$d\,(\mathcal{T} - \frac{N_D}{H})/dH = \frac{N_D}{H^2} > 0 \tag{IA.10}$$

$$d\,E_0\left[\min\left(H, \tilde{R}_V\right)\right]/dH = (1 - p_{def}) > 0 \tag{IA.11}$$

[2]To see this, note that $HP_0 = E_0\left[\min\left(H, \tilde{R}_V\right)\right] < E_0[\tilde{R}_V] = 1$.

Demonstrating the equivalence in the second line is straightforward, as shown in Appendix IA.1.10. We refer to (IA.10) as increasing the *dilution* of existing bondholders' claim, since the increase in H reduces the share of tax revenues that goes to the holders of the existing debt, N_D. We refer to (IA.11) as reducing either the *default buffer* or *precautionary taxation*, since by increasing H, it increases the probability that $\tilde{R}_V < H$, in which case the government defaults. Hence, (IA.10) and (IA.11) show that increasing H (while holding $\mathcal{T}$ constant) increases T_0. It immediately follows that $d\mathcal{G}/dH > 0$ and there is a benefit to increasing H. Substituting in for dT_0/dH, the first-order condition becomes:

$$\frac{d\mathcal{G}}{dT_0}\left(\frac{N_D}{H^2}E_0\left[\min\left(H,\tilde{R}_V\right)\right]+(\mathcal{T}-\frac{N_D}{H})(1-p_{def})\right)-D\frac{dp_{def}}{dH}=0$$

Appendix IA.1.10 also shows that as H increases, raising it further becomes decreasingly effective at increasing T_0:

$$\frac{d^2T_0}{dH^2}=\frac{-2N_D}{H^3}\int_0^H xp_{\tilde{R}_V}(x)dx-(\mathcal{T}-\frac{N_D}{H})p_{\tilde{R}_V}(H)<0$$

where $p_{\tilde{R}_V}(x)$ denotes the probability density of $\tilde{R}_V$ evaluated at x. In other words, T_0 is concave in H. Together with the concavity of $\mathcal{G}$ in T_0, this implies that $\mathcal{G}$ is concave in H, e.g., $d^2\mathcal{G}/dH^2$.[3] The implication is that while increasing H provides a benefit to the government by increasing the transfer through dilution and reduction of precautionary taxation, the marginal benefit is decreasing. Meanwhile, the government bears a cost for increasing H; the resulting increased likelihood of default increases the expected deadweight cost of default.

We assume that at the optimal choice of H, $d^2 p_{def}/d^2H \geq 0$.

[3]Note that in the first-order conditions, we have assumed that the government takes into account the (negative) impact of higher H on prices. Thus, we have *NOT* treated the government here as a price-taker. If we instead treat the government as a price-taker, the resulting conditions are simpler: $dT_0/dH = P_0\mathcal{T}$ (as dP_0/dH is omitted due to the price-taking assumption) and the first-order condition is: $d\mathcal{G}/dT_0(P_0\mathcal{T}) - Dd\,p_{def}/dH = 0$. In this case, concavity of $\mathcal{G}$ in H still holds because $\mathcal{G}$ is concave in T_0.

IA.1.10 Uncertainty Calculations

To derive $d\,E_0\left[\min\left(H, \tilde{R}_V\right)\right]/dH$, rewrite the expectation as:

$$E_0\left[\min\left(H, \tilde{R}_V\right)\right] = \int_0^H x\, p_{\tilde{R}_V}(x)dx + H\int_H^\infty p_{\tilde{R}_V}(x)dx$$

Now taking the derivative with respect to H, one obtains:

$$\begin{aligned} d\,E_0\left[\min\left(H, \tilde{R}_V\right)\right]/dH =& Hp_{\tilde{R}_V}(H) - Hp_{\tilde{R}_V}(H) + \int_H^\infty p_{\tilde{R}_V}(x)dx \\ =& \int_H^\infty p_{\tilde{R}_V}(x)dx \\ =& (1 - p_{def}) \end{aligned}$$

The first line is just the derivative, while the last line follows by definition of p_{def}.

Using this result we have that:

$$\frac{dT_0}{dH} = \frac{N_D}{H^2}E_0\left[\min\left(H, \tilde{R}_V\right)\right] + \left(\mathcal{T} - \frac{N_D}{H}\right)(1 - p_{def})$$

Substituting in the expression above for $E_0\left[\min\left(H, \tilde{R}_V\right)\right]$, taking the derivative with respect to T_0, and simplifying gives:

$$\begin{aligned} \frac{d^2T_0}{dH^2} =& \frac{-2N_D}{H^3}\left[\int_0^H x\, p_{\tilde{R}_V}(x)dx + H\int_H^\infty p_{\tilde{R}_V}(x)dx\right] + \frac{N_D}{H^2}(1 - p_{def}) \\ &+ \frac{N_D}{H^2}(1 - p_{def}) - \left(\mathcal{T} - \frac{N_D}{H}\right)p_{\tilde{R}_V}(H) \\ =& \frac{-2N_D}{H^3}\left[\int_0^H x\, p_{\tilde{R}_V}(x)dx\right] - \left(\mathcal{T} - \frac{N_D}{H}\right)p_{\tilde{R}_V}(H) \end{aligned}$$

Since $(\mathcal{T} - N_D/H) = N_T/H > 0$, it is clear that $d^2T_0/dH^2 < 0$.

IA.1.11 Proof of Proposition 2

The starting point are the first-order conditions for $\mathcal{T}$ and for H, given by (IA.8) and (IA.9), respectively. Substituting out $\frac{d\mathcal{G}}{dT_0}$ and rearranging gives the relation

$$-\frac{d\mathcal{L}}{d\mathcal{T}}\frac{dT_0}{dH} = HP_0D\frac{dp_{def}}{dH} = 0 \tag{IA.12}$$

Differentiating with respect to L_1 gives on the left-hand side:

$$-\frac{d^2\mathcal{L}}{d\mathcal{T}^2}\frac{d\mathcal{T}}{dL_1}\frac{dT_0}{dH} - \frac{d\mathcal{L}}{d\mathcal{T}}\frac{d^2T_0}{d\mathcal{T}dH}\frac{d\mathcal{T}}{dL_1} - \frac{d\mathcal{L}}{d\mathcal{T}}\frac{d^2T_0}{dH^2}\frac{dH}{dL_1}$$

and on the right-hand side:

$$(1-p_{def})D\frac{dp_{def}}{dH}\frac{dH}{dL_1} + HP_0D\frac{d^2p_{def}}{dH^2}\frac{dH}{dL_1}$$

Combining the terms in $\frac{d\mathcal{T}}{dL_1}$ gives:

$$\frac{d^2\mathcal{L}}{d\mathcal{T}^2}\frac{dT_0}{dH} - \frac{d\mathcal{L}}{d\mathcal{T}}\frac{d^2T_0}{d\mathcal{T}dH}$$

and it is not difficult to see that each term has a positive sign. Combining the terms in $\frac{dH}{dL_1}$ gives:

$$\frac{d\mathcal{L}}{d\mathcal{T}}\frac{d^2T_0}{dH^2} + (1-p_{def})D\frac{dp_{def}}{dH} + HP_0D\frac{d^2p_{def}}{dH^2}$$

and again each term is positive. Thus, we see that at the optimal values, $\text{sgn}\left(\frac{d\mathcal{T}}{dL_1}\right) = \text{sgn}\left(\frac{dH}{dL_1}\right)$. It remains to show that both of these signs are indeed *positive*.

To that end, let V represent the objective function of the government with the first-order conditions given by (IA.8) and (IA.9). Let $X = [\mathcal{T}, H]$ be the vector of the two controls. Then the first order conditions can be written as just $dV/dX = 0$. Differentiating this with respect to L_1 then gives

$$\frac{dV}{dL_1dX} + \frac{d^2V}{dX^2}\frac{dX}{dL_1} = 0 \quad .$$

By assumption, the optimal X is internal and so d^2V/dX^2 is negative definite. Isolating

dX/dL_1 then gives

$$\frac{dX}{dL_1} = -\left(\frac{d^2V}{dX^2}\right)^{-1} \frac{dV}{dL_1 dX} \quad .$$

Premultiplying by $\frac{dV^T}{dL_1 dX}$ we obtain

$$\frac{dV^T}{dL_1 dX}\frac{dX}{dL_1} = -\frac{dV^T}{dL_1 dX}\left(\frac{d^2V}{dX^2}\right)^{-1} \frac{dV}{dL_1 dX} > 0$$

where the sign follows since the Hessian is negative definite. Since

$$\frac{d^2\mathcal{G}}{dL_1 d\mathcal{T}} > 0$$

it is straightforward to see that $\frac{dV}{dL_1 dX} > 0$, i.e., both terms in the vector are positive. Hence, we must have that $dX/dL_1 > 0$ as well since both terms in this vector are of the same sign. Similar steps prove the result for ϑ.

IA.1.12 Proposition 3

Below we derive the return on financial sector equity, debt, and the sovereign bond. A complication created by the guarantee is that the number of outstanding sovereign bonds is state contingent, since it depends on the realization of $\tilde{A}_1$. Let $N_G(\tilde{A}_1)$ denote the number of new bonds issued towards the guarantee. This means there will also be a different price for sovereign bonds contingent on the realization of $\tilde{A}_1$. Hence, P_0 will now depend on $\tilde{A}_1$, as will T_0. This state-contingency is implicit below but will be omitted to avoid excessive notation.

Assume that $\tilde{A}_1 \sim U[A_{min}, A_{max}]$ and consider two types of shocks. The first is a shock to the value of the risky asset held by the financial sector. This shock changes the mean of $\tilde{A}_1$ by shifting the support of $\tilde{A}_1$ by an amount dA. Thus, $\tilde{A}_1$ remains uniformly distributed with the same dispersion, but a different mean. The second shock affects the sovereign bond price by changing the expected growth rate of future output by dR. For $\tilde{R}_V$ uniformly distributed this corresponds to a dR shift in its support.

From the model we have that the value of financial sector equity is given by

$$E = \int_{\underline{A}_1}^{A_{max}} (\tilde{A}_1 + T_0 - L_1) p(\tilde{A}_1) d\tilde{A}_1$$

where $p(\tilde{A}_1)$ is the uniform probability density. Calculating the change in E induced by a shock dA gives

$$\frac{d\,E}{d\,A} = p_{solv} + \frac{T_0(A_{max}) - T_0(\underline{A}_1)}{A_{max} - A_{min}} = p_{solv} \quad .$$

The second equality follows by the fact that there is no change in the guarantee once $\tilde{A}_1 > \underline{A}_1$ because at this point the financial sector is solvent. Calculating the change in E due to a shock dR gives

$$\frac{d\,E}{d\,R} = \frac{dP_0(\underline{A}_1)}{dR} N_T p_{solv}$$

Note that since there is no change in the guarantee for $\tilde{A}_1 > \underline{A}_1$, the quantity dP_0/dR is the same for any $\tilde{A}_1 > \underline{A}_1$.

Next, we have that the value of financial sector debt is given by

$$D = \int_{\underline{A}_1}^{A_{max}} L_1 p(\tilde{A}_1) d\tilde{A}_1 + \int_{A_{min}}^{\underline{A}_1} (\tilde{A}_1 + T_0) p(\tilde{A}_1) d\tilde{A}_1 + \int_{A_{min}}^{\underline{A}_1} (L_1 - \tilde{A}_1 - T_0) P_0 p(\tilde{A}_1) d\tilde{A}_1$$

The last term gives the value of the guarantee. Differentiating, simplifying, and combining terms gives that the change in D induced by a shock dA is

$$\frac{d\,D}{d\,A} = (1 - p_{solv})(1 - P_0(A_{min})) + \frac{T_0(\underline{A}) - T_0(A_{min})}{A_{max} - A_{min}} (1 - P_0(A_{min}))$$

The change in D due to a shock dR is given by

$$\frac{d\,D}{d\,R} = \int_{A_{min}}^{\underline{A}_1} \frac{dP_0}{dR} N_T (1 - P_0) p(\tilde{A}_1) d\tilde{A}_1 + \int_{A_{min}}^{\underline{A}_1} (L_1 - \tilde{A}_1 - T_0) \frac{dP_0}{dR} p(\tilde{A}_1) d\tilde{A}_1$$

The second term represents the change in value of the existing guarantee due to the change in the sovereign bond price. The first term incorporates both the change in the value of the existing transfer plus the change in the 'amount' of guarantee. That is, if dR is positive, the transfer increases in value by dT_0/dR, but this reduces the amount of guarantee given by the government for each realization by that same amount. This is true for each realization of $\tilde{A}_1$ under the integral sign.

We now approximate these values by by ignoring the state-dependence of P_0 on $\tilde{A}_1$ in the above expressions. This simplifies them to:

$$\begin{aligned}\frac{dE}{dA} &= p_{solv}\\ \frac{dE}{dR} &\approx \frac{dP_0}{dR}N_T p_{solv}\end{aligned}$$

and

$$\begin{aligned}\frac{dD}{dA} &\approx (1-p_{solv})(1-P_0)\\ \frac{dD}{dR} &\approx \frac{dP_0}{dR}N_T(1-p_{solv})(1-P_0)+\frac{1}{2}\frac{dP_0}{dR}(1-p_{solv})(\underline{A}_1-A_{min})\end{aligned}$$

By inspection one can then see that the following relation holds for these approximations:

$$dD \approx \frac{1-p_{solv}}{p_{solv}}(1-P_0)\,dE+\frac{1}{2}(1-p_{solv})(\underline{A}_1-A_{min})\,dP_0$$

Simple algebra and a substitution then give (12),

$$\frac{d\,D}{D} \approx \frac{(1-p_{solv})(1-P_0)}{p_{solv}}\frac{E}{D}\frac{dE}{E}+\frac{(1-p_{solv})^2(A_{max}-A_{min})}{2}\frac{P_0}{D}\frac{dP_0}{P_0} \quad .$$

Hence, we have

$$\frac{d\,D}{D} \approx \beta_E\frac{dE}{E}+\beta_g\frac{dP_0}{P_0} \quad .$$

where

$$\begin{aligned}\beta_E &= \frac{(1-p_{solv})(1-P_0)}{p_{solv}}\frac{E}{D}\\ \beta_g &= \frac{(1-p_{solv})^2(A_{max}-A_{min})}{2}\frac{P_0}{D}\end{aligned}$$

IA.2 Model with State-Contingent Taxation

This section extends the model in the main text so that the government sets the tax rate at time 2, thereby making the tax rate fully state contingent. Let ω denote the state realized at $t = 2$, and let $\tilde{\theta}(\omega)$ and $\tilde{V}(K_1)(\omega)$ be the state-contingent tax rate and the realized value of output. The following proposition gives for the optimal state-contingent tax policy.

Proposition IA.1. *Let T_0 be the government's desired transfer and p_{def} be the maximum probability of default it is willing to tolerate to achieve it. Assuming the (T_0, p_{def}) pair is feasible, the optimal debt issuance $\hat{N}_T$ and state-contingent tax rate $\tilde{\theta}(\omega)$ implementing it are given by:*

$$\hat{N}_T = \min\left(\frac{T_0}{1-p_{def}}, \max\left\{V : prob\left(\tilde{V}(K_1) \geq V\right) = (1-p_{def})\right\} - N_D\right)$$

and

$$\begin{aligned}
\tilde{\theta}(\omega) = \frac{\hat{N}_T + N_D}{\tilde{V}(K_1,\omega)} \quad & \textit{on any } \; C \subset \{\omega : \tilde{V}(K_1, w) \geq \hat{N}_T + N_D\} \;\; \textit{where } \; prob(C) = (1-p_{def}) \\
& \textit{and } \left\{\theta(\omega) : \omega \in \overline{C}\right\} \textit{ is chosen in any way that satisfies} \\
& E_0\left[\tilde{\theta}(\omega)\tilde{V}(K_1,\omega)\mathbf{1}_{\omega\in\overline{C}}\right] = (\hat{N}_T + N_D)\left(\frac{T_0}{\hat{N}_T} - (1-p_{def})\right)
\end{aligned}$$

Proof: The proof is instructive in clarifying the tax policy given in the Proposition. We prove its optimality by showing that among all policies that achieve the transfer T_0 with probability of default (no greater than) p_{def}, it requires the minimum expected tax revenue. Since underinvestment is a function only of expected tax revenue, the policy induces the minimum possible underinvestment distortion and is therefore optimal.

To see that the policy given in the Proposition requires the minimum expected tax revenue, note that for any tax policy, the required expected tax revenue $\mathcal{T}$ is bounded below by the following:

$$\begin{aligned}
\mathcal{T} &\geq (N_T + N_D)P_0 \\
&= T_0 + N_D P_0 \quad .
\end{aligned}$$

To avoid excess taxation, the government sets $\mathcal{T} = T_0 + N_D P_0$. This is always possible with state-contingent taxation since the tax rate can be adjusted state-by-state. Furthermore, to minimize $T_0 + N_D P_0$ given T_0 and N_D, the government must minimize P_0. As $P_0 = T_0/N_T$, this is equivalent to maximizing N_T. There are two restrictions on the maximal value of N_T,

$$(1) \qquad N_T \leq \frac{T_0}{1 - p_{def}}$$

$$(2) \qquad N_T + N_D \leq \max\{V : \text{prob}\,(V(K_1) \geq V) = 1 - p_{def}\}$$

Restriction (1) follows from the fact that $P_0 \geq 1 - p_{def}$. Restriction (2) follows directly from the requirement that the probability of default be less than or equal to p_{def}. The value of $\hat{N}_T$ given in the Proposition is the minimum of (1) and (2). The optimal tax policy $\hat{\theta}(\omega)$ then follows directly from T_0, p_{def} and the choice of $\hat{N}_T$. QED.

For an illustration of Proposition IA.1, consider the optimal policy when

$$\hat{N}_T = \frac{T_0}{1 - p_{def}}$$

holds, which we refer to as **case 1**. We call the alternative possibility **case 2**. Under case 1, the optimal tax policy simplifies to

$$\tilde{\theta}(\omega) = \frac{\hat{N}_T + N_D}{\tilde{V}(K_1, \omega)} \text{ with probability } 1 - p_{def}$$
$$\text{and } \tilde{\theta}(\omega) = 0 \text{ otherwise (i.e., with probability } p_{def}\text{)}$$

As required, the probability of default is p_{def}. Moreover, note that $P_0 = 1 - p_{def}$ and hence, as required, the transfer is $P_0 \hat{N}_T = T_0$. Finally, the expected tax revenue raised by the policy is $\hat{\mathcal{T}} = (\hat{N}_T + N_D)(1 - p_{def}) = T_0 + N_D(1 - p_{def})$. To see that this the minimal expected tax revenue necessary required by T_0 and p_{def}, note that for any tax policy, the required expected tax revenue $\mathcal{T}$ is bounded below by the following:

$$\begin{aligned} \mathcal{T} &\geq (N_T + N_D)P_0 \\ &= T_0 + N_D P_0 \\ &\geq T_0 + N_D(1 - p_{def}) = \hat{\mathcal{T}} \quad . \end{aligned}$$

The first inequality is an equality if there is never any surplus tax revenue. The second inequality follows from the fact that $P_0 \geq (1 - p_{def})$. Under the policy given in Proposition IA.1, both inequalities are in fact equalities.

Note that optimal state-contingent taxation does *not* eliminate the possibility of default. If anything, it makes it clear that the sovereign uses the possibility of default to dilute existing bondholders and thereby increase the transfer to the banks without increasing the underinvestment distortion. This is clearly demonstrated by the expression for the expected tax revenue under case 1, which can be rewritten as:

$$\mathcal{T} = T_0 + N_D - \underbrace{p_{def} N_D}_{\text{dilution}}$$

with the dilution term indicated. The expression indicates how increasing the probability of default allows the government to reduce the expected tax revenue necessary to support a transfer of T_0. This highlights the trade-off faced by the sovereign between creditworthiness and underinvestment.

Hence, the ability to make taxes state contingent does not change the fundamental trade-off between the size of the bailout, underinvestment, and the probability of default. What it does give the sovereign is the ability to eliminate any excess taxation and thereby minimize the amount of underinvestment incurred for any level of transfer and probability of default.

We now conclude the analysis of the optimal taxation policy by showing how to check the feasibility of a pair (T_0, p_{def}). This can be checked as follows. First find $\hat{\mathcal{T}}$ corresponding to the optimal $\hat{N}_T$. Note that $\hat{N}_T$ is itself a function of $\hat{\mathcal{T}}$ since the expected tax revenue determines investment and hence output, i.e., $V(K_1(\mathcal{T}))$. This means that $P_0 = T_0/\hat{N}_T$ is also a function of $\hat{\mathcal{T}}$. Therefore, $\hat{\mathcal{T}}$ is a solution to the equation

$$\hat{\mathcal{T}} = T_0 + P_0(\hat{\mathcal{T}}) N_D \quad ,$$

which holds under the optimal policy since there is no excess taxation. If $\mathcal{T} > \mathcal{T}^{max}$ (the Laffer limit on tax revenues) then (T_0, p_{def}) is infeasible. Otherwise, if $\hat{N}_T$ corresponds to case 1, then (T_0, p_{def}) *is* feasible. If $\hat{N}_T$ corresponds to case 2, then (T_0, p_{def}) is feasible if and only if

$$E_0\left[\tilde{V}(K_1,\omega)\mathbf{1}_{\omega\in\overline{C}}\right] \geq \hat{\mathcal{T}} - (\hat{N}_T + N_D)(1 - p_{def}) \quad .$$

In words, the maximum tax revenue that can be raised in the default states must be sufficient to cover the difference between expected total tax revenue and the tax revenue raised in the non-default states, $(\hat{N}_T + N_D)(1 - p_{def})$.

IA.2.1 The Optimal Probability of Default and Transfer

We now prove analogs to Propositions 1 and 2 in the main text. It now becomes natural to take T_0 and p_{def} as the controls instead of $\mathcal{T}$ and H. Note that for any feasible pair (T_0, p_{def}) there is a corresponding unique pair $(\mathcal{T}, H)$. For simplicity, we only consider an open region of the parameter space in which case 1 holds for the optimal T_0 and p_{def}. In this case we have that

$$
\begin{aligned}
P_0 &= 1 - p_{def} \\
\hat{N}_T &= T_0/(1 - p_{def}) \\
\mathcal{T} &= T_0 + N_D(1 - p_{def}) \quad .
\end{aligned}
$$

The first-order condition for T_0 is similar to that for the model in the main text, save for the change of variable,

$$\frac{d\mathcal{G}}{dT_0} + \frac{d\mathcal{L}}{d\mathcal{T}} = 0 \tag{IA.13}$$

where the gain $\mathcal{G}$ and loss $\mathcal{T}$ functions are the same as in the text and we use the fact that $d\mathcal{T}/dT_0 = 1$ under case 1. Note also that since there is no excess taxation,

$$H = \frac{N_T + N_D}{\mathcal{T}} = \frac{1}{P_0} \quad .$$

There are three possible cases for the first-order condition for p_{def}:

$$-\frac{d\mathcal{L}}{d\mathcal{T}} N_D - D \leq 0 \qquad \text{when } p_{def} = 0 \tag{IA.14}$$

$$-\frac{d\mathcal{L}}{d\mathcal{T}} N_D - D = 0 \qquad \text{when } 0 < p_{def} < 1 \tag{IA.15}$$

$$-\frac{d\mathcal{L}}{d\mathcal{T}} N_D - D > 0 \qquad \text{when } p_{def} = 1 \tag{IA.16}$$

where we use the fact that $d\mathcal{T}/dp_{def} = -N_D$ under case 1. The first region for the FOC corresponds to $p_{def} = 0$. This occurs when the benefit of increasing p_{def} is low and attains when the optimal taxation level $\hat{\mathcal{T}}$ is low, resulting in low marginal loss from underinvestment. The second region corresonds to when the optimal probability of default is internal, and hence the first-order condition holds with equality. Finally, it is possible to have $p_{def} = 1$, in which case the third region holds. Note that for the first and third regions, any increase in the transfer T_0 must come from an increase in tax revenues $\mathcal{T}$ since p_{def} is not changing.

The second-order conditions for T_0 and for p_{def} when $0 < p_{def} < 1$ are as follows,

$$\frac{d^2\mathcal{G}}{dT_0^2} + \frac{d^2\mathcal{L}}{d\mathcal{T}^2} < 0$$
$$\frac{d^2\mathcal{L}}{d\mathcal{T}^2} N_D^2 < 0$$

while the cross-partial is

$$-\frac{d^2\mathcal{L}}{d\mathcal{T}^2} N_D$$

The determinant of the Hessian matrix is therefore

$$\frac{d^2\mathcal{G}}{dT_0^2} \frac{d^2\mathcal{L}}{d\mathcal{T}^2} N_D^2 > 0 \quad .$$

and hence the Hessian is negative definite in this region.

The following proposition shows that the sovereign keeps the probability of p_{def} at zero so long as financial sector debt-overhang L_1 is low, and increases p_{def} in L_1 when L_1 is high.

Proposition IA.2. *(1) If financial sector debt-overhang L_1 is low, the optimal probability of default is $\hat{p}_{def} = 0$. If L_1 is sufficiently high, $\hat{p}_{def}$ is increasing in L_1. (2) The optimal transfer $\hat{T}_0$ is increasing in L_1.*

Proof: When $0 < \hat{p}_{def} < 1$, both first-order conditions hold. Substituting the first-order condition for T_0 into that for p_{def} gives

$$\frac{d\mathcal{G}}{dT_0} N_D - D = 0$$

Taking the derivative of this equation with respect to L_1 implies

$$\frac{d^2\mathcal{G}}{dL_1 dT_0} + \frac{d^2\mathcal{G}}{dT_0^2} \frac{dT_0}{dL_1} = 0$$
$$\Rightarrow \frac{dT_0}{dL_1} = -\frac{d^2\mathcal{G}}{dL_1 dT_0} / \frac{d^2\mathcal{G}}{dT_0^2} > 0$$

The last inequality follows from the fact that an increase in L_1 increases the marginal gain from the transfer: $\frac{d^2\mathcal{G}}{dL_1 dT_0} > 0$.

Taking the derivative of the first-order condition for p_{def} with respect to p_{def} gives

$$\frac{d^2\mathcal{L}}{d\mathcal{T}^2}\frac{d\mathcal{T}}{dL_1}N_D = 0$$

which implies that when $0 < \hat{p}_{def} < 1$ (i.e., the optimal choice is interior), $d\mathcal{T}/dL_1 = 0$, i.e., total tax revenues are constant in L_1. It follows from $\mathcal{T} = T_0 + N_D(1 - p_{def})$ that

$$0 = \frac{dT_0}{dL_1} - N_D\frac{dp_{def}}{dL_1}$$
$$\Rightarrow \frac{dp_{def}}{dL_1} = \frac{1}{N_D}\frac{dT_0}{dL_1} > 0$$

Now consider the case of $\hat{p}_{def} = 0$. The first-order condition for T_0 is unchanged when $p_{def} = 0$. Taking its derivative with respect to L_1 and rearranging gives

$$\frac{dT_0}{dL_1} = -\frac{d^2\mathcal{G}}{dL_1 dT_0}/\left(\frac{d^2\mathcal{G}}{dT_0^2} + \frac{d^2\mathcal{L}}{d\mathcal{T}^2}\right) > 0$$

The first-order condition p_{def} is

$$-\frac{d\mathcal{L}}{d\mathcal{T}}N_D - D < 0 \quad .$$

So long as it is negative, $\hat{p}_{def} = 0$. Taking the derivative of this quantity with respect to L_1 gives

$$-\frac{d\mathcal{L}^2}{d\mathcal{T}^2}\frac{dT_0}{L_1}N_D > 0 \quad .$$

Hence, the benefit to increasing p_{def} increases in L_1 and can become positive for a sufficiently large value of L_1. QED.

The following proposition looks at the effect of existing sovereign debt N_D on the sovereign's optimal policy. For clarity in interpreting this comparative static, we assume that changing the stock of existing government debt does not change the value of $\tilde{A}_1 + A_G$. Since an increase in N_D of dN_D induces an increase in A_G, the bank's holdings of a fraction k_A of existing government debt, of $dA_G = k_A dN_D$, we assume an offsetting unifom shift of $-k_A dN_D$ in the distribution of $\tilde{A}_1$. Hence, any change in p_{solv} is due to the change in the endogenous optimal transfer $\hat{T}_0$.

Proposition IA.3. *(1) When existing government debt, N_D, is low, the optimal probability of default is $\hat{p}_{def} = 0$. If N_D is sufficiently high then $\hat{p}_{def}$ is* increasing *in N_D. (2) The optimal transfer $\hat{T}_0$* decreases *in N_D when $\hat{p}_{def} = 0$, and* increases *in N_D when $0 < \hat{p}_{def} < 1$.*

Proof: When $0 < \hat{p}_{def} < 1$ both first-order conditions hold and substituting the condition for T_0 into the one for p_{def} we again have

$$\frac{d\mathcal{G}}{dT_0}N_D - D = 0$$

Taking the derivative of this equation with respect to N_D implies

$$\begin{aligned}
&\frac{d^2\mathcal{G}}{dT_0^2}\frac{dT_0}{dN_D}N_D + \frac{d\mathcal{G}}{dT_0} = 0 \\
\Rightarrow &\frac{dT_0}{dN_D} = -\frac{d\mathcal{G}}{dT_0}/\frac{d^2\mathcal{G}}{dT_0^2}\frac{1}{N_D} > 0
\end{aligned}$$

Taking the derivative of the first-order condition for $\hat{p}_{def}$ gives

$$\begin{aligned}
&\frac{d^2\mathcal{L}}{d\mathcal{T}^2}\frac{d\mathcal{T}}{dN_D}N_D + \frac{d\mathcal{L}}{d\mathcal{T}} = 0 \\
\Rightarrow &\frac{d\mathcal{T}}{dN_D} = -\frac{d\mathcal{L}}{d\mathcal{T}}/\frac{d^2\mathcal{L}}{d\mathcal{T}^2}\frac{1}{N_D} < 0
\end{aligned}$$

It then follows from $\mathcal{T} = T_0 + N_D(1 - p_{def})$ that

$$\begin{aligned}
&\frac{d\mathcal{T}}{dN_D} = \frac{dT_0}{dN_D} + (1 - p_{def}) - N_D\frac{dp_{def}}{dN_D} \\
&\Rightarrow \frac{dp_{def}}{dN_D} = -\frac{1}{N_D}\left(\frac{d\mathcal{T}}{dN_D} - \frac{dT_0}{dN_D} - (1 - p_{def})\right) > 0
\end{aligned}$$

Now consider the case of $\hat{p}_{def} = 0$. The first-order condition for T_0 is unchanged. Taking its derivative with respect to N_D and rearranging gives

$$\begin{aligned}
&\frac{d^2\mathcal{G}}{dT_0^2}\frac{dT_0}{dN_D} + \frac{d^2\mathcal{L}}{dN_D d\mathcal{T}} + \frac{d^2\mathcal{L}}{dT_0^2}\frac{dT_0}{dN_D} = 0 \\
&\frac{dT_0}{dN_D} = -\frac{d^2\mathcal{L}}{dN_D d\mathcal{T}}/\left(\frac{d^2\mathcal{G}}{dT_0^2} + \frac{d^2\mathcal{L}}{d\mathcal{T}^2}\right) < 0
\end{aligned}$$

since $\frac{d^2\mathcal{L}}{dN_D d\mathcal{T}} = \frac{d^2\mathcal{L}}{d\mathcal{T}^2}\frac{d\mathcal{T}}{dN_D} = \frac{d^2\mathcal{L}}{d\mathcal{T}^2} < 0$. Substituting the first-order condition for T_0 into that for

p_{def} gives

$$\frac{d\mathcal{G}}{dT_0} N_D - D < 0 \quad ,$$

which is the marginal benefit to increasing p_{def}. So long as it is negative, $\hat{p}_{def} = 0$. Taking the derivative of this quantity with respect to N_D gives

$$\frac{d\mathcal{G}^2}{dT_0^2} \frac{dT_0}{N_D} N_D + \frac{d\mathcal{G}}{dT_0} > 0 \quad .$$

Hence, the benefit to increasing p_{def} increases in N_D and can become positive for a sufficiently large value of N_D.