ONLINE APPENDIX

Unintended Consequences of LOLR Facilities: The Case of Illiquid Leverage

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Online Appendix A: Solution to the Bank's Problem (Exogenous Price Setting)

Lemma 1: Unless $\alpha^* = 0$, α^* is interior and the SOC is satisfied.

Proof:

Define $u_B^0 \equiv \frac{1}{e^I}[B - e^L - e^I(\bar{x}_1 + lx_2)]$.

1. Suppose $e^L + e^I\bar{x}_1 > B$. Then,
   $e^L + e^I(\bar{x}_1 + lx_2) > B \quad \forall l$,
   and $e^L + e^I p > B$ since $p \geq \bar{x}_1 \Rightarrow u_B^0 < 0$. Also,
   $\frac{\partial u_B}{\partial \alpha} < 0 \Rightarrow u_B < 0 \quad \forall \alpha \in [0, e^I]$.
2. Now, suppose $\alpha = e^I$.
   Then $e^L + p\alpha \geq B$, so that $u_B \to -\infty$. In turn,
   $\alpha \in [0, e^I]$ maps one-for-one onto $u_B \in [-\infty, u_B^0 < 0]$
3. For $u \sim \mathcal{N}(0, \sigma^2)$,
   $g(u) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-u^2}{2\sigma^2}}$, and
   $g'(u) = \frac{1}{\sqrt{2\pi\sigma^2}} (\frac{-u}{\sigma^2}) e^{\frac{-u^2}{2\sigma^2}}$.
   Then $(1-l)x_2 g'(u_B) - g(u_B) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-u^2}{2\sigma^2}} \left[(1-l)x_2(\frac{-u_B}{\sigma^2}) - 1\right] > 0$ iff $(1-l)x_2(-u_B) > \sigma^2$
   $\Leftrightarrow (-u_B) > \frac{\sigma^2}{(1-l)x_2}$
   $\Leftrightarrow (-u_B) < \bar{u}_B(l) \equiv \frac{-\sigma^2}{(1-l)x_2}$, or equivalently,
   $\alpha > \bar{\alpha}(l) \equiv u_B^{-1}(\bar{u}_B(l))$, where $u_B(u_B^{-1}(x)) = x$.
   (a) Suppose $\bar{u}_B(l) > u_B^0$.
       Then, $\frac{\partial^2 E}{\partial \alpha^2} < 0$ for $\alpha \in [\bar{\alpha}(l) < 0, e^I]$.
   (b) Suppose $\bar{u}_B(l) < u_B^0$.
       Then $\frac{\partial^2 E}{\partial \alpha^2} < 0$ for $\alpha \in [\bar{\alpha}(l) > 0, e^I]$ and
       $\frac{\partial^2 E}{\partial \alpha^2} > 0$ for $\alpha \in [0, \bar{\alpha}(l)]$.
4. Note that $E(\alpha = e^I) = e^L + e^I p - B$ and $E(\alpha = 0) = \int_{u_B^0}^{\infty} [e^L + e^I(x_1 + x_2) - B]$. And,
   $\frac{\partial E}{\partial \alpha} = \int_{u_B^0}^{\infty} [(u - u_B) + (1-l)x_2] g(u) du + \frac{(B - e^L - e^I p)}{(e^I - \alpha)} [1 - G(u_B) + (1-l)x_2 g(u_B)]$
   $\to -\infty$ as $\alpha \to e^I$. It follows then that α^*, the solution to the FOC, satisfies $\alpha^* < e^I$.

We now consider α over the following cases (see Figures 11 (a) - (f)):

<u>Case I:</u> $\bar{u}_B(l) > u_B^0$.

From (3) above,

$\frac{\partial^2 E}{\partial \alpha^2} < 0$ for $\alpha \in [\bar{\alpha}(l) < 0, e^I]$. Then we have two sub-cases:

Case Ia: If $\alpha^* > 0$, then α^* is the global maximum.

Case Ib: If $\alpha^* < 0$, then $\frac{\partial E}{\partial \alpha} < 0 \quad \forall \alpha > \alpha^*$ since $\frac{\partial^2 E}{\partial \alpha^2} < 0$

$\Rightarrow \alpha = 0$ is the global maximum.

<u>Case II:</u> $\bar{u}_B(l) < u_B^0$.

Then $\frac{\partial^2 E}{\partial \alpha^2} < 0$ for $\alpha \in [\bar{\alpha}(l) > 0, e^I]$ and $\frac{\partial^2 E}{\partial \alpha^2} > 0$ for $\alpha \in [0, \bar{\alpha}(l)]$. Again, we have two sub-cases:

Case IIa: $\frac{\partial E}{\partial \alpha}\big|_{\alpha=0} > 0$.

Then, $\frac{\partial E}{\partial \alpha}\big|_{\alpha=\bar{\alpha}(l)} > 0$ since $\frac{\partial^2 E}{\partial \alpha^2} > 0 \quad \forall \alpha \in [0, \bar{\alpha}(l)]$.

$\therefore \alpha^* \in [\bar{\alpha}(l), e^I]$ since $\frac{\partial E}{\partial \alpha}\big|_{\alpha=e^I} < 0$.

Then, α^* is the global maximum.

Case IIb: $\frac{\partial E}{\partial \alpha}\big|_{\alpha=0} < 0$.

Case IIbi: Then if $\frac{\partial E}{\partial \alpha}\big|_{\alpha=\bar{\alpha}(l)} < 0$, $\alpha^* \notin [\bar{\alpha}(l0 < e^I]$, and $\alpha = 0$ is the global maximum.

Case IIbii: Or, if $\frac{\partial E}{\partial \alpha}\big|_{\alpha=\bar{\alpha}(l)} > 0$, then $\alpha^* \in [\bar{\alpha}(l), e^I]$ and $\alpha = \alpha^*$ or 0 based on

whether $E(\alpha^*) > E(\alpha = 0)$ or vice-versa.

Thus, the only possible solutions are $\alpha = 0$ or $\alpha = \alpha^*$, the solution to the FOC. $\square$

Proposition A1: As B increases, there exists a value $\hat{B}$ such that $\alpha = 0$ is the gloal optimum.

Proof:

From equation (6), we have

$$\frac{\partial E}{\partial \alpha} = \frac{1}{(e^I - \alpha)} \left[-E + (e^L + e^I p - B)(1 - G(u_B) + (1-l)x_2 g(u_B)) \right] = 0.$$

Thus,

$$\frac{\partial^2 E}{\partial \alpha \partial B} = \frac{1}{(e^I - \alpha)} \left[-\frac{\partial E}{\partial B} - [1 - G(u_B) + (1-l)x_2 g(u_B)] + (e^L + e^I p - B)(-g(u_B) + (1-l)x_2 g'(u_B)) \frac{\partial u_B}{\partial B} \right]$$

Since

$$\frac{\partial u_B}{\partial B} = \frac{1}{(e^I - \alpha)} \quad \text{and}$$
$$\frac{\partial E}{\partial B} = -[1 - G(u_B)] - (1-l)x_2 g(u_B),$$

we have that

$$\frac{\partial^2 E}{\partial \alpha \partial B} = \frac{(e^L + e^I p - B)((1-l)x_2 g'(u_B) - g(u_B))}{(e^I - \alpha)^2}.$$

For all interior solutions,

$$\text{sign}\left(\frac{d\alpha^*}{dB}\right) = \text{sign}\left(\frac{\partial^2 E}{\partial\alpha\partial B}\right).$$

From the proof of Lemma 1, the SOC is satisfied for all interior solutions, and $(1-l)x_2 g'(u_B) > g(u_B)$. By the solvency condition, $e^L + e^I p - B > 0$. Therefore, $\frac{d\alpha^*}{dB} > 0$ for interior solutions.

In general, as B increases, the interior $\alpha^* \to e^I$. We have shown, however, that $E(\alpha = 0) > E(\alpha^* \to e^I)$. Therefore, as B increases, there exists a threshold value $\hat{B}$ such that $\alpha = 0$ is the global optimum. $\square$

Proposition A2: When $\alpha^* = 0$, $\frac{du_B}{dB} > 0$.

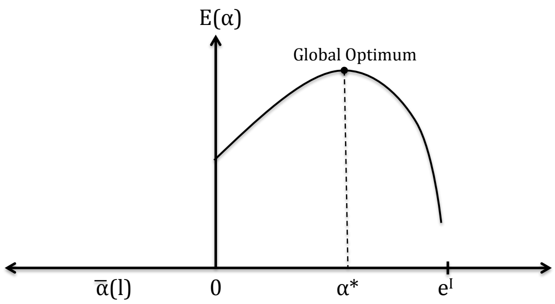
Proof:

For all interior solutions,

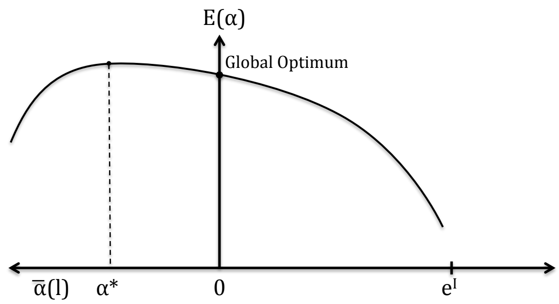
$$\frac{du_B}{dB} = \underbrace{\frac{\partial u_B}{\partial B}}_{>0} + \underbrace{\frac{\partial u_B}{\partial \alpha}}_{<0}\underbrace{\frac{d\alpha^*}{dB}}_{>0}$$

(For signs, see Equation 7 and Online Appendix A, Proposition 1).

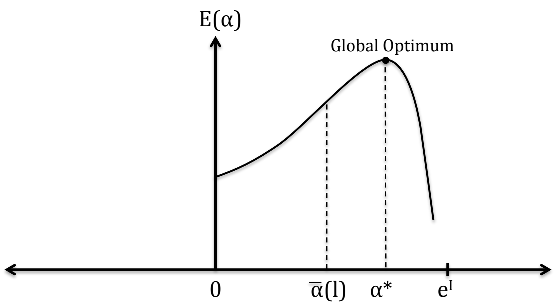
In general, $\frac{du_B}{dB}$ cannot be signed. When the solution to the bank's problem is at the $\alpha^* = 0$ corner, however, the rightmost terms will disappear, and therefore $\frac{du_B}{dB} > 0$ and the probability of default is also increasing. $\square$



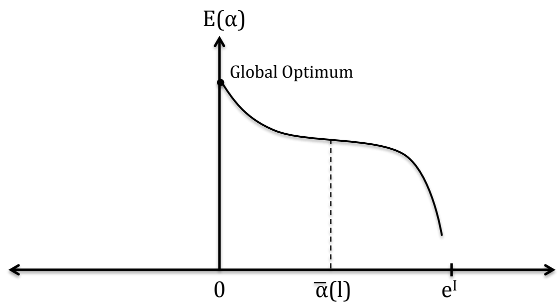
(a) Case Ia from Appendix A, Lemma 1



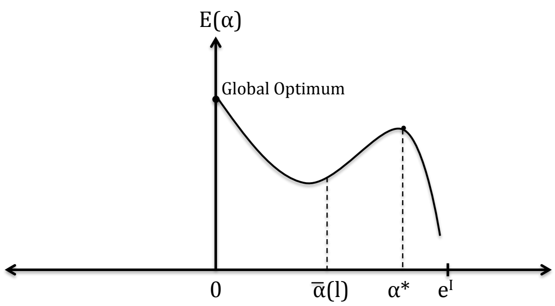
(b) Case Ib from Appendix A, Lemma 1



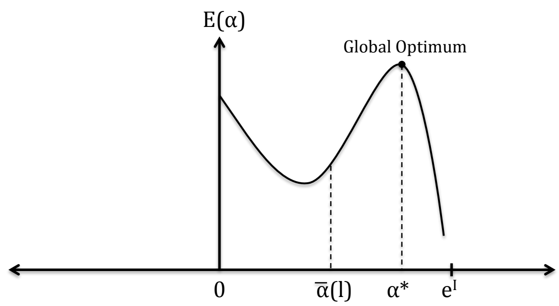
(c) Case IIa from Appendix A, Lemma 1



(d) Case IIbi from Appendix A, Lemma 1



(e) Case IIbii(1) from Appendix A, Lemma 1



(f) Case IIbii(2) from Appendix A, Lemma 1

Online Appendix A Figures 1 (a)-(f): Demonstration of Cases in Lemma 1

Online Appendix B: Equity Issuance Model

Suppose that instead of implementing asset-based deleveraging, the broker-dealer engages in financial deleveraging by issuing equity at date 0 and retaining the proceeds as a cash buffer for repaying debt in the future. We assume for tractability and for obtaining an interior solution that there is a convex cost of issuing debt. These costs can represent direct underwriting costs as well as indirect costs resulting from adverse selection problems in issuance (Myers and Majluf, 1984).

Formally, the firm raises an amount of equity k at issuance cost $c(k) = \frac{1}{2}\gamma k^2$, $\gamma > 0$. Then the no-default condition at date 1 is givey by

$$e^L + e^I(\bar{x}_1 + u) + e^I l x_2 + k \geq B, \quad \text{or}$$
$$u \geq u_B(k) \equiv \frac{1}{e^I}[B - k - e^L - e^I\bar{x}_1 - e^I l x_2].$$

As is clear, we can either assume that equity raised is retained as cash or alternatively that it is used to reduce leverage by repurchasing debt (or calling it) at face value.

Suppose that the equity value post recapitalization is E(k). Then raising equity of k would require selling a stake θ of existing shareholder wealth to new shareholders, such that

$$\theta E(k) = k.$$

Then, the current shareholders maximize their share of equity value $(1-\theta)E(k) = E(k) - k$, where

$$E(k) \equiv \int_{u_B}^{\infty} [e^L + e^I x_1 + e^I x_2 + k - B] g(u) du - \frac{1}{2}\gamma k^2.$$

Thus, the equity issuance problem is

$$\max_k \hat{E}(k) \equiv E(k) - k.$$

The FOC is given by

$$\int_{u_B}^{\infty} g(u)du - [e^L + e^I(\bar{x}_1 + u_B) + e^I x_2 + k - B]\frac{\partial u_B}{\partial k} - \gamma k - 1 = 0.$$

Noting that $\frac{\partial u_B}{\partial k} = -\frac{1}{e^I}$ and substituting for $u_B(k)$, we obtain the simplified FOC

$$\int_{u_B}^{\infty} g(u)du + (1-l)x_2 g(u_B) - \gamma k - 1 = 0, \quad \text{or}$$
$$PR(u \leq u_B(k)) = (1-l)x_2 g(u_B) - \gamma k.$$

The SOC is given by

$$
\begin{aligned}
&(-g(u_B) + (1-l)x_2 g'(u_B))\frac{\partial u_B}{\partial k} - \gamma \\
&= \frac{g(u_B) - (1-l)x_2 g'(u_B)}{e^I} - \gamma.
\end{aligned}
$$

We refer the reader to Online Appendix A for proof that either the SOC holds or the solution is at a corner.[1] For interior solutions, k^*, the optimal equity issuance, is given by

$$\frac{\partial \hat{E}}{\partial k} = 0.$$

In turn,

$$\frac{\partial^2 \hat{E}}{\partial k \partial l} + \frac{\partial^2 \hat{E}}{\partial k^2}\frac{dk^*}{dl} = 0.$$

Provided that the SOC is interior, then $\frac{\partial^2 \hat{E}}{\partial k^2} < 0$, so

$$\text{sign}\left(\frac{dk^*}{dl}\right) = \text{sign}\left(\frac{\partial^2 \hat{E}}{\partial k \partial l}\right).$$

Differentiating the FOC with respect to l, we obtain $\frac{\partial^2 \hat{E}}{\partial k \partial l} = -(1-l)x_2^2 g'(u_B)$. For $u \sim \mathcal{N}(0, \sigma^2)$ and $u_B < 0$, $g'(u_B) > 0$ and so $\frac{\partial^2 \hat{E}}{\partial k^2} < 0$. This, in turn, implies that $\frac{dk^*}{dl} < 0$.

In other words, as the LOLR becomes more generous, the extent of financial deleveraging by issuing equity capital declines. The intuition is the same as before. While a complete analysis of the problem would require considering corner solutions that arise whenever the SOC does not hold, and also requires ensuring that the required equity stake to sell $\gamma = \frac{k}{E(k)} < 1$, this sketch of the model with equity issuance highlights that asset-based and recapitalization-based deleveraging behave similarly in the strength of the LOLR. Since the lack of recapitalization by banks has been the focus of other studies (Acharya et al. 2014 and Acharya et al. 2011b, among others), we focus in this paper on asset sales.

References

[1] Myers, S. and N. S. Majluf, Corporate Financing and Investment Decisions When Firms Have Information That Investors Do Not Have, *Journal of Financial Economics, 13 (2)*, 187-221.

[1] Although the range of parameters is different, the proof follows a similar set of steps.