Capacity and Inventory Management in the Presence of a Long-term Channel and a Spot Market

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Abstract

Manufacturers often sell their products both to customers acquired over time and sustained through contractual agreements and to new businesses through electronic markets. These two sales channels (contract markets and exchanged based sales and procurement) coexist for several capital-intensive industries such as the semiconductor and chemical industries (Kleindorfer and Wu 2003). The two sales channels can enable the manufacturer to better utilize the available capacity if the manager can optimally allocate resources. In this paper, we establish an optimal production and inventory allocation policy for a periodic-review, finite-horizon, capacity-constrained manufacturing system. In particular, we show that a policy with two thresholds, *produce-up-to* and *sell-down-to/buy-up-to* is optimal. We also provide some insights into the manufacturer's dynamic pricing policy for the long-term channel.

1 Introduction

Today, many commodity chip buyers meet their requirements through both long-term contracting with the manufacturer and purchasing from a DRAM spot market. For example, Hewlett Packard procures chips through long-term contracts and spot markets (Billington 2002). Contracts often cover a preset planning horizon (such as 12 months). While these contracts provide price stability, they often require the buyer to agree on a quantity flexibility up front. On the other hand, the spot market does not require either the chip buyer or the manufacturer to commit any quantity upfront. However, this flexibility comes at a risk due to volatile spot market value. Converge.com is one of several spot

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markets available for trading memory chips. The copper industry is another example, in which a fixed price contract often governs the trade between the manufacturer and the buyer. Yet, buyers can also procure from commodity exchanges such as the London metal exchange. Since the early 90's several spot exchanges have existed for commodity chemicals, plastics and metals, such as www.metalsite.com (Glen and Weiner 1989).

In this paper, we study a capacity constrained manufacturer's periodic review production and inventory allocation problem. The manufacturer produces to satisfy uncertain demand received through two different channels. The first channel constitutes long-term customer demand such as distributors or wholesalers. The trade in this channel is based on contractual agreements. Examples of such agreements are the price only contracts or the quantity discount contract structured for multiple periods. The manufacturer may incur a penalty cost if he cannot deliver the orders requested through this first channel. Unmet orders can be lost or backlogged. Our analysis will address both scenarios. The second channel is the spot market or an exchanged based business-to-business market. At the beginning of each period, the manufacturer decides on the quantity to sell through the exchange. A clearance mechanism, such as an auction, determines the value of products offered on the spot market. The final value is determined at the end of each period. The manufacturer produces and sells his product through these two channels concurrently. By deciding on the production quantity, the manufacturer consumes also from the remaining capacity that is available through the planning horizon. His objective is to optimally decide how much to produce given available capacity and how to allocate inventory across the two sales channels.

Contract markets and spot exchanges are inherently related. An increase in a product's popularity or a shortage would probably affect both sales channels. For example, the 1999 earthquake in Taiwan caused an immediate increase in global computer memory (DRAM) price. The spot price for DRAM went up five times in one week and the contract price paid by major producers increased by 25% in the same week (Papadakis and Ziemba 2002). When long-term buyers cannot satisfy their requirement through contractual agreements, they may move to the spot market to supplement their contracted orders, which can also shift the spot market value. To capture the non-stationarity of long-term demand, the spot market value and the correlation between these two channels, we introduce the state of the world modeled with a Markov process. This process governs both the distribution of long-term demand and the spot market value.

We do not assume here a specific contract for the long-term market, and we do not specify an auction mechanism for the spot market. We take these markets as given and characterize an optimal production and inventory policy to maximize the manufacturer's total profit. In particular, we show that a *produce-up-to* and *sell-down-to/buy-up-to* policy is optimal when the long-term profit function is concave and the spot market has a diminishing value to the quantity traded (i.e., increasing and concave). At the beginning of each period, the manufacturer observes the state of the world, the net inventory, and the available production capacity. Based on the state of the system, the policy triggers a production order to increase the net inventory to a level as close as possible to that period's *produce-up-to* threshold given available capacity. If after the production decision the net inventory exceeds (resp., is below) the *sell-down-to* (resp., *buy-up-to*) threshold, then the policy requires the manufacturer to allocate (resp., buy) the difference between the threshold and the net inventory on (resp., from) the spot market. Note that both produce-up-to and sell-down-to levels are functions of the state of the world, which equilibrates the importance of both channels. For instance, if the spot market is more attractive than the long-term channel, this may translate into a lower sell-down-to level. We also show that when the spot market value is linear in the number of units traded, the structure of the optimal policy could be simplified further. In that case, the production manager produces such that the capacity level for the next period is consumed down to a threshold. This threshold is state dependent only through the state of the world. To provide additional insights, we also establish monotonicity results and show how policy parameters change with respect to the state of the system. We also provide a numerical example to illustrate the optimal policy's structure. Finally, we characterize the manufacturer's optimal pricing for the long-term sales channel and his optimal total capacity prior to the start of the planning horizon. For this case, we restrict the mean of the long-term demand to a linear function of the unit price. By comparing this setting to the one where no spot market is available, we show, for example, that the existence of a spot market induces a higher unit price. Hence, the long-term customer is worse off with a spot market.

2 Literature Review

The present paper focuses on the operational aspect of the problem, that is, on how to optimally manage a periodic-review, capacity constrained manufacturing system that satisfies customers through long-term channel (contract market) and the spot market over a finite horizon. Hence, the paper is closely related to three streams of research: capacity constrained production or inventory control problems, supply chain contracting and spot markets. Here, we focus on closely related papers and refer the reader to others who provide comprehensive coverage on each of these topics.

The capacity constrained inventory control problem poses significant challenges. For example, when the manager incurs a *positive* fixed cost for a production decision, no one knows what the optimal policy is for this system. Researchers, however, have been able to partially characterize or provide optimal production control policies for special cases and different demand models (Shaoxiong and Lambretch 1996, Gallego and Scheller-Wolf 2000, and Özer and Wei 2004). On the other hand, for a *zero* fixed cost *stationary* production system, Federgruen and Zipkin (1986) show that a *modified* base stock policy is optimal. We refer the reader to Özer and Wei (2004) for a more comprehensive coverage of capacity constrained production control problems.

Another important aspect of the capacity constrained production problem is the decision regarding the expansion and contraction of capacity. Luss (1982) provides a comprehensive review of this topic. Recently, Angelus and Porteus (2002) characterize an optimal policy for simultaneous capacity and production management when the inventory is not carried from one period to the next. Bradley and Glynn (2003) study a continuous version of a simultaneous capacity and inventory decision problem. In the present paper, we take the total capacity over the entire horizon as given. However, we assume that the manufacturer can decide how much of the available production capacity to use for the current period and how much to reserve for future periods. These capacity dynamics are different from the classical literature, which assumes production capacity for each period to be known and predetermined. In this sense, our capacity constrained production control problem has the flavor of the revenue management problem (Gallego and van Ryzin 1994 and Bitran and Mondschein 1997).

Many companies allocate a large fraction of their procurement budget to long-term contracts (Billington 2002). Contract markets are an integral part of today's supply chains. Among the most commonly used contracts are fixed price and quantity discounts contracts. A large group of researchers study the interaction between two firms focusing on coordinating contracts, such as buyback contracts (Pasternack 1985) and quantity flexibility contracts (Tsay 1999). Cachon (2003) and Chen (2003) provide an extensive survey of this literature. The difficulty of administering a portfolio of contracts popularized the use of simple contracts whose impact on supply chain profits is studied by others, such as the price only contract in Lariviere and Porteus (2001), the quantity discount in Monahan (1984) and advance purchase contract in Özer and Wei (2002).

The third line of research is the extensive literature on spot markets and electronic business to business exchanges. Kleindorfer and Wu (2003) provides a comprehensive review of this topic. The research on spot markets addresses topics such as the viability of spot exchanges (Kaplan and Sawhney 2000), the existence of equilibrium (Mendelson and Tunca 2002), pricing mechanisms such as auctions (Chen 2001), relationships between agents (Laffont and Tirole 1998), and finally operations decisions (Seifert et al. 2002 or Araman et al 2002). In the latter works, the spot market is often considered as a means to salvage left over inventory. In our setting, we provide a necessary assumption ensuring that both long-term channel and spot market are equally important. We characterize an optimal policy under both cases.

Kleindorfer and Wu (2003) point out that the integration of long-term markets and spot markets for multi-period, state dependent demand analysis is an open question. In this paper, we address exactly this problem and characterize an optimal policy for a manufacturer who sells through these channels. We introduce a Markov process that governs the distribution of the long-term channel's demand as well as the spot market value. To the best of our knowledge, Song and Zipkin (1993) and Sethi and Cheng (1997) are the first to introduce a Markov modulated demand process into single location inventory control problems and to characterize the optimality of *state dependent* base stock and (s,S) policies.

Relatively few papers address simultaneously contracting markets and spot markets in a multi-period inventory control problem. Those who study this problem often establish effective procurement strategies for a *buyer*. For example, Martinez-de-Albeniz and Simchi-Levi (2003) study the problem of a buyer who *procures* supplies through long-term contracts and a spot market. On the other hand, we study the problem of a capacity constrained manufacturer who *sells* through these channels. Our work complements Martinez-de-Albeniz and Simchi-Levi (2003). Kai and Lee (2003) also address the manufacturer's problem. They study the effect of pricing and supply chain design strategies, such as postponement, for a manufacturer who can sell customized products through Internet and a traditional retail store. In contrast to this dual sales channel literature, there is an extensive literature on production control policies for a manufacturer who can replenish from two distinct supply sources (see for an example, Sethi, Yan and Zhang 2003). Finally, one paper similar in spirit to ours is Caldentey and Wein (2004). In an infinite horizon and a continuous time setting, the manufacturer, who manages a single server queue under heavy-traffic, responds to both demand from the long-term buyer and the spot market. The long-term demand is a deterministic fluid, while the price of the spot market follows a geometric regulated Brownian motion. Suboptimal but simple inventory policies are obtained in this context.

We organize the rest of the paper as follows. In § 3, we present the model and our notation. In § 4, we characterize an optimal production and inventory allocation policy. Furthermore, we consider the special case of a linear spot market value function and discuss the behavior of the myopic policy. We also provide additional structures to shed some light on the problem. In § 5, we provide insights on the optimal long-term price-only contract and total capacity. In § 6, we provide a numerical example to illustrate the optimal policy structure. In § 7, we conclude the paper.

3 Model Formulation

Consider a capacity constrained manufacturer who produces and sells through two channels over a finite horizon. The manufacturer's objective is to maximize his profit by optimally deciding on (i) how much to produce given limited production capacity and (ii) how to allocate the finished products between two sales channels. Through the first channel, the manufacturer sells the product to his long-term buyers. During each period t, the manufacturer faces a non-negative random demand d_t, the probability density function of which is f_{d_t}. Through the second channel, the spot market, the manufacturer sells his product to spontaneous buyers. A clearance mechanism determines the *spot market value*, which is the manufacturer's profit (cost) from selling on (buying from) the spot market.

An exogenous random variable ω_t, the state of the world, governs the system. In particular, this variable governs both the distribution of long-term demand and the spot market value. It essentially models the non-stationarity of the long-term demand, the spot market value, and the correlation between these two channels. This variable evolves based on a discrete time Markov chain and takes values in the finite set $\mathcal{S} = \{1, ..., S\}$. We denote by A its $S \times S$ transition matrix. In other words, both the spot market value and the long-term demand are Markov modulated processes generated by the same Markov chain.

On the production side, the manufacturer relies on a subcontractor. At the beginning of the planning horizon $t = 0$, the subcontractor and the manufacturer agree on the total quantity Q that the manufacturer can consume. The manufacturer pays in advance the cost of reserving and producing Q units. At the beginning of each period t, the manufacturer decides on the capacity to use from the remaining available capacity Q_t.

The sequence of events is as follows. At the beginning of period $t \in \{0, 1, ..., T\}$, the manufacturer observes the amount of on-hand inventory I_t, the total backorder B_t, and the state of the world w_t. When backordering is not allowed we set $B_t \equiv 0$. He decides on (i) the production quantity q_t and (ii) the allocation of inventory between the two channels. After production completion, the manufacturer assigns s_t units on the spot market, while the remaining inventory faces the long-term demand. Next, the long-term customer demand d_t and the spot market value function are simultaneously realized. Demand is satisfied through on hand inventory, otherwise it is either backlogged or lost. The manager receives the payment from the long-term channel and the spot market transaction and he incurs holding and shortage costs based on the end-of-period net inventory.

At the beginning of period t, let

$$\begin{aligned} x_t \quad &: \quad \text{be the net inventory } \underline{\text{before}} \text{ the production and spot market transactions} \\ &= \quad I_t - B_t, \\ z_t \quad &: \quad \text{be the net inventory } \underline{\text{after}} \text{ the production and spot market decisions} \\ &= \quad x_t + q_t - s_t. \end{aligned} \tag{1}$$

We refer to z_t as the *modified net inventory.*

The state of the system is given by (x_t, Q_t, w_t). After observing d_t, the distribution of which depends on w_t, the manufacturer meets the long-term demand from modified net inventory z_t. At the end of period t, he updates net inventory by $x_{t+1} = g(z_t - d_t)$. If demand is lost, the update is $g(z_t - d_t) = [z_t - d_t]^+$, where $[x]^+ = \max\{0, x\}$ and $[x]^- = \max\{0, -x\}$. If full backlogging is allowed, the update is $g(z_t - d_t) = z_t - d_t$. In general, we assume that g is sublinear (i.e. $g(x+y) \leq g(x) + g(y)$ for all (x, y) on the real line), so that $g(0) = 0$, g is non-decreasing, concave and $|g'(x)| \leq 1$. In summary, the state space updates are

$$\begin{aligned} x_{t+1} &= g(z_t - d_t), &(2)\\ Q_{t+1} &= Q_t - q_t, &(3)\\ \mathrm{P}(w_{t+1} = j | w_t = i) &= \mathrm{P}_i(w_{t+1} = j) = A_{ij}, &(4) \end{aligned}$$

for $t \in \{0, 1, ..., T-1\}$ and $i, j \in \mathcal{S}$.

The following are the constraints on the production quantity and on the quantity sold in the spot market.

$$\begin{cases} 0 \leq q_t \leq Q_t, \\ s_t \leq [x_t + q_t]^+. \end{cases} \tag{5}$$

The first is the production capacity constraint. The manufacturer can produce at most Q_t units at each period t. The second constraint is to ensure that the manufacturer can sell in the spot market only when he has positive on-hand inventory. Note also that this second constraint can be equivalently replaced with $-[x_t + q_t]^- \leq z_t$. Once the production decision is made, the manufacturer allocates his inventory between the spot market and the long-term channel. Hence, deciding on the quantity s_t to sell on the spot market is equivalent to deciding on the net inventory that faces the long-term demand, that is z_t. The spot market decision can be interpreted as an inventory allocation decision among sales channels. We refer to this constraint set by $Y(x, Q) \triangleq \{(q_t, z_t) \in \mathbb{R}^2,$ such that $q_t \in [0, Q]$ and $-[x + q_t]^- \leq z_t\}$.

Next, we define the manufacturer's expected total profit at the beginning of each period. His profit has two components: the profit from spot market transactions and the profit from the long-term channel.

The profit from the spot market Π_t depends on the state of the world $w_t \in \mathcal{S}$ and the number of units s_t the manufacturer trades. We assume the following condition.

A1. For all $w_t \in \mathcal{S}$ and $t \in \{0, 1, \ldots, T\}$, the spot market value function $\Pi_t(\cdot, w_t)$ is increasing and concave in the number of units $s_t \in \mathbb{R}$ traded on the spot market such that $\Pi_t(0, w_t) \equiv 0$.

The concavity on the positive line of the spot market value function models economies of scale from a buyer perspective, which is expected from a spot market relying on a clearance mechanism. Intuitively, concavity means that every additional unit put in the market has a diminishing return for the buyers. Works related to spot markets, in which the exchange occurs as a result of a clearance mechanism, usually assume that the final unit price is linear in the amount traded (Mendelson and Tunca 2003). Spot markets are often run by auctions. Maskin and Riley (1989) show that total price paid to acquire units through an efficient auction mechanism is concave in the quantity sold for valuations following increasing hazard rate distributions. Concerning the concavity on the negative line, it is equivalent to a convex payment function for the manufacturer. This assumption relates to the manufacturer experiencing an increasing marginal cost for every additional unit bought from the spot market. Such behavior is typical for instance in a capacity constrained spot market. Furthermore, the concavity on the negative line assumption is also consistent with our model where the manufacturer's main activity is selling; while buying from the spot market (comparable to a subcontracting activity) is left as a means to hedge against a high penalty cost from the long-term contract.

The expected profit from the long-term channel Γ_t depends on the state of the world $w_t \in \mathcal{S}$ and the modified net inventory z_t. We assume the following condition.

A2. For all $w_t \in \mathcal{S}$ and $t \in \{0, \dots, T\}$, the expected single-period profit from the long-term channel $\Gamma_t(\cdot, w_t)$ is twice differentiable in the modified net inventory z_t, concave and coercive[1] such that $\partial_z \Gamma_t(0, w_t) \geq 0$.

This assumption is satisfied, for example, when the manufacturer charges a wholesale price $p_t > 0$ for each unit sold through the long-term channel and incurs a penalty cost b_t per unit of unmet demand, and a holding cost h_t per unit of on-hand inventory at the end of the period. In this example, when loss of sales is assumed, the expected profit from the long-term channel is

$$\Gamma_t(z_t, w_t) = \mathbb{E}_t\{p_t \min\{d_t, z_t\} - h_t[z_t - d_t]^+ - b_t[d_t - z_t]^+\}. \tag{6}$$

When backlogging is allowed, the expected profit function is given by

$$\Gamma_t(z_t, w_t) = \mathbb{E}_t\{p_t d_t - h_t[z_t - d_t]^+ - b_t[d_t - z_t]^+\}. \tag{7}$$

Payment is made at the time of the order. In both cases the expectation is taken with respect to long-term demand d_t whose distribution depends on the realization of the state of the world w_t.

Notice that the wholesale prices are pre-specified at the beginning of the planning horizon. A particular setting is the case where the manufacturer and his long-term buyers agree on a fixed wholesale price p per unit sold during the entire planning horizon. Assumption A2 is also satisfied when the manufacturer provides quantity discounts. In this case, we replace the unit price p_t with sum of concave price functions whose arguments are the number of units purchased by long-term buyers. In the last section, we discuss the case where the price p_t is also a control variable and the manufacturer dynamically chooses the prices along with the initial net inventory and the starting capacity.

We conclude this section with a condition on the spot market and the long-term contract marginal profits at point zero.

[1] We call a function $g : \mathbb{R} \to \mathbb{R}$ coercive if $\lim_{|x| \to \infty} g(x) = -\infty$

A3. For all $w_t \in \mathcal{S}$ and $t \in \{0, \ldots, T\}$, $\Pi_t'(0, w_t) \leq \Gamma_t'(0, w_t)$.

In the lost sales case described earlier, A3 translates into $\Pi_t'(0, w_t) \leq b_t + p_t$. Condition A3 basically links both channels and creates a balance between them. Indeed, it implies that the long-term contract is more profitable than the spot market when small quantities are traded. On the other hand, recall that the spot market value function is increasing in the quantity traded while the long-term contract is coercive and eventually decreasing to $-\infty$. We will show later in Theorem 1 that this condition results in short-selling being naturally eliminated, or equivalently that the second constraint in (5) is redundant.

At the beginning of any period t, the manufacturer's total expected profit is given by $\Gamma_t(z_t, w_t) + \Pi(s_t, w_t)$.

In the proofs, we drop the state of the world w_t and the time t from profit functions or variables when this omission does not cause any confusion. We denote the partial derivatives $\frac{\partial^2 F}{\partial x_i \partial x_j}$ either by $\partial_{x_i, x_j} F$ or $\partial_{i,j} F$, where $F(x_1, x_2, ..x_n)$ is a function of n variables. The choice will be clear from the context. Finally, when F is said to be increasing (decreasing) we mean weakly increasing (decreasing), and when concave, we mean the function is jointly concave in all its arguments.

4 An Optimal Production and Inventory Allocation Policy

Next, we study the problem in which the total production capacity Q, the spot market value function Π_t, and the long-term profit function Γ_t are given.

4.1 An Optimal Policy

At every period t, the manufacturer decides how much to produce and how much to sell on the spot market. The dynamics are given by equations (2), (3) and (4). Given the state of the system (x_t, Q_t, w_t), total profit function at time t, with $T - t$ periods remaining to termination, is determined by the following recursive equation

$$\begin{aligned} J_t(x_t, Q_t, w_t) &= \max_{(q_t, z_t) \in Y(x_t, Q_t)} V_t(q_t, z_t | x_t, Q_t, w_t) \qquad \text{where,} \\ V_t(q_t, z_t | x_t, Q_t, w_t) &= \Gamma_t(z_t, w_t) + \Pi_t(x_t + q_t - z_t, w_t) + \mathbb{E} J_{t+1}(x_{t+1}, Q_{t+1}, \omega_{t+1}) \end{aligned} \tag{8}$$

and $J_{T+1}(x_{T+1}, \cdot, \cdot) \equiv K(x_{T+1})$ and $K(\cdot)$ is a concave terminal profit function.

We will prove through Theorem 1 and under assumption A3 that the second constraint in Equation (5) is always satisfied. Hence, the set of constraints would be replaced by $Y(x_t, Q_t) = [0, Q_t] \times \mathbb{R}$.

In addition, and for technical reasons, we need to define the function V_t as a function of q_t on the entire real line. This results in Q_{t+1} taking eventually negative values. For this reason, we let $Y(x_t, Q_t) = \{Q_t\} \times \mathbb{R}$, when $Q_t < 0$. We refer to $\bar{J}$ as the function defined by the maximization problem in (8) without any constraint on (q_t, z_t). We use the notation " $^-$ " to refer to the variables that correspond to this unconstrained optimization problem. We sometimes refer to the term $\mathbb{E}J_{t+1}(x_{t+1}, Q_{t+1}, \omega_{t+1})$ by $R(z_t, Q_t - q_t, \omega_t)$ where R is a real function from $\mathbb{R}^2 \times \mathcal{S}$. Next, we characterize the optimal production and inventory allocation policy. We start by stating two lemmas that simplify the presentation of the main result.

Lemma 1 *Consider a function $J : \mathbb{R}^2 \to \mathbb{R}$, increasing in both variables and concave. Let $g : \mathbb{R} \to \mathbb{R}$ also be an increasing concave function, and d be a random variable with a finite mean. Then, $(z, u) \mapsto \mathbb{E}_d J(g(z - d), u)$ is also increasing and jointly concave in z and u.*

Proof. The increasing property is clearly maintained through composition. We will provide a proof for the concavity conclusion. Consider, (u, z) and $(u', z') \in \mathbb{R}^2$ and $\theta \in (0, 1)$. We write

$$\begin{aligned}
\theta J(g(z-d), u) + (1-\theta) J(g(z'-d), u') &\le J(\theta\, g(z-d) + (1-\theta) g(z'-d), \theta\, u + (1-\theta) u') \\
&\le J(g(\theta\,(z-d) + (1-\theta)(z'-d)), \theta\, u + (1-\theta) u') \\
&= J(g(\theta\, z + (1-\theta) z' - d), \theta\, u + (1-\theta) u').
\end{aligned}$$

The first inequality is due to the concavity of J. The second one is the result of both the concavity of g and the monotonicity of J. Under mild assumptions (see page 481 of Durrett), we can interchange expected values and derivatives and thus the results are preserved under expectation. ■

Lemma 2 *Based on (8), we drop t and w_t and get*

$$\begin{aligned}
J(x, Q) &= \max_{(q,z) \in Y(x,Q)} V(q, z, x, Q), \text{ where} \\
V(q, z, x, Q) &= \Gamma(z) + \Pi(x + q - z) + R(z, Q - q).
\end{aligned}$$

(i) *If the functions Γ, Π and R are concave and Y has a convex graph[2], then V and J are also concave.*

(ii) *If Γ and Π are strictly concave and $Y(x, Q)$ is as defined above for all x and Q, then V is strictly concave in (q, z, x) and (q, z, Q) if and only if R is strictly concave in its second variable. In this case, the function J is strictly concave as well but separately in x and Q.*

[2] Y is said to have a convex graph if for all $Z \in Y(X)$, $Z' \in Y(X')$, and $\alpha \in [0, 1]$, we have $\alpha\, Z + (1-\alpha)\, Z' \in Y(\alpha X + (1-\alpha) X')$

Proof. We provide a proof based on the Hessian of V. This proof also gives a better understanding of how the variables interact in this multidimensional problem. To simplify exposition, we suppress the points at which the functions are evaluated.

$$H = \begin{bmatrix} \Pi'' + \partial_{2,2}R & -\Pi'' - \partial_{1,2}R & \Pi'' & -\partial_{2,2}R \\ -\Pi'' - \partial_{1,2}R & \Gamma'' + \Pi'' + \partial_{1,1}R & -\Pi'' & \partial_{1,2}R \\ \Pi'' & -\Pi'' & \Pi'' & 0 \\ -\partial_{2,2}R & \partial_{1,2}R & 0 & \partial_{2,2}R \end{bmatrix}.$$

In order to obtain the formulation of H, we use $\partial_q R(z, Q-q) = -\partial_Q R(z, Q-q) = -\partial_2 R$ and $\partial_z \Pi(x+q-z) = -\partial_q \Pi(x+q-z) = -\partial_x \Pi(x+q-z) = -\Pi'$.

Next, we show that H is negative semi-definite; that is, $x^T H x \leq 0$ for all vectors x or equivalently all the eigenvalues of H are non-positive. Now notice that the first column (respectively, row) is a linear combination of the third and the fourth columns (respectively, rows). Applying a Gauss elimination on H we obtain

$$H' = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \bar{H}' \end{bmatrix}$$

where,

$$\bar{H}' = \begin{bmatrix} \Gamma'' + \Pi'' + \partial_{1,1}R & -\Pi'' & \partial_{1,2}R \\ -\Pi'' & \Pi'' & 0 \\ \partial_{1,2}R & 0 & \partial_{2,2}R \end{bmatrix}.$$

Note that H and H' have the same eigenvalues. Hence, it suffices to show that sub-matrix $\bar{H}'$ is negative semi-definite. We apply another Gauss elimination by replacing the second column (resp. row) of $\bar{H}'$ with the sum of the first and the second column (resp. row), and changing the order of the variables. We obtain the following matrix

$$\bar{H}'' = \begin{bmatrix} \Pi'' & 0 & 0 \\ 0 & \Gamma'' + \partial_{1,1}R & \partial_{1,2}R \\ 0 & \partial_{1,2}R & \partial_{2,2}R \end{bmatrix}$$

The result follows because all the functions are concave, that is, all the second derivatives are non-positive and the determinant of the Hessian of R is non-negative, $\partial_{2,2}R \cdot \partial_{1,1}R - (\partial_{1,2}R)^2 \geq 0$. By taking the maximum of V when (q, z) belong to the convex graph $Y(x, Q)$, we get a concave function in (x, Q) (see page 227, Theorem A4, of Porteus 2002). Hence, J is concave in (x, Q).

To prove part (ii), we consider V first as a function of (q, z, Q). By carrying out a similar Gauss elimination as before, we transform the Hessian of V into $\bar{H}''$. If Π and Γ are strictly concave, then strict concavity of V is guaranteed by the strict concavity of R only with respect to its second variable. We can observe these results by computing the determinant of $\bar{H}''$,

$$\det \bar{H}'' = \Gamma''\Pi''\partial_{2,2}R + \Pi''(\partial_{1,1}R \cdot \partial_{2,2}R - (\partial_{1,2}R)^2).$$

With regard to the strict concavity of J, recall here that $Y = [0, Q] \times \mathbb{R}$. If q belongs to the interior of the set Y, then the strict concavity of J is maintained through maximization (the proof follows exactly the same steps of Theorem A4, of Porteus 2002). Otherwise, either $q = 0$, or $q = Q$. In both cases, the function V is strictly concave in (z, Q), so that J is concave in Q. Similar proof holds for the strict concavity with respect to x. ■

Now we are ready to state the main result which fully characterizes an optimal policy.

Theorem 1 *The following holds for any state of the world w_t under A1, A2, and A3:*

(i) *J_t is increasing in x_t and Q_t,*

(ii) *J_t is concave in (x_t, Q_t) and strictly and separately concave in x_t and Q_t,*

(iii) *V_t is concave in (q_t, z_t, x_t, Q_t) and strictly concave in (q_t, z_t, Q_t) and (q_t, z_t, x_t),*

(iv) *An optimal policy is defined by a state dependent $(\bar{q}_t, \bar{z}_t)$-policy, where the produce-up-to $\bar{q}_t$ and sell-down/buy-up-to $\bar{z}_t$ thresholds are the solution to the following system of equations*

$$\begin{cases} \partial_q V_t(q_t, z_t | x_t, Q_t, w_t) = 0, \\ \partial_z V_t(q_t, z_t | x_t, Q_t, w_t) = 0. \end{cases} \tag{9}$$

Under this policy, optimal production and inventory allocation are given by

$$(q_t^*, \, z_t^*) = \begin{cases} (0, \, \bar{z}_t(0)) & \text{if } \bar{q}_t \leq 0 \\ (\bar{q}_t, \, \bar{z}_t) & \text{if } \bar{q}_t \in (0, Q_t) \\ (Q_t, \bar{z}_t(Q_t)) & \text{if } \bar{q}_t \geq Q \end{cases} \tag{10}$$

where $\bar{z}_t(q)$ is a solution to $\partial_z V_t(q, z_t | x_t, Q_t, w_t) = 0$.

Proof. The proof is based on an induction argument. The result for part (i) trivially holds for $t = T + 1$. Assume that it holds for $t + 1$, and let $Q_t^1 < Q_t^2$,

$$\begin{aligned} V_t(q_t, z_t | x_t, Q_t^1, w_t) &= \Gamma_t(z_t, w_t) + \Pi_t(x_t + q_t - z_t, w_t) + \mathbb{E} J_{t+1}(g(z_t - d_t), Q_t^1 - q_t, w_{t+1}) \\ &\leq \Gamma_t(z_t, w_t) + \Pi_t(x_t + q_t - z_t, w_t) + \mathbb{E} J_{t+1}(g(z_t, d_t), Q_t^2 - q_t, w_{t+1}) \\ &= V_t(q_t, z_t | x_t, Q_t^2, w_t). \end{aligned}$$

By taking the maximum on both sides and noticing that $[0, Q_t^1] \subset [0, Q_t^2]$, we conclude that J_t is increasing in Q_t. A similar induction argument shows that J_t is also increasing in x_t.

Before we start the proof for remaining parts we observe the following. Consider the function V_t as a function of z_t. We compute its first derivative and evaluate it at the point $-[x_t + q_t]^- \leq 0$. We obtain

$$\begin{aligned}
&\partial_z V_t(q_t, -[x_t+q_t]^-|x_t, Q_t, w_t) \\
&\quad = \Gamma'(-[x_t+q_t]^-, w_t) - \Pi'(x_t+q_t+[x_t+q_t]^-, w_t) \\
&\qquad + g'(-[x_t+q_t]^- - d_t)\mathbb{E}\partial_x J_{t+1}(g(-[x_t+q_t]^- - d_t), Q_t - q_t, w_{t+1}) \\
&\quad \geq \Gamma'(0, w_t) - \Pi'(0, w_t) + g'(0)\mathbb{E}\partial_x J_{t+1}(0, Q_t - q_t, w_{t+1}) \\
&\quad \geq 0.
\end{aligned}$$

The first inequality is due to the concavity of all functions. The last inequality is due to A3 and the fact that J and g are increasing functions (by (i) and the definition of g). Based on this observation we conclude that $\bar{z}_t \geq -[x_t + \bar{q}_t]^-$. Hence, we conclude that $\text{argmax}_{(q,z)\in[0,Q]\times\mathbb{R}} V_t(q, z|x_t, Q_t) \in Y(x, Q)$ and hence the second constraint in (5) is always satisfied. In the remaining we redefine $Y(x, Q) = [0, Q] \times \mathbb{R}$. To initiate the induction argument for Part (ii), note that $J_{T+1}(x, Q_{T+1}) = K(x)$ is concave in (x, Q_{T+1}) and strictly concave in x. To initiate the induction argument for the strict concavity in Q_t consider the last period, in which all the remaining capacity will be produced, so that $J_T(x_T, Q_T) = \max_{z_T \in Y(x_T, Q_T)} \Gamma_T(z_T) + \Pi_T(x_T + Q_T - z_T) + \mathbb{E}K(g(z_T - d_T))$. The inside of the maximum is concave in (z_T, Q_T) and strictly concave in Q_T. So, part (ii) is satisfied for $t = T+1$ or $t = T$. Next, we define $R_T(z, Q_T - q_T) \equiv \mathbb{E}J_{T+1}(g(z - d_T), Q_{T+1})$. Lemma 1 implies that R_T is concave. The functions Π_T and Γ_T are also both concave. Hence, Lemma 2 implies Part (ii) and (iii) for $t = T$. Now assume for an induction argument that Part (ii) is true for $t+1$, that is $J_{t+1}(x, Q_{t+1})$ is concave in (x, Q_{t+1}) and strictly and separately concave in both arguments. From Lemma 1, $R_t(z, Q_t - q_t) \equiv \mathbb{E}J_{t+1}(g(z - d_t), Q_{t+1})$ is concave. Hence Lemma 2 implies that Part (ii) and (iii) are true for t, concluding the induction argument. Finally, Part (iv) is a result of Part (iii), and the strict concavity of V_t in (q_t, z_t). We write

$$J_t(x_t, Q_t) = \max_{q\in[0,Q_t]} \max_z V_t(q, z|x_t, Q_t),$$

and note that the the inside maximum in the previous equation is a concave function of q. Hence, $q_t^* = \text{argmax}_{q\in[0,Q_t]} \max_z V_t(q, z|x_t, Q_t)$ is the closest to $\bar{q}_t$ (unconstrained maximizer) in $[0, Q_t]$. If $\bar{q}_t \in (0, Q_t)$, then clearly (q_t^*, z_t^*) are the solutions of (9). Otherwise, $\bar{q}_t \in \{0, Q_t\}$ and $\bar{z}_t(q)$ is the corresponding maximizer of $V_t(q, z|x_t, Q_t)$ when q is respectively 0 or Q_t. This completes the proof. ■

The first part of the previous theorem is intuitive. The optimal profit J_t increases with Q_t. Larger capacity yields larger profits. Increasing capacity is equivalent to a relaxation in our problem because the manufacturer can always sell inventory on the spot market. Similarly, the profit increases with the initial net inventory. The manufacturer can always sell the additional units on the spot market based on the spot market price $\Pi_t(\cdot, \omega_t)$ without incurring any additional holding cost. Hence, by the end of the planning horizon the manufacturer will always produce the remainder of his initially capacity Q.

Theorem 1 characterizes the optimal policy. From the system of equations given in (10) we observe that both the optimal production quantity and the modified net inventory are functions of the state of the system (x_t, Q_t, ω_t). Therefore, the policy is clearly state dependent. If the state of the system is such that $\bar{q}_t(x_t, Q_t, \omega_t) \leq 0$, then the manufacturer produces nothing. Otherwise, he produces $\min\{\bar{q}_t, Q_t\}$ units; the manufacturer produces up to $\bar{q}_t$ given available production capacity Q_t.

In each of the previous cases, the manufacturer brings the net inventory after production to the level $\bar{z}_t$. In particular, if the amount of inventory after production is lower than $\bar{z}_t$, then the manufacturer is better off buying up-to $\bar{z}_t$ from the spot market. If the amount of inventory after production is higher than $\bar{z}_t$, then the manufacturer is better off selling on the spot market down-to the threshold $\bar{z}_t$. Note that the target level $\bar{z}_t$ depends both on the long-term channel expected profit Γ_t and on the spot market value function Π_t in *current* and *future* periods. Hence, the optimal policy captures the speculative behavior of delaying production when the manufacturer expects to observe a high future spot market value.

The policy partitions the state space into three easily identifiable regions. Each region provides a recipe for production and inventory allocation targets. The policy is state dependent through the state of the world w_t and also the state space x_t and Q_t. By taking a closer look at the objective function, however, we show that the policy parameters have a simpler structure. Note that we can rewrite the objective function as

$$\begin{aligned} V(q_t, z_t|x_t, Q_t, w_t) &= \Gamma_t(z_t, w_t) + \Pi_t(x_t + Q_t - (Q_t - q_t) - z_t, w_t) + R(z_t, Q_t - q_t, w_t), \\ &= \tilde{V}(Q_t - q_t, z_t|x_t + Q_t, w_t). \end{aligned} \tag{11}$$

Now it is easy to see that $Q_t - \bar{q}_t$ and $\bar{z}_t$ are only functions of the total units available in the supply chain i.e. $x_t + Q_t$, along with the state of the world w_t. The quantity $Q_t - \bar{q}_t$ is a policy parameter that we can denote by $\bar{Q}_{t+1}$. It is the capacity level remaining for period $t+1$ that the manufacturer is targeting by producing $\bar{q}_t$ in period t. What matters is the total units $x_t + Q_t$ owned by the manufacturer. The production cost is incurred at the start of the horizon and hence units can, in the *unconstrained* problem, move freely between post- and pre-production sites.

The previous observation, has an important impact on the computations of the policy parameters and the value function. It enables a state space reduction with regard to the values of the policy parameters. The value function itself was not altered, but one can start computing $\bar{J}_t$ (which is one dimension lower), and then conclude on the value of J_t by checking if the optimal value is an interior point or not.

Proposition 1 *The unconstrained profit function $\bar{J}_t$ and the policy parameters, $Q_t - \bar{q}_t$ and $\bar{z}_t$ are only functions of $x_t + Q_t$.*

In particular, the proposition shows that $\bar{J}_t$ is *submodular* (Topkis 1978). This result implies that the net inventory level x_t and the remaining capacity Q_t are economic substitutes, that is, when the net inventory increases, the manufacturer needs less capacity and vice-versa.

We recall that based on the concavity of the function V, if $\bar{z}_t$ and $\bar{q}_t$ are finite, then they define the unique solution of the system given by (9). In the next Theorem, we show that both values must be finite under the assumptions stated earlier, which is not necessarily obvious in this multidimensional setting.

Theorem 2 *Under A1 with strict concavity, A2 and A3, the solutions to the system of Equations (9) are finite; specifically*

$$\bar{z}_t \leq z_t^0 < \infty \quad \textit{for} \quad t \in \{0, 1, ..., T\} \qquad \textit{while} \qquad |\bar{q}_t| < \infty \quad \textit{for} \quad t \in \{0, 1, ..., T-1\},$$

where z_t^0 is the unique maximizer of Γ_t.

Proof. We first show the following relation.

$$\text{If,} \quad x + Q \leq x' + Q' \quad \text{and} \quad x \geq x', \qquad \text{then} \qquad J(x, Q) \leq J(x', Q'). \tag{12}$$

Consider, (x, Q) and (x', Q') as in Equation (12). We necessarily have $Q' \geq Q$. Let z^* and q^* be such that $J(x, Q) = \Gamma(z^*) + \Pi(x + q^* - z^*) + R(z^*, Q - q^*)$. In particular, $0 \leq q^* \leq Q$. We define $q' = x + q^* - x'$ so that $0 \leq q' \leq x + Q - x' \leq Q'$ and $Q' - q' \geq Q - q^*$. We observe that $(z^*, q') \in Y(x', Q')$, and conclude that

$$J(x', Q') \geq \Gamma(z^*) + \Pi(x' + q' - z^*) + R(z^*, Q - q') \geq J(x, Q).$$

Now consider any $z \geq z_t^0$ and q, by coerciveness of Γ_t there exists a $z' \leq z_t^0$ such that $\Gamma_t(z) = \Gamma_t(z')$. Let $q' = q - z + z' \leq q$. By sublinearity of g we have that $g(z - d) + Q - q \leq g(z' - d) + z - z' + Q - q$ and we can conclude that

$$\Gamma(z) + \Pi(x + q - z) + \mathbb{E}J(g(z - d), Q - q) \leq \Gamma(z') + \Pi(x + q' - z') + \mathbb{E}J(g(z' - d), Q - q').$$

This concludes the proof for the first part of the Theorem. To prove the second part, first notice that $\bar{q}_T = \infty$. However, for any $t < T$ we have already proved that $\bar{z}_t$ is finite, hence, if $\bar{q}_t$ is infinite it implies that the number of units sold on the spot market is infinite. Recall from the definition of $Y(x_t, Q_t)$, that $q_t = Q_t$ when $Q_t \leq 0$. Hence, if $\bar{q}_t > Q_t$, then $\bar{q}_{t+1} = Q_t - q_t$. Therefore, for a certain $z_{t+1} \leq z_{t+1}^0$, we have

$$\begin{aligned}
&\limsup_{q_t \to \infty} [\Pi_t(x_t + q_t - z_t) + R(z_t, Q_t - q)] \\
&\qquad \leq \Gamma_t(z_{t+1}^0) + \limsup_{q_t \to \infty} \left[\Pi_t(x_t + q_t - z_t) + \mathbb{E}\left\{ \Pi_{t+1}(x_{t+1} + Q_t - q_t - z_{t+1}) + R(z_{t+1}, 0) \right\} \right].
\end{aligned}$$

By concavity of the function Π_t, we can conclude that for any $x_t, z_t, Q_t, x_{t+1}, z_{t+1}$,

$$\lim_{q_t \to \infty} \Pi(x_t + q_t - z_t) + \Pi(x_{t+1} + Q_t - q_t - z_{t+1}) = -\infty.$$

To see this equality, note that for $u \geq 1$ the strict concavity property translates into

$$\Pi(u) + \Pi(-u) < u \cdot \Pi'(1) + \Pi(1) - u \cdot \Pi'(0) = u \cdot (\Pi'(1) - \Pi'(0)) + \Pi(1).$$

Hence, $\bar{V}_t(z_t, q_t = \infty | x_t, Q_t) = -\infty$. A similar reasoning holds if $q_t = -\infty$. This constitutes proof that $\bar{V}_t$ is coercive in q_t for $t < T$. ■

The first part of the previous result shows that the inventory threshold, $\bar{z}_t$ is upper bounded by the myopic inventory threshold when the spot market is not available to the seller, z_t^0. More generally, one expects that the inventory threshold is lower in the dual channel case than selling only through the long-term channel case. In other words, the presence of a spot market results in a decrease in the level of inventory and by the same token in a decrease in the holding costs. Intuitively, the spot market is a source of an additional demand. Hence, by being able to sell on the spot market, the capacity constrained manufacturer needs to devote less to the long-term contract.

4.2 Linear Spot Market Case

We replace assumption A1 with

A1'. The spot market value function is linear on the real line, $\Pi_t(u, w_t) \equiv \pi_t(w_t) \cdot u$, for all $w_t \in \mathcal{S}$ and $t \in \{0, \ldots, T\}$.

Under this new assumption, the spot market has a constant unit price π_t, that is revealed at the end of each period. As we mentioned above, a strict concave spot market value function translates into a decreasing marginal gain and thus restrains the manufacturer from trading a high amount of units on the spot market ($\bar{q}_t < \infty$). When the spot market value function is considered linear in the number of units traded, $\bar{q}_t$ could be infinite and one might want to add a constraint C_t on the number of units that can be bought from the spot market. The set of constraints $Y(x_t, Q_t)$ should then be adjusted accordingly, but remains a convex graph. Clearly, such linear spot market value function satisfies A1, and hence Theorem 1 still holds. For the remaining, and without loss of generality, we assume $C_t = \infty$. The next proposition provides additional structure on the optimal policy when the random spot market value function is linear.

Proposition 2 *Under the same conditions of Theorem 1, replacing A1 with A1', we obtain the following*

(i) $\bar{J}_t$ *is linear in* $x_t + Q_t$,

(ii) *the policy parameters* $Q_t - \bar{q}_t$ *and* $\bar{z}_t$ *are independent of* x_t *and* Q_t *and are only functions of the state of the world* w_t.

Proof. The proof is straightforward when considering A1' and Equation (11) ■

We refer to $\bar{Q}_{t+1} = Q_t - \bar{q}_t$ as the *consume-down-to* level. Under this policy, the manufacturer will review the state of the world w_t and if he produces, he will produce up to $\bar{q}_t$ such that the remaining capacity for the next period is consumed down to $\bar{Q}_{t+1}$. Recall from Proposition 1 that this quantity is independent of the state (x_t, Q_t), while the production parameter $\bar{q}_t$ is independent of x_t and linear in Q_t. Similarly, the inventory facing the long-term demand follows an order-to policy independent of the state of the system (except again through the state of the universe). Intuitively, the constant profit margin of the spot market makes the spot market less relevant from an optimization point of view. The objective is essentially to manage the long-term channel with respect to the remaining inventory units and capacity; the spot market is used as a secondary sales channel.

These observations lead us to conclude that when the spot market is considered to be as important as the long-term channel, the spot market value should *not* be modeled as a linear function of the quantity traded. However, in the case where the spot market is indeed considered as a salvage channel (secondary market) the linear framework seems adequate. The manufacturer can employ such simple policies to his advantage. Next, we discuss myopic policies.

4.3 A Myopic Policy

To obtain additional insights, we study myopic policies that maximize current period's profit without consideration of future ones. These types of sub-optimal solutions are attractive because they are simple and can be good approximations for optimal policies. In fact, we show below that a myopic policy is asymptotically optimal in initially reserved capacity per period, that is Q/T.

In a myopic approach, the maximization problem reduces to

$$\hat{J}_t(x_t, Q_t, w_t) = \max_{(z_t, q_t) \in Y(x_t, q_t)} [\Gamma_t(z_t, w_t) + \Pi_t(x_t + q_t - z_t, w_t)]. \tag{13}$$

From assumptions A1 and A2, we conclude that $z \mapsto \Gamma_t(z, w_t) + \Pi_t(x_t + q_t - z, w_t)$ is concave and coercive. Hence, this function admits a unique maximizer z_t^m. It is easy to observe that this function is also increasing in q_t: the manufacturer will myopically choose to produce all the capacity Q at once, in the first period. A probably better and more realistic myopic policy is one that assumes a pre-scheduled production scheme: a fixed amount every period, say $q_m = Q/T$. The optimal inventory level is then the solution to the following equation,

$$\Gamma_t'(z, w_t) = \Pi_t'(x_t + q_m - z, w_t). \tag{14}$$

Consider the problem in which the spot market value function is strictly concave and the profit margin tends to zero as more products are sold through the spot market. In such an environment, we show that the myopic policy is asymptotically optimal when both the initial capacity and long-term horizon become large but where $Q/T \to \infty$; (this includes the case where T is constant and the capacity gets large). We let $J_0(x, Q, T, w)$ and $J_0^m(x, Q, T, w)$ denote the value functions at time 0, when adopting the optimal and the myopic policy, respectively. We also denote by r the discount factor, crucial as we let T go to infinity.

Theorem 3 *Assume that* $\sup_t\{\Pi_t(\infty)\} < \infty$. *Then,*

$$\lim_{Q/T\to\infty} J_0(x, Q, T, w)/J_0^m(x, Q, T, w) = 1,$$

where $\Pi_t(\infty) \stackrel{D}{=} \lim_{s\to\infty} \Pi_t(s)$

Proof. Without loss of generality, we assume that $\Pi_t(\infty)$ is independent of t. Let $\varepsilon > 0$, and consider the following specific myopic policy which relies on a constant production Q/T per period and a modified net inventory $z_t = z_t^m \le z_t^0$, where z_t^0 is the maximizer of Γ_t and z_t^m is the solution to the myopic problem given by (13). From Equation (14), we have that $z_t^m \to z_t^0$, as $Q/T \to \infty$. Furthremore, the amount traded on the spot market, given by $s_t = x_t + Q/T - z_t$, increases to infinity with Q/T. We now let the periodic production Q/T be large enough so that the probability that at all times $z_t \ge z_t^0 - \varepsilon$ and $\Pi_t(s_t) > \Pi(\infty) - \varepsilon$ is bigger than $1 - \varepsilon$. Conditioning on this event, we write

$$\begin{aligned}(1-\varepsilon)(\int_{t=0}^{T} \exp(-rt)\Gamma_t(z_t^0 - \varepsilon) + &\int_0^T \exp(-rt)(\Pi(\infty) - \varepsilon)) \\ &\le J_0^m(x, Q, T) \le J_0(x, Q, T) \\ &\qquad \le \int_0^T \exp(-rt)\Gamma_t(z_t^0) + \int_0^T \exp(-rt)\Pi(\infty).\end{aligned}$$

The profit function based on the myopic policy described earlier is clearly bounded by the optimal myopic profit function, which in turn is bounded by the optimal value function J_0. For the last inequality note that the right hand side is an absolute upper bound on the profits of both channels. Now we let ε go to zero which completes the proof. ■

As the amount produced per period, that is Q/T, goes to infinity, we make it more likely to have positive on-hand inventory at any period. If at each period there are enough units to achieve any policy, then the profits realized during each period can be decoupled and a myopic policy becomes optimal.

4.4 Monotonicity Results

We recall that the general policy parameters described earlier are state dependent. They are simplified in the linear spot market case. To get a better general sense of the policy parameters, we devote the rest of this section to analyzing their behavior and the resulting optimal actions; that is, how z^*, q^*, $\bar{z}$ and $\bar{q}$ change as functions of state variables. Based on the setting we are in, (i.e. a two-dimension control problem), we will make use of a particular version of the Implicit Function Theorem which we begin with below.

Lemma 3 *(Implicit Function Theorem) Let T be a function from $\mathbb{R}^3$ into $\mathbb{R}^2$, such that for all $(x, y, z) \in \mathbb{R}^3, T(x, y, z) = (\partial_x f(x, y, z), \partial_y f(x, y, z))$, where f is a real valued function twice continuously differentiable such that $\partial_{x,x} f(x, y, z)\partial_{y,y} f(x, y, z) - \partial_{x,y} f^2(x, y, z) \neq 0$.*

Assume that $\exists\ (\bar{x}_0, \bar{y}_0, z_0) \in \mathbb{R}^3$ such that $T(\bar{x}_0, \bar{y}_0, z_0) = (0, 0)$. Then there exists a continuous and differentiable function $\phi : \mathbb{R} \to \mathbb{R}^2$, such that $T(\phi(z), z) = (0, 0)$ for all z and its gradient is given by:

$$\bar{x}'(z) = \frac{\partial_{y,z} f \partial_{x,y} f - \partial_{x,z} f \partial_{y,y} f}{\partial_{y,y} f \partial_{x,x} f - \partial_{x,y} f^2}$$

and

$$\bar{y}'(z) = -\frac{\partial_{y,z} f \partial_{x,x} f - \partial_{x,z} f \partial_{x,y} f}{\partial_{y,y} f \partial_{x,x} f - \partial_{x,y} f^2}$$

where, $\bar{x}$ and $\bar{y}$ are the components of ϕ (i.e. $\phi(z) = (\bar{x}(z), \bar{y}(z))$.

Proof. The existence, uniqueness and differentiability of the function ϕ follows from the general statement of the Implicit Function Theorem (see for instance page 138 of Cheney 2001). Concerning the specific formulation of the gradient of ϕ, recall that $T(\phi(\cdot), \cdot)$ is identically zero for all z and thus the derivative with respect to z of each component is zero:

$$\frac{\partial}{\partial z} \partial_x f(\bar{x}, \bar{y}, z) = \bar{x}'(z)\partial_{x,x} f + \bar{y}'(z)\partial_{x,y} f + \partial_{x,z} f = 0$$

and,

$$\frac{\partial}{\partial z} \partial_y f(\bar{x}, \bar{y}, z) = \bar{x}'(z)\partial_{x,y} f + \bar{y}'(z)\partial_{y,y} f + \partial_{y,z} f = 0.$$

Solving these two equations gives us the result. ■

Theorem 4 *Under assumptions A1, A2 and A3, we have*

(i) $0 < \partial_x(\bar{z}_t - \bar{q}_t) \leq 1$, $0 < \partial_Q(\bar{q}_t - \bar{z}_t) \leq 1$, $\partial_x \bar{z}_t = \partial_Q \bar{z}_t$ *and* $1 + \partial_x \bar{q}_t = \partial_Q \bar{q}_t$. *In particular,* $x_t + \bar{q}_t - \bar{z}_t$ *and* $Q_t - \bar{q}_t + \bar{z}_t$ *are increasing in* x_t *and* Q_t.

(ii) $\partial_x \bar{J}_t(x_t, Q_t, w_t) = \Pi'(x_t + \bar{q}_t - \bar{z}_t, w_t) = \partial_Q \bar{J}_t(x_t, Q_t, w_t) = \mathbb{E}\partial_2 \bar{J}_{t+1}(g(z_t - d_t), Q_t - \bar{q}_t, w_{t+1}) \geq 0$ *and* $\partial_{x,Q} \bar{J}_t(x_t, Q_t, w_t) \leq 0$*.*

Proof. For part (i), we apply Lemma 3 where f is taken equal to V considered as a function of three variables q, z and Q. We write for all w and t, $V(q, z, Q|x, w) = \Gamma(z, w) + \Pi(x + q - z, w) + R(z, Q - q, w)$, where R is jointly concave and x is fixed. Define $\bar{z}(Q)$ and $\bar{q}(Q)$ such that $\partial_q V(\bar{q}, \bar{z}, Q|x, w) = \partial_z V(\bar{q}, \bar{z}, Q|x, w) = 0$. Applying Theorem 3 we get,

$$\begin{aligned}\partial_Q \bar{q} =& \frac{\partial_{2,2}R(\Gamma'' + \Pi'' + \partial_{1,1}R) - (\Pi'' + \partial_{1,2}R)\partial_{1,2}R}{\partial_{2,2}V\partial_{1,1}V - \partial_{1,2}V^2} \\ =& \frac{\partial_{2,2}R\,\Gamma'' + \Pi''(\partial_{2,2}R - \partial_{1,2}R) + \partial_{2,2}R\,\partial_{1,1}R - (\partial_{1,2}R)^2}{\partial_{2,2}V\partial_{1,1}V - \partial_{1,2}V^2}.\end{aligned} \tag{15}$$

Similarly, we write

$$\begin{aligned}\partial_Q \bar{z} =& \frac{-\partial_{2,2}R(-\Pi'' - \partial_{1,2}R) - \partial_{1,2}R(\Pi'' + \partial_{2,2}R)}{\partial_{2,2}V\partial_{1,1}V - \partial_{1,2}V^2} \\ =& \frac{\Pi''(\partial_{2,2}R - \partial_{1,2}R)}{\partial_{2,2}V\partial_{1,1}V - \partial_{1,2}V^2}.\end{aligned} \tag{16}$$

By taking the difference, we obtain

$$\partial_Q(\bar{z} - \bar{q}) = -\frac{\partial_{2,2}R\Gamma'' + (\partial_{2,2}R\partial_{1,1}R - (\partial_{1,2}R)^2)}{\partial_{2,2}V\partial_{1,1}V - \partial_{1,2}V^2}. \tag{17}$$

Note that the function V is strictly concave, which results in the denominator of Equation (17) being strictly positive. The numerator is non-positive again by the concavity of the functions involved. A similar proof shows that $\bar{z}_t - \bar{q}_t$ is increasing in x_t. For that consider V function of (q, z, x). Applying again Lemma 3 we can write

$$\partial_x \bar{q} = -\frac{\Pi''(\Gamma'' + \partial_{1,1}R - \partial_{1,2}R)}{\partial_{2,2}V\partial_{1,1}V - \partial_{1,2}V^2}, \qquad \text{and} \qquad \partial_x \bar{z} = \frac{\Pi''(\partial_{2,2}R - \partial_{1,2}R)}{\partial_{2,2}V\partial_{1,1}V - \partial_{1,2}V^2}.$$

We conclude by taking the difference of the previous expressions, and noticing that $(1,1)^T R(1,1) = \partial_{1,1}R + \partial_{2,2}R - 2\partial_{1,2}R \leq 0$ by concavity of R. We develop the denominator in Equation (17) which is equal to

$$\Gamma''\Pi'' + \Gamma''\partial_{2,2}R + \Pi''(\partial_{2,2}R + \partial_{1,1}R - 2\partial_{1,2}R) + (\partial_{1,1}R\partial_{2,2}R - (\partial_{1,2}R)^2).$$

By the concavity of the functions involved all the terms of the denominator are non-negative. Now consider the numerator of Equation (17) and notice that it is smaller than or equal to the denominator. This shows the upper bound, $\partial_Q(\bar{z} - \bar{q}) \leq 1$. In a similar way we prove that $\partial_x(\bar{z} - \bar{q}) \leq 1$.

For part (ii), we write $\bar{J}(x, Q, w) = V(\bar{z}, \bar{q}|x, Q, w)$ and take first the derivative with respect to x. We get for all $w \in \mathcal{S}$

$$\begin{aligned}
\partial_x \bar{J}(x,Q,w) &= \partial_x \bar{z}\,\Gamma'(\bar{z},w) + (1+\partial_x \bar{q} - \partial_x \bar{z})\Pi'(x+\bar{q}-\bar{z},w) \\
&\qquad + \partial_x \bar{z}\,\partial_1 R(\bar{z},Q-\bar{q},w) - \partial_x \bar{q}\,\partial_2 R(\bar{z},Q-\bar{q},w) \\
&= \partial_x \bar{z}\,\partial_z V(\bar{z},\bar{q}|x,Q,w) + \partial_x \bar{q}\,\partial_q V(\bar{z},\bar{q}|x,Q,w) + \Pi'(x+\bar{q}-\bar{z}) \\
&= \Pi'(x+\bar{q}-\bar{z},w).
\end{aligned}$$

The last equality is due to $\partial_{\bar{z}} V(\bar{z},\bar{q},Q|x,w) = \partial_{\bar{q}} V(\bar{z},\bar{q},Q|x,w) = 0$. Recall also, that $\bar{J}$ is only a function of the sum $x_t + Q_t$ and hence $\partial_x \bar{J} = \partial_Q \bar{J}$. In a similar way we have

$$\begin{aligned}
\partial_Q \bar{J}(x,Q,w) &= \partial_Q \bar{z}\Gamma'(\bar{z},w) + (\partial_Q \bar{q} - \partial_Q \bar{z})\Pi'(x+\bar{q}-\bar{z},w) \\
&\qquad + \partial_Q \bar{z}\partial_1 R(\bar{z},Q-\bar{q},w) + (1-\partial_Q \bar{q})\partial_2 R(\bar{z},Q-\bar{q},w) \\
&= \partial_Q \bar{z}\partial_z V(\bar{z},\bar{q}|x,Q,w) + \partial_Q \bar{q}\partial_q V(\bar{z},\bar{q}|x,Q,w) + \partial_2 R(\bar{z},Q-\bar{q},w) \\
&= \partial_2 R(\bar{z},Q-\bar{q},w).
\end{aligned}$$

Taking now the derivative with respect to x and Q, based on our previous computations we obtain

$$\partial_{x,Q} \bar{J}(x,Q,w) = \partial_Q(\bar{q}-\bar{z})\Pi''(x+\bar{q}-\bar{z},w). \tag{18}$$

By recalling (i) and A1, we complete the proof. ■

Note that $x_t + \bar{q}_t - \bar{z}_t$ relates to the amount sold on the spot market. Through part (i) of the previous Theorem, we know that this quantity is increasing in both x_t and Q_t. The quantity $Q_t - \bar{q}_t + \bar{z}_t$ behaves exactly as $Q_t - \bar{q}_t + \bar{z}_t - d_t$. When backlogging is allowed, the latter quantity relates to the total units remaining in the system at the end of the period, $\bar{x}_{t+1} + \bar{Q}_{t+1}$. The monotonicity of these two quantities means not only that if more units are available either through x_t or Q_t, then more units will be sold on the spot market, but also that more units will be kept for the next period.

We now define the following quantities. Let $q_t^*(x_t, Q_t|z_t, w_t) \equiv \arg\max_{0 \le q_t \le Q_t} V(q_t, z_t|x_t, Q_t, w_t)$ be the optimal production quantity when the modified net inventory is given in advance. Let $z_t^*(x_t, Q_t|q_t, w_t) \equiv \arg\max_{-[x_t+q_t]^- \le z_t} V_t(q_t, z_t|x_t, Q_t, w_t)$ be the optimal modified net inventory when the production quantity for each period is determined in advance, (e.g., $q_t \equiv Q/T$ for all $t \le T$, which can also be imposed by the manufacturer's supplier). We similarly define the corresponding policy parameters $\bar{q}_t(x_t, Q_t|z_t, w_t)$ and $\bar{z}_t(x_t, Q_t|q_t, w_t)$. Next, we state some monotonicity results for the production and the inventory allocation policy.

Theorem 5 *Under the conditions of Theorem 1 the following holds.*

(i) *For a given production quantity q_t, the optimal modified net inventory, $z_t^*(x_t, Q_t|q_t, w_t)$, is increasing in x_t and decreasing in Q_t.*

(ii) *For a given modified net inventory z_t, the optimal production quantity, $q_t^*(x_t, Q_t|z_t, w_t)$ is decreasing in x_t and increasing in Q_t.*

(iii) *For a given state (x_t, Q_t), the optimal modified net inventory, $z_t^*(q_t|x_t, Q_t, w_t)$, is increasing in q_t. Similarly, the optimal production quantity, $q_t^*(z_t|x_t, Q_t, w_t)$ is increasing in z_t.*

(iv) *Similar results hold for $\bar{q}_t(x_t, Q_t|z_t, w_t)$ and $\bar{z}_t(x_t, Q_t|q_t, w_t)$.*

Proof. We define $R_t(z_t, Q_t - q_t, w_t) \equiv E_{w_t} J_{t+1}(x_{t+1}, Q_{t+1}, w_{t+1})$. Note that $R(\cdot, \cdot, w_t)$ is a function of two variables jointly concave as a result of Lemma 1 and Theorem 1. We first show the result for $\bar{z}$ and $\bar{q}$ and we argue at the end that it indeed holds for z^* and q^*. For Part (i), note that Part (ii) of Theorem 1 allows us to write $\bar{z}$ as the unique solution to the following equation

$$\Gamma'(z) + \partial_1 R(z, Q - q) = \Pi'(x + q - z, w). \tag{19}$$

Hence, we can conclude that $\bar{z}$ is increasing in x by noticing that the LHS of Equation (19) is decreasing in z and independent of x, while the RHS is increasing in z and decreasing in x. Next, we apply a similar proof to show that $\bar{z}$ as a function Q is weakly decreasing. From Theorem 4 (ii), we have that $\partial_{1,2} R(z, Q - q) \leq 0$ and thus the LHS is decreasing in Q, while the RHS is constant. Part (ii) can be shown exactly the same way. For part (iii), we fix the state of the system (x, Q, w), and identify first how the optimal inventory level, $z^*(q|x, Q, w)$ behaves as the production quantity changes (note that q is not necessarily the optimal production value). We apply the basic version of the Implicit function Theorem (see Bertsekas 1985) with V considered as a function of z and q when fixing x and Q. Recall that $\partial_z V(q, \bar{z}|x, Q, w) = 0$, hence, the existence of a differentiable function of q, $\bar{z}(\cdot|x, Q, w)$, such that

$$\partial_q \bar{z}(q|x, Q, w) = -\frac{\partial_{1,2} V}{\partial_{1,1} V} = \frac{\Pi'' + \partial_{1,2} R}{\Gamma'' + \Pi'' + \partial_{1,1} R} \geq 0. \tag{20}$$

A similar argument holds for $\partial_z \bar{q}(z|x, Q, w)$. In order to conclude the result for z^* and q^*, recall the system of equations given by (10). Observe that all the possibilities of z^* and q^* in part (i), (ii) and (iii) behave similarly respectively to $\bar{z}$ and $\bar{q}$. ■

This theorem suggests that the optimal capacity and the initial inventory are economic substitutes: as one increases, less is needed from the other. However, the optimal modified net inventory and the remaining available capacity are not economic substitutes.

Theorem 6 *At each period t, the amount sold on the spot market s_t^*, is increasing in x_t and Q_t.*

Proof. The case where $\bar{q}_t$ is an interior point is dealt with through Theorem 4. We consider here the other cases. If $q^* = 0$, $s_t^* = x_t - \bar{z}_t$. The quantity $\bar{z}_t$ is by definition the solution to the equation $\Gamma'(\bar{z}) + \Pi'(x_t - \bar{z}_t) + \partial_1 R(\bar{z}_t, Q_t) = 0$. We take the derivative with respect first to x_t and obtain that

$$0 \leq \bar{z}_t'(x_t) = \frac{\Pi''(x_t - \bar{z}_t)}{\Gamma''(\bar{z}_t) + \Pi''(x_t - \bar{z}_t) + \partial_{1,1} R(\bar{z}_t, Q)} < 1.$$

Taking the derivative with respect to Q_t, we obtain

$$\bar{z}_t'(Q_t) = -\frac{\partial_{1,2}R(z_t, Q_t)}{\Gamma''(\bar{z}_t) + \Pi''(x_t - \bar{z}_t) + \partial_{1,1}R(\bar{z}_t, Q)} \leq 0.$$

We can conclude that s_t^* is increasing in x_t and Q_t. We carry out a similar analysis when $q^* = Q$. ■

The last Theorem shows that more units available on hand leads to more units sold on the spot market. It is not clear that such a result holds as well for the q^* and z^*.

5 Long-term Channel Pricing Policy

Thus far we have taken the long-term channel profit function and total capacity as given. We then characterized the manufacturer's optimal production and inventory allocation policy. Here, we propose to determine the optimal long-term contract parameters and the total capacity Q. In particular, we assume the long-term profit function Γ_t to be governed by a parameter p_t at every period t. For the sake of clarity, we consider the parameter p_t as the unit price that the manufacturer decides and reveals at the beginning of each period to his long-term channel. This scenario includes the setting where the wholesale price is constant and the manufacturer decides on a constant unit price p, for the entire horizon. In the general case, the setting enables the manufacturer to dynamically decide on the long-term price, the quantity to produce, and how to allocate the inventory every period. Note that the manufacturer can influence long-term demand by deciding on the price. We model this relationship with a multiplicative model and a linear mean-demand-price curve. Let $w_t \in \mathcal{S}$ be the state of the world. We define

$$\mu_p(w_t) = \alpha_{w_t} p_t + \beta_{w_t}, \tag{21}$$

as the mean of demand from the long-term channel where α_{w_t} and β_{w_t} are two real numbers with $\alpha_{w_t} \leq 0$. Demand for the long-term channel is given by $\mu_p(w_t)d_1(w_t)$, where d_1 is a random variable with mean 1 and a density function $f(\cdot, w_t)$ defined on the positive line. We observe that the non-negativity of the mean demand enforces the following constraint on p_t,

$$0 \leq p_t \leq p_{max} \triangleq \min_{w_t} \beta_{w_t}/|\alpha_{w_t}|. \tag{22}$$

We assume that A2 still holds when we consider Γ_t as a function of (z_t, p_t, w_t). In order to observe whether this assumption is still meaningful, we consider the lost sales and backlog cases discussed earlier and given in Equations (6) and (7). We show that the single period long-term profit function Γ_t is indeed concave.

Proposition 3 *Under the setting described above the single period profit from the long-term channel, Γ_t, given by either equations (6) and (7) is concave in (z_t, p_t).*

Proof. We will consider once more the Hessian of the function involved, namely Γ. Recall that

$$\Gamma(z,p) = p\,\mu_p - h\int_0^{z/\mu_p}(z-\mu_p\,u)f(u)du - b\int_{z/\mu_p}^{\infty}(\mu_p\,u - z)f(u)du.$$

The first derivatives of the function Γ with respect to p and to z are

$$\partial_z\Gamma(z,p,w) = b\int_{z/\mu_p}^{\infty} f(x)dx - h\int_0^{z/\mu_p} f(x)dx$$

and,

$$\partial_p\Gamma(z,p,w) = 2\alpha_w p + \beta_w - b\int_{z/\mu_p}^{\infty}\alpha_w x f(x)dx + h\int_0^{z/\mu_p}\alpha_w x f(x)dx.$$

Hence,

$$\partial_{z,z}\Gamma(z,p,w) = -1/\mu_p f(z/\mu_p)(b+h) < 0$$

while,

$$\begin{aligned}\partial_p^2\Gamma(z,p,w) &= 2\alpha_w - b\frac{\alpha_w z}{(\alpha_w p+\beta)^2}\frac{\alpha_w z}{\alpha_w p+\beta}f(z/\mu_p) - h\frac{\alpha_w z}{(\alpha_w p+\beta)^2}\frac{\alpha_w z}{\alpha_w p+\beta}f(z/\mu_p)\\ &= 2\alpha_w - \frac{z^2\alpha_w^2}{\mu_p^3}f(z/\mu_p)(b+h) < 0.\end{aligned}$$

Finally,

$$\partial_{z,z}\Gamma(z,p,w)\partial_{p,p}\Gamma(z,p,w) - (\partial_{z,p}\Gamma(z,p,w))^2 = -2\alpha_w/\mu_p f(z/\mu_p)(b+h) > 0.$$

A similar proof holds for the lost sales case. ∎

The main result of this section is given by the following theorem.

Theorem 7 *Under the same conditions as in Theorem 1 while considering Γ_t function of z and p, we have that parts* (i) *and* (iv) *of Theorem 1 hold in this case as well. Furthermore,* (ii) *and* (iii) *are replaced with*

(ii) *J_t is jointly concave in (x_t, Q_t, p_t), strictly concave in (x_t, p_t) and (Q_t, p_t),*

(iii) *V_t is concave in $(q_t, z_t, x_t, Q_t, p_t)$.*

Proof. The proof of this Theorem follows the same steps of Theorem 1. The latter relied mainly on Lemma 2. Therefore, we start by proving an equivalent result to Lemma 2 by considering p as an additional variable. We show the concavity of $\Gamma + \Pi + R$, when Γ and R are also functions of p. Note that without loss of generality, we can combine the functions $\Gamma + R$ into one function that we denote by R. As in the proof of Lemma 2, we compute the Hessian of the function of interest. We denote it by

$$H_p = \begin{bmatrix} & & & & -\partial_{2,3}R \\ & & & & \partial_{1,3}R \\ & & \mathbf{H} & & 0 \\ & & & & \partial_{2,3}R \\ -\partial_{2,3}R & \partial_{1,3}R & 0 & \partial_{2,3}R & \partial_{3,3}R \end{bmatrix}.$$

The matrix H is the one defined in Lemma 2. By applying a Gauss elimination, we convert H_p into $\bar{H}_p$.

$$\bar{H}_p = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & & & & \partial_{1,3}R \\ 0 & & \bar{\mathbf{H}}' & & 0 \\ 0 & & & & \partial_{2,3}R \\ 0 & \partial_{1,3}R & 0 & \partial_{2,3}R & \partial_{3,3}R \end{bmatrix}.$$

We let again $\bar{H}_p'$ be the 4×4 non-zero sub-matrix of $\bar{H}_p$. Again, using a Gauss elimination, we can easily convert $\bar{H}_p'$ into

$$\begin{bmatrix} \Pi'' & 0 \\ 0 & \mathbf{R} \end{bmatrix},$$

where $\mathbf{R}$ is the Hessian of the function R. From its Hessian, we conclude that the function V_t is concave (resp., strictly concave) if and only if R is concave (resp., strictly concave). The rest of the proof is exactly the same as the one for Theorem 1. ■

Consider, for instance, the setting where the manufacturer decides before the start of the horizon on setting a fixed price p_0 for the long-term unit demand and reserving a total capacity Q. The previous Theorem shows that there exists a unique way of doing so, and the optimal price and capacity are given by solving the system of equations when the initial inventory $x = 0$,

$$\begin{cases} \partial_p J_0(0, Q, p) = 0 \\ \partial_Q J_0(0, Q, p) = 0. \end{cases} \tag{23}$$

If we let $\bar{p}$, $\bar{Q}$, the respective solutions of those equations take the value infinity when no solution exists, then the optimal long-term price and capacity are given respectively by $p^* = \min\{\bar{p}, p_{max}\}$ and $Q^* = \min\{\bar{Q}, Q_{max}\}$, where Q_{max} is the maximum capacity the supplier can allocate to the manufacturer and p_{max} is defined in Equation (22).

Although a fixed price contract is quite common in manufacturing, in some industries (such as airlines, hotel and more recently retail) dynamic pricing policies are often used. As we saw for the previous Theorem, the problem becomes less tractable with three control variables (q, z, p). However, convinced of the importance of dynamic pricing, we obtain with little effort additional results that depict some of the main characteristics of the optimal pricing strategy.

We start by studying some monotonicity results related to the price p_t. First, we show that $\bar{z}_t$ decreases with the value of the price p_t. The intuition true in the myopic case, where by increasing the price the demand decreases and hence less inventory is needed, is therefore true as well in the dynamic setting.

Proposition 4 *The optimal sell-down-to/buy-up-to level $\bar{z}_t$ is decreasing with the unit long-term price p_t.*

Proof. To show this, it is enough to prove that $\partial_{p,z} V \leq 0$ (see Topkis (1978)). Recalling that $\bar{q}$, being the optimal capacity level, is a solution to $\Pi'(x+q-z) - \partial_2 R(z, Q-q, p) = 0$. We consider the expression

$$\begin{aligned}
&\partial_{p,z} V(\bar{q}, z, p|x, Q) \\
&\quad = \partial_{z,p}\Gamma(z,p) + \partial_z[\bar{q}'(p)(\Pi'(x+q-z) - \partial_2 R(z, Q-q, p)) + \partial_3 R(z, Q-q, p)] \\
&\quad = -\alpha\, \partial_{z,z}\Gamma(z,p) - \alpha\, \partial_{z,z} R(z, Q-q, p) \\
&\quad \leq 0.
\end{aligned}$$

The last equality is obtained by observing (see proof of Proposition 3) that $\partial_p \Gamma(z,p) = 2\alpha p + \beta - \alpha\, \partial_z \Gamma(z,p)$. Similarly, we have $\partial_p \mathbb{E} J(z-d, Q-q) = \partial_p \int_0^\infty J(z-\mu_p u) f(u) du = -\alpha\, \partial_z \mathbb{E} J(z-d, Q-q)$. The final inequality is due to the concavity of the functions involved. ■

Other monotonicity results can be obtained for the price p_t. For instance, in the linear spot market case, it is easy to show again that $\bar{p}_t$ is independent of x and Q as well. One interesting question is to study the monotonicity of $\bar{p}_t$ as a function of Q_t. The traditional revenue management theory predicts that the price is decreasing with the total capacity (Bitran and Mondshein (1997) or Gallego and van Ryzin (1994)). In our context, the quantity is connected with $\bar{z}_t$, which is not clearly monotonic in the capacity Q_t, as we discussed in the previous section. In order to facilitate the analysis, we consider the same problem but where the modified net inventory $\bar{z}_t = S_t$ is fixed and pre-determined. This case is plausible, for example, when the manufacturer has committed a fixed pre-specified quantity to face the long-term demand every period. We show the following result.

Proposition 5 *Under the setting described above, the optimal dynamic price $\bar{p}_t$ is decreasing in the total capacity Q_t available at time t.*

Proof. To prove this result, we show that $\partial_{p,Q} V(\bar{q}, p|S, x, Q) \leq 0$ for a given S. Taking the derivative of V with respect to p and then Q as in the previous proof leads to $-\alpha(1-\bar{q}'(Q))\, \partial_{1,2} R(S, Q-q, p)$. From Theorem 4, we conclude that $1 - \bar{q}'(Q) \geq 0$ and so $\partial_{p,Q} V(\bar{q}, p|S, x, Q) \leq 0$ ■

We end this section by considering the myopic policy. We follow the same steps as in Proposition 3, to show that in the dynamic pricing context a myopic policy is still asymptotically optimal with $\bar{p}_t^m = \text{argmax}_p \ \max_z \Gamma_t(z, p, w_t)$. In particular, in a stationary environment (i.e. Γ_t is independent of t) $\bar{p}_t^m \equiv p^m$ is then independent of t.

Proposition 6 *In a stationary environment a constant price is asymptotically optimal as the initial capacity Q increases.*

Finally, in the myopic setting we study how the presence of the spot market affects the price the manufacturer offers to the long-term demand. We show that as the spot market channel becomes more profitable the manufacturer tends to increase the price on the long-term customer. Indeed, to obtain a bigger share from the spot market demand, the manufacturer needs to devote a bigger part of his capacity to the spot market. Hence, less inventory is devoted to the long-term channel; in order to minimize the impact of the penalty cost, the manufacturer needs to decrease the long-term demand rate. He does so by increasing the unit price p_t.

To study this problem, we parameterize the spot market value by a multiplicative scalar ϵ, so that the value obtained from selling u units is $\epsilon\,\Pi(u)$. Note that for $\epsilon = 0$, the spot market is not an available option. We also assume a price only contract where the manufacturer offers a fixed unit price $p_t \equiv p$ for every unit sold to the long-term demand.

Proposition 7 *Under a myopic policy, the long-term unit price p is increasing with ϵ.*

Proof. For clarity, we assume a linear spot market function, although the same proof holds true for any concave spot market value function. To obtain the optimal price p and the policy parameter $\bar{z}$, the manufacturer solves both equations concurrently: $\partial_p \Gamma(z, p) = 0$ and $\partial_z \Gamma(z, p) - \epsilon\,\pi = 0$. We have that

$$\partial_p \Gamma(z, p, w) = 2\alpha_w p + \beta_w - b \int_{z/\mu_p}^{\infty} \alpha_w x f(x) dx + h \int_0^{z/\mu_p} \alpha_w x f(x) dx$$

so that

$$z/\mu_p = G^{-1}\left(\frac{2p + \beta_w/\alpha_w - b}{b + h}\right) \tag{24}$$

and $G(z) = \int_0^z u f(u) du$ is increasing in z. Basically, the price z/μ_p decreases with p. On the other hand, $\partial_z \Gamma(z, p) = b - (b + h) F(z/\mu_p) = \epsilon\,\pi$,
or equivalently

$$z/\mu_p = F^{-1}\left(\frac{b - \epsilon\pi}{b + h}\right). \tag{25}$$

We conclude that the ratio z/μ_p decreases with ϵ while it is decreasing in p. This completes the proof. ■

From Equation (25) note also that the order-down-to level z decreases as the spot market becomes more profitable (ϵ increases). If ϵ is large enough, one will shift completely to the spot market.

6 A Numerical Example

The purpose of this section is to illustrate numerically the optimal policy structure. We use a backward induction algorithm to solve the functional Equation (8). The long term channel profit function is assumed stationary over the planning horizon, and is as given in Equation (7). Backlogging is allowed. We set the parameters to $p = 10$, $h = 2$, $b = 18$, $T = 6$. The market is governed by three states of the world: "High", "Medium" and "Low". The state transition matrix is

$$A_2 = \begin{pmatrix} \frac{3}{4} & \frac{1}{5} & \frac{1}{20} \\ \frac{1}{10} & \frac{4}{5} & \frac{1}{10} \\ \frac{1}{20} & \frac{1}{5} & \frac{3}{4} \end{pmatrix}$$

In all world states, long term demand is normally distributed with a standard deviation, σ, of 1. The expected demand μ in a given period is determined by the state of the world. We use $\mu = 20$, 10, and 5 for high, medium and low demand states, respectively. Demand is truncated to ensure positive values. We consider a logarithmic spot market value function, that is $\Pi(s_t, w_t) = k_w p \log(1 + s_t)$, where k_w is one for low demand, four for medium demand, and eight for high demand.

Figure 1 depicts the profit function J_1 under medium demand. Notice that optimal profit is increasing and strictly and separately concave in x_1 and Q_1 (Theorem 1). Figure 2 illustrates the

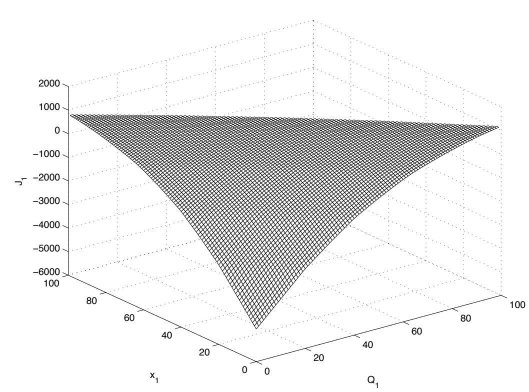

Figure 1: Optimal Profit J_1

optimum production quantity q_t^* for three values of x_t. For each instance, when sufficient capacity is available, a produce-up-to threshold is sought; otherwise, producing all remaining capacity is optimal (Theorem 1). In cases where we have ample remaining capacity, we also produce units destined for the spot market. Figure 2 illustrates the optimum modified net inventory z_t^* for three values of x_t. Consider for example $x_1 = 10$ and $Q_1 = 40$. From these graphs, it is optimal to produce $q_1^* = 12$ units and bring net inventory to 32 units; and sell 11 units in the spot market to bring the modified net inventory down to $z_1^* = 21$ units. Note also that the sell-down-to level (21) matches the expected demand for the present state (20) and a small margin of safety stock (1).

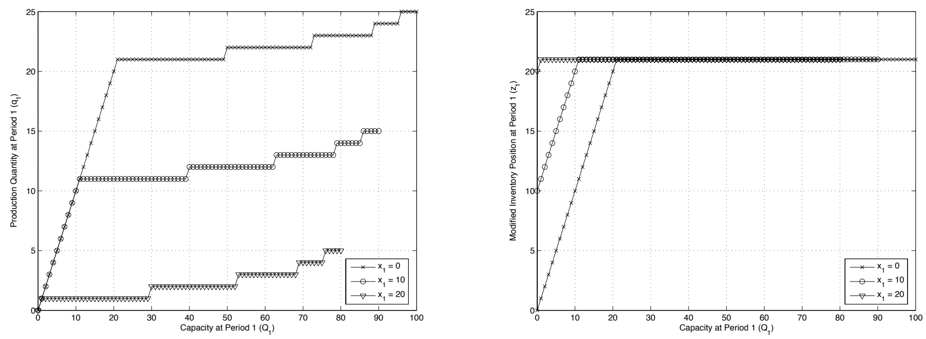

Figure 2: Production quantity q_1^* (left) and modified net inventory z_1^* (right)

7 Conclusion

In this paper, we studied a capacity constrained manufacturer who sells to a long-term channel and trades with a spot market over a finite horizon. We established optimal policies to maximize the system's profit in a general case when profit value functions (from the long-term channel and the spot market) are concave. In the special case of a linear spot market the policy is state independent. In general, following such a policy, the manufacturer optimally produces from total capacity available and decides on how to allocate inventory between the contract market and spot market. Production managers and decision makers are often interested in understanding qualitatively how they should respond to changes in the environment of the problem. The structural results in this paper provide ways for developing insights as well as tools for designing efficient algorithms for large scale settings. For example, we show that the number of units sold on the spot market increases with both net inventory and remaining capacity available. We also show that myopic policies can define very good approximations when the expected capacity per period (Q/T) is large. The model also provides a framework to quantify the performance of a capacity constrained production system with respect to, for example, capacity and long-term pricing policy in the presence of a spot market. From a dynamic pricing perspective, our study confirms results obtained in the literature under simpler settings. For instance, we show that the price is monotone with the total capacity and that a constant price is asymptotically optimal when the total capacity becomes large. Finally, we observe that the presence of a spot market induces the manufacturer to increase the long-term channel unit price, making the long-term customer worse-off.

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