Online Appendix
Disasters with Unobservable Duration and Frequency: Intensified Responses and Diminished Preparedness

A Proofs and Derivations

A.1 Full Information

To prove Proposition 1, we first treat the case of full-information in which the only state variables are $s \in \{0,1\}$ and q. For ease of notation, define the following combination of preference parameters:

$$e_0 \equiv \frac{\theta}{\psi}\,\rho^{\psi} \qquad \text{and} \qquad e_1 \equiv -\frac{\psi}{\theta}. \tag{A.1}$$

Also define $\lambda(0) = \eta, \lambda(1) = \lambda$.

Lemma *Denote*

$$g(s) \equiv \theta\,\rho - (1-\gamma)\left(\mu(s) - \frac{1}{2}\gamma\sigma(s)^2\right) - \zeta(s)\ \left([1-\chi(s)]^{1-\gamma} - 1\right) \tag{A.2}$$

for $s \in \{0,1\}$. Let $H(s)$'s denote the solution to the following system of recursive equations:

$$g_0 \equiv g(0) = e_0\,(H(0))^{e_1} + \eta\left[\frac{H(1)}{H(0)} - 1\right] \tag{A.3}$$

$$g_1 \equiv g(1) = e_0\,(H(1))^{e_1} + \lambda\left[\frac{H(0)}{H(1)} - 1\right] \tag{A.4}$$

Assuming the solutions are positive, optimal consumption in state s is

$$C(s) = \rho^{\psi}\ (H(s))^{e_1}\,q, \tag{A.5}$$

and the value function of the representative agent is

$$\mathbb{J}(s) \equiv \frac{H(s)q^{1-\gamma}}{1-\gamma}. \tag{A.6}$$

Proof. Using the evolution of capital stock for the representative agent (1) the Hamilton-

Jacobi-Bellman (HJB) equation for each state s can be written:

$$\begin{aligned}0 = \max_C \Big[& f(C, \mathbb{J}(s)) + \mathbb{J}_q(s)(q\mu(s) - C) \\ & + \frac{1}{2}\mathbb{J}_{qq}(s) q^2 \sigma(s)^2 + \zeta(s)\left[\mathbb{J}(s, q(1-\chi(s))) - \mathbb{J}(s,q)\right] \\ & + \lambda(s)\left[\mathbb{J}(s') - \mathbb{J}(s)\right]\Big]\end{aligned} \tag{A.7}$$

for $s = \{0, 1\}$ and $s' = \{1, 0\}$.

Taking the first-order condition with respect to $C(s)$ in (A.7), we obtain

$$f_c(C, \mathbb{J}(s)) - \mathbb{J}_q(s) = 0. \tag{A.8}$$

Using $f(C, \mathbb{J})$ from (4) and taking the derivative with respect to C, we obtain

$$f_c = \frac{\rho C^{-\psi^{-1}}}{[(1-\gamma)\mathbb{J}(s)]^{\frac{1}{\theta}-1}}. \tag{A.9}$$

Substituting the conjecture $\mathbb{J}(s)$ in equation (7) yields

$$f_c = \frac{\rho C^{-\psi^{-1}}}{H(s)^{\frac{\gamma-\psi^{-1}}{1-\gamma}} q^{\gamma-\psi^{-1}}}. \tag{A.10}$$

Then, for state s, we obtain by substituting $\mathbb{J}_q(s) = H(s)q^{-\gamma}$ in (A.8), and simplifying:

$$C(s) = \frac{H(s)^{-\psi/\theta} q}{\rho^{-\psi}} \tag{A.11}$$

which agrees with (6) using the definitions of the constants in (A.1).

To verify the conjectured form of the value function, we plug it in to the HJB equation (A.7) and reduce it to the recursive system in the proposition via the following steps:

1. substitute the optimal policy $C(s)$ into the HJB equation (A.7);
2. cancel the terms in q which have the same exponent; and
3. group constant terms not involving Hs and define them to be $g(0)$ for state 0 and $g(1)$ for state 1.

The third step yields the system of recursive equations A.3, A.4. □

Regularity Conditions

The functions $H(s)$ are necessarily bounded by the limiting solutions in which the economy is never in a disaster, H_0^{min}, or is always in a disaster, H_1^{max}. It is straightforward to show that these constants are given by

$$H_0^{min} = \left(\frac{g_0}{e_0}\right)^{1/e_1} \qquad \text{and} \qquad H_1^{max} = \left(\frac{g_1}{e_0}\right)^{1/e_1}.$$

These quantities are real and positive if g_0, g_1, and e_0 all have the same sign. Given this, it can be shown that a necessary and sufficient condition for existence of a unique solution is that $g_1 < g_0$.

A.2 Proposition 1: HJB System with Parameter Uncertainty

Proof. As noted in the text, the model can be parameterized in terms of the state variables $M, \hat{\eta}, \hat{\lambda}$, and q, where $M = M_t$ is an integer counter that increases on a state switch such that $M_0 = 0$ and even numbered states are the non-disaster epochs and odd numbered states are the disasters. Also, in the non-disaster states, $\hat{\lambda}$ is constant, while $\hat{\eta}$ is constant in disasters. As a consequence, compared with the derivation above for the full-information case, there is now only one additional source of variability in each regime. The dynamics of $\hat{\eta}$ are given in (5) with an analogous expression for and $\hat{\lambda}$. And note that, under the agents' information set, the dynamics of the wealth variable q are identical to the full information dynamics.

As a result, the HBJ equations under partial information are the same as (A.7) above (with state 0 and state 1 being replaced by M and $M+1$) with the addition of a single term on the right side:

$$-\frac{(\hat{\eta})^2}{a^{\eta}} \frac{\partial \mathbb{J}(0)}{\partial \hat{\eta}} \tag{A.12}$$

for $s = 0$, and

$$-\frac{(\hat{\lambda})^2}{a^{\lambda}} \frac{\partial \mathbb{J}(1)}{\partial \hat{\lambda}} \tag{A.13}$$

for $s = 1$. Since, under the agent's information set, the state switches are a point-process with instantaneous intensities $\hat{\eta}$ and $\hat{\lambda}$, these quantities also replace their full information counterparts, η and λ, in multiplying the jump terms in the respective equations.

The next steps in the derivation involving the first order condition for optimal consumption are unchanged from the full-information case. This follows because consumption does not enter into any of the new terms involving the information variables. Replace $\mathbb{J}$ by the conjecture $\frac{q^{1-\gamma}}{1-\gamma} H(\hat{\eta}, \hat{\lambda}, M)$, then a common power of q term is cancelled, and the whole equation is divided by H. These manipulations lead to the above two terms showing up on the right hand side, in a system that is otherwise identical to the full-information system (A.3) and (A.4).

$$g_0 = e_0 H_M^{e_1} + \hat{\eta} \left(\frac{H_{M+1}}{H_M} - 1 \right) - \frac{(\hat{\eta})^2}{a^\eta H_M} \frac{\partial H_M}{\partial \hat{\eta}} \tag{A.14}$$

$$g_1 = e_0 H_{M+1}^{e_1} + \hat{\lambda} \left(\frac{H_{M+2}}{H_{M+1}} - 1 \right) - \frac{(\hat{\lambda})^2}{a^\lambda H_{M+1}} \frac{\partial H_{M+1}}{\partial \hat{\lambda}} \tag{A.15}$$

where the constants g_0 and g_1 are as defined in Lemma 1 above.

□

A.2.1 Solution Algorithm

In the full information case, solution of the algebraic system over a grid in the $(\hat{\eta}, \hat{\lambda})$ plane is straightforward. The unknown constants $H(s)$ are bounded by the limiting solutions in which the economy is never in a disaster, H_0^{min}, or is always in a disaster, H_1^{max}. The former corresponds to $\eta = 0$ and the latter to $\lambda = 0$.

For the general case, we pick a large even integer M^{max} and assume that the economy has converged to the full information solution with $s = 0$ at M^{max} and $s = 1$ at $M^{max} - 1$. Given these solutions, the HBJ system for $M = M^{max} - 2$ is just a first order ODE, since the jump terms in (A.14)-(A.15) can be explicitly evaluated. For even values of M, the boundary condition at $\hat{\eta} = 0$ is again the full-information solution because the posterior standard deviation $\sqrt{a^\eta} \hat{\eta}$ is also zero. (Note that the value of $\hat{\lambda}$ is immaterial if disasters cannot arise.) Likewise, for odd values of M, the boundary condition at $\hat{\lambda} = 0$ is given by the full-information solution. Hence, the first-order ODEs can be explicitly solved in alternating directions. The procedure is then repeated for all lower values of M.

A.3 Pricing Kernel, Riskless Rate and Proposition 2

This section first derives the pricing kernel and riskless rate under partial information. The results are then used to prove Proposition 2 Section 3.2 which describes the pricing equation of insurance against a disaster.

Under stochastic differential utility, the kernel can be represented as

$$\Lambda_t = e^{\int_0^t f_{\mathbb{J}} du} f_C \tag{A.16}$$

where the aggregator function is given in (4). With the form of the value function and the optimal consumption rule from Proposition 1, evaluating the partial derivatives yields (after some rearrangement)

$$\Lambda_t = q^{-\gamma}\ H(\hat{\eta}, \hat{\lambda}, M)\ e^{\int_0^t [c_u\ (\theta-1) - \rho\theta] du} \tag{A.17}$$

where $c = c(\hat{\eta}, \hat{\lambda}, M) \equiv C/q$ is the marginal propensity to consume.

The riskless rate is minus the expected rate of change of $d\Lambda_t/\Lambda_t$ under the agents' information set. Applying Itô's lemma, for even values of M, the expected change is

$$c\ (\theta - 1) - \rho\theta\ - \gamma(\mu - c) + \frac{1}{2}\gamma(\gamma+1)\sigma^2$$

$$-\frac{(\hat{\eta})^2}{a^\eta}\ \frac{1}{H}\frac{\partial H}{\partial \hat{\eta}}\ + \hat{\eta}\left(\frac{H(M+1)}{H(M)} - 1\right).$$

A key simplification is to observe that, by the HJB equation derived above (see (A.14)), the latter two terms in this expression can be replaced by $g_0 - \frac{\theta}{\psi}c$. This causes all of the terms involving c to exactly cancel. Using the definition of g_0 in (A.2), the remaining terms are just $-\mu + \gamma\sigma^2$. Hence we have shown

$$r_0 = \mu - \gamma\sigma^2.$$

Repeating the above steps for odd values of M and applying the same trick yields

$$r_1 = \mu - \gamma\sigma^2 - \zeta\chi(1-\chi)^{-\gamma}.$$

Turning to the insurance claim, the asset is assumed to make a terminal payout of 1.0 upon the occurrence of the next disaster. Proposition 3 characterizes its price in normal-times prior to that disaster.

Proof. We conjecture that the price, P, of the insurance is not a function of wealth, q. Moreover, when $s = 0$, the state variables a^η, a^λ, and $\hat{\lambda}$ are all fixed, and $\hat{\eta}$ evolves deterministically according to (5).

By the definition of the pricing kernel, for any claim in the economy, its instantaneous payout per unit time (in this case, zero) times Λ must equal minus the expected change of the product process $P\Lambda$, or

$$\mathcal{L}(\Lambda(q_t, s_t, \hat{\eta}_t)\ P(s_t, \hat{\eta}_t))/\Lambda_t = 0, \tag{A.18}$$

where $\mathcal{L}(X)$ is the drift operator $\mathrm{E}[dX]/dt$ under the agents' information set.

Using Itô's lemma for jumping processes to expand the expected change,

$$-\frac{(\hat{\eta})^2}{a^\eta}\frac{\partial P}{\partial \hat{\eta}} + \mu_\Lambda P + \hat{\eta}\left(\frac{H(M+1)}{H(M)} - P\right) = 0$$

where we have written μ_Λ for the deterministic terms in $d\Lambda_t/\Lambda_t$ and used the fact that $P(M+1) = 1$.

Next, add and subtract $\hat{\eta}(\frac{H(M+1)}{H(M)} - 1)P$ and use the fact that the expected growth rate of the pricing kernel is minus the riskless rate:

$$r_0 = -\mu_\Lambda - \hat{\eta}\left(\frac{H(M+1)}{H(M)} - 1\right)$$

to get (8):

$$-\frac{(\hat{\eta})^2}{a^\eta}\frac{\partial P}{\partial \hat{\eta}} - r_0 P + \hat{\eta}\frac{H(M+1)}{H(M)}(1-P) = 0.$$

□

A.4 Real Options

A.4.1 Mitigation

The text in Section 3 describes endowing the model economy with a one-time real option to invest in a mitigation technology to alter a structural parameter, χ, via $\chi = g(i)$ where I is a lump-sum investment and $i = I/q$. Since the option is a one-shot decision, the post-investment economy is identical to the original model (without the technology) and hence its value function is as derived in the main propositions.

Then, the assertion is that, for two otherwise equal economies E1 and E2, if the sensitivity of the value function, H, to χ is weaker in E1 than in E2, then, if a solution to the real-options problem exists in E2, a solution also exists in E1 with smaller optimal investment.

To see this, view H as a function of χ, and the problem is to choose i to maximize the $H(g(i))(1-i)^{1-\gamma}/(1-\gamma)$ with first order condition $-g'(i)\,\partial \log H(g(i))/\partial\chi = (\gamma - 1)/(1-i)$. Assume $\gamma > 1$. Then the right side (the marginal cost) is an unbounded increasing function of i on $[0,1)$ which is the same for both economies. Call it $RHS(i)$. On the left side (the marginal benefit), the first term is the same for both economies. The hypothesis is that $\partial \log H(\chi)/\partial\chi$ is smaller in E1 than in E2 for all χ implying that the second term is smaller. Hence $LHS1(i) < LHS2(i)$ for all i. Assume $LHS2$ is continuous and declining. Then, if an interior solution, i_2^*, exists, it follows that on $[i_2^*,1)$ we have $LHS1 < LHS2 < RHS$, meaning that there cannot be a solution for E1 in this region. Hence, either there is a solution $i_1^* < i_2^*$ or no interior optimum exists and $i_1^* = 0$ in E1.

A.4.2 Information Production

The top panel of the table below presents the optimal information investment as a fraction of wealth when the economy contains a technology allowing agents to purchase a realization of N transitions of the disaster process, e.g., in a laboratory. The realization increases a^{λ} by N but also alters the mean $\hat{\lambda}$ depending on the (random) time-length of the realization.[1] The table assumes that the information production function is $N = 200\,i$, where $i = I/q$ is the lump-sum investment. The option to make this investment is a one-time occurrence at the on-set of a disaster.

The lower panel reports the welfare gain, in units of wealth, of the investment. The difference between the respective panels can be interpreted as the value-added of the technology.

[1]The time-length is a virtual output. The experiment is assumed to be atemporal. Agents receive the results immediately.

Table A.1: Information Production

(A) Optimal Investment

Benchmark

		$\hat{\lambda}$		
		0.2	0.5	1.0
$\hat{\eta}$	0.01	0.015	0.015	0.015
	0.05	0.015	0.015	0.015

$\psi = 0.20$

		$\hat{\lambda}$		
		0.2	0.5	1.0
$\hat{\eta}$	0.01	0.030	0.035	0.040
	0.05	0.035	0.030	0.030

$\gamma = 2$

		$\hat{\lambda}$		
		0.2	0.5	1.0
$\hat{\eta}$	0.01	0.015	0.015	0.010
	0.05	0.010	0.015	0.015

$\rho = 0.02$

		$\hat{\lambda}$		
		0.2	0.5	1.0
$\hat{\eta}$	0.01	0.030	0.035	0.035
	0.05	0.025	0.030	0.03015

(B) Welfare Gain

Benchmark

		$\hat{\lambda}$		
		0.2	0.5	1.0
$\hat{\eta}$	0.01	0.038	0.046	0.047
	0.05	0.032	0.042	0.042

$\psi = 0.20$

		$\hat{\lambda}$		
		0.2	0.5	1.0
$\hat{\eta}$	0.01	0.314	0.346	0.343
	0.05	0.284	0.320	0.343

$\gamma = 2$

		$\hat{\lambda}$		
		0.2	0.5	1.0
$\hat{\eta}$	0.01	0.031	0.032	0.020
	0.05	0.021	0.030	0.026

$\rho = 0.02$

		$\hat{\lambda}$		
		0.2	0.5	1.0
$\hat{\eta}$	0.01	0.114	0.164	0.188
	0.05	0.085	0.130	0.143