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# FRACTIONAL BROWNIAN MOTION WITH $H < 1/2$ AS A LIMIT OF SCHEDULED TRAFFIC

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**Abstract**

This paper shows that fractional Brownian motion with $H < 1/2$ can arise as a limit of a simple class of traffic processes that we call "scheduled traffic models". To our knowledge, this paper provides the first simple traffic model leading to fractional Brownnian motion with $H < 1/2$. We also discuss some immediate implications of this result for queues fed by scheduled traffic, including a heavy-traffic limit theorem.

## 1. Introduction

There is an extensive literature justifying the use of fractional Brownian motion (and, more generally, fractional Levy motion) as a mathematical description of the complex aggregate traffic that is carried by data networks; see, for example, (Kurtz (1996), Gurin et al.(1999) Mikosh et al.(2002), Pipiras et al. (2004), Kaj (2005) and Kaj and Taqqu (2008)). One can support the use of such models either on the basis of statistical analysis, or on the basis of limit theorems that establish that such processes arise naturally as asymptotic descriptions of physically realistic models that characterize network traffic at less aggregated

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scales (say, that of packets in the network). For example, Mikosh et al. (2002) shows that fractional Brownian motion can arise as a limit of a superposition of "on-off" source models with appropriately heavy-tailed inputs. However, one common characteristic of these limit theorems is that the limit processes that arise always exhibit non-negative auto-correlation structure. In particular, the fractional Brownian motions that arise as such limits have associated Hurst parameters $H \geq 1/2$.

In this paper, we propose a simple traffic model that has the property that, when appropriately re-scaled, convergence to a fractional Brownian motion (fBm) with $H < 1/2$ ensues. Our main result (Theorem 1) provides a queueing level/point process level interpretation of such fBm's. The model that we consider is one that we call a "scheduled traffic" model; its origin goes back at least as far as Cox and Smith (1961), in which such a point process is termed a "regular arrival process with unpunctuality". Customers are scheduled to arrive to the system at regular (say, unit) intervals. So, customer $j$ is scheduled to arrive at time $j$. However, because of random effects experienced along the path traveled to the system, customer $j$'s actual arrival time is $j + \xi_j$. As a consequence, the number $N_n$ of arrivals to the system in $(0, n]$ is given by

$$N_n = \sum_{j=-\infty}^{\infty} I(\xi_j + j \in (0, n]),$$

where the customer index set is taken, for convenience, to be doubly infinite. Customers with $\xi_j$ negative arrive "early" and customers with "perturbations" $\xi_j$ that are positive arrive "late". In this paper, we (reasonably) assume that the sequence of perturbations $(\xi_j : -\infty < j < \infty)$ is a family of independent and identically distributed (iid) random variables (rv's). Under this assumption, $(N_n : n \geq 0)$ has stationary increments (in discrete time), in the sense that $N_{n+m} - N_m \stackrel{\mathcal{D}}{=} N_n - N_0$ for $n, m \geq 0$ (where $\stackrel{\mathcal{D}}{=}$ denotes "equality in distribution"), and $\mathbb{E}N_1 = 1$.

We show elsewhere that there exists a rv $\Gamma$ such that $N_n - n \Rightarrow \Gamma$ as $n \to \infty$ if (and only if) $\mathbb{E}|\xi_1| < \infty$. In order that we obtain a functional limit theorem for $(N_n : n \geq 0)$ in which the limit process is a fBm, we shall therefore consider heavy-tailed perturbations with $\mathbb{E}|\xi_1| = \infty$. In particular, we shall assume that the perturbations are non-negative and satisfy

$$\mathbb{P}(\xi_0 > x) \sim c\, x^{-\alpha} \tag{1}$$

as $x \to \infty$, for $0 < c < \infty$ and $0 < \alpha < 1$. In the presence of (1), we establish a Donsker-type functional limit theorem for the above scheduled traffic model in which the limit process is a fBm with $H = (1-\alpha)/2$; see Section 2 for a full description of the result. Thus, such a scheduled traffic process exhibits a negative dependency structure. This is intuitively reasonable, as a scheduled traffic process has the characteristic that if one observes more arrivals than normal in one interval, this likely has occurred because either future customers have arrived early or because previously scheduled customers arrived late (thereby reducing the number of arrivals to either past or future intervals). We note also that $H \downarrow 0$ as $\alpha \uparrow 1$ (so that the level of negative dependence increases as the perturbations exhibit smaller fluctuations), and $H \uparrow 1/2$ (the Brownian motion case) as $\alpha \downarrow 0$ (so that the perturbations are "more random").

This paper is organized as follows, Section 2 states and proves the main result of the paper (our functional limit theorem for scheduled traffic), while Section 3 describes the implications in the queueing context. Specifically, the workload process for a single server queue fed by scheduled traffic is studied in the "heavy traffic" setting.

## 2. The Main Result

For $t \geq 0$, let $X_n = (X_n(t) : t \geq 0)$ be defined via

$$X_n(t) = \frac{N_{\lfloor nt \rfloor} - \lfloor nt \rfloor}{n^{(1-\alpha)/2}}.$$

Also, for $H \in (0,1)$, let $B_H = (B_H(t) : t \geq 0)$ be a mean zero Gaussian process with covariance function given by

$$\mathbb{C}\mathrm{ov}(B_H(s), B_H(t)) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H})$$

for $s, t \geq 0$. The process $B_H$ is a continuous path process with stationary increments for which $B_H(0) = 0$, and is the fBm (with zero mean and unit variance parameter) having Hurst parameter $H$.

**Theorem 1.** *Suppose that $(\xi_j : -\infty < j < \infty)$ is an iid sequence of positive rv's satisfying (1). Then,*

$$X_n \Rightarrow \sqrt{2c(1-\alpha)^{-1}} B_H$$

*as $n \to \infty$, where $H = (1-\alpha)/2$ and $\Rightarrow$ denotes weak convergence on $D[0,\infty)$.*

As is common in proving such results, our proof comes in two parts: convergence of the finite-dimensional distributions and verification of tightness.

**Proposition 2.1.** *Under the conditions of Theorem 1,* $X_n \overset{fdd}{\to} \sqrt{2c(1-\alpha)^{-1}}B_H$ *as* $n \to \infty$, *where* $\overset{fdd}{\to}$ *denotes weak convergence of the finite-dimensional distributions.*

*Proof.* For notational simplicity, we prove convergence of the finite-dimensional distributions only for two time epochs; the general case is essentially identical. We start by observing that for $t \geq 0$,

$$\begin{aligned} X_n(t) &= n^{-H}\big(\sum_{j=1}^{\lfloor nt \rfloor} I(j+\xi_j \in (0, \lfloor nt \rfloor]) - \lfloor nt \rfloor \\ &\qquad + \sum_{j\leq 0} I(j+\xi_j \in (0, \lfloor nt \rfloor])\big) \\ &= n^{-H}\big(-\sum_{j=1}^{\lfloor nt \rfloor} I(j+\xi_j > \lfloor nt \rfloor) \\ &\qquad + \sum_{j\leq 0} I(j+\xi_j \in (0, \lfloor nt \rfloor])\big). \end{aligned}$$

For $0 \leq t_1 < t_2$ and $\theta_1, \theta_2 \in \mathbb{R}$, set $n_1 = \lfloor nt_1 \rfloor$, $n_2 = \lfloor nt_2 \rfloor$, $\tilde{\theta}_1 = n^{-H}\theta_1$, and $\tilde{\theta}_2 = n^{-H}\theta_2$. Then,

$$\begin{aligned} &\theta_1 X_n(t_1) + \theta_2 X_n(t_2) \\ = &-(\tilde{\theta}_1 + \tilde{\theta}_2)\sum_{j=1}^{n_1} I(\xi_j + j > n_2) - \tilde{\theta}_1 \sum_{j=1}^{n_1} I(\xi_j + j \in (n_1, n_2]) - \tilde{\theta}_2 \sum_{j=n_1+1}^{n_2} I(\xi_j + j > n_2) \\ &+ (\tilde{\theta}_1 + \tilde{\theta}_2)\sum_{j\leq 0} I(\xi_j + j \in (0, n_1]) + \tilde{\theta}_2 \sum_{j \leq 0} I(\xi_j + j \in (n_1, n_2]). \end{aligned}$$

Put $\bar{F}(j) \overset{D}{=} \mathbb{P}(\xi_0 > j)$ for $j \geq 0$. The i.i.d. structure of the $\xi_j$'s establishes that the log-moment generating function of $(X_n(t_1), X_n(t_2))$ (evaluated at $(\theta_1, \theta_2)$) is given by

$$\begin{aligned} &\sum_{j=1}^{n_1} \log(1 + (e^{-\tilde{\theta}_1 - \tilde{\theta}_2} - 1)\bar{F}(n_2 - j) + (e^{-\tilde{\theta}_1} - 1)(\bar{F}(n_1 - j) - \bar{F}(n_2 - j))) \\ &+ \sum_{j=n_1+1}^{n_2} \log(1 + (e^{-\tilde{\theta}_2} - 1)\bar{F}(n_2 - j)) \\ &+ \sum_{j\leq 0} \log\big(1 + (e^{\tilde{\theta}_1 + \tilde{\theta}_2} - 1)(\bar{F}(-j) - \bar{F}(n_1 - j)) + (e^{\tilde{\theta}_2} - 1)(\bar{F}(n_1 - j) - \bar{F}(n_2 - j))\big). \end{aligned}$$

Because, $\tilde{\theta}_i \to 0$ as $n \to \infty$ and $\log(1+x) = x(1+o(1))$ as $x \to 0$, it follows that

$$\begin{aligned}
&\log \mathbb{E} \exp(\theta_1 X_n(t_1) + \theta_2 X_n(t_2)) \\
&= \big((e^{-\tilde{\theta}_1-\tilde{\theta}_2} - e^{-\tilde{\theta}_1}) \sum_{j=1}^{n_1} \bar{F}(n_2 - j) + (e^{-\tilde{\theta}_1} - 1) \sum_{j=1}^{n_1} \bar{F}(n_1 - j) + (e^{-\tilde{\theta}_2} - 1) \sum_{j=n_1+1}^{n_2} \bar{F}(n_2 - j) \\
&\quad + (e^{\tilde{\theta}_1+\tilde{\theta}_2} - 1) \sum_{j=0}^{\infty} (\bar{F}(j) - \bar{F}(n_1 + j)) + (e^{\tilde{\theta}_2} - 1) \sum_{j=0}^{\infty} (\bar{F}(n_1 + j) - \bar{F}(n_2 + j))\big)(1 + o(1))
\end{aligned}$$

as $n \to \infty$. For $0 \le k_1 \le k_2$ and $r \ge k_2 - k_1$,

$$\sum_{j=0}^{r} (\bar{F}(k_1 + j) - \bar{F}(k_2 + j)) = \sum_{j=k_1}^{k_2-1} \bar{F}(j) - \sum_{j=k_1+r+1}^{k_2+r} \bar{F}(j).$$

Because, $\sum_{j=k_1+r+1}^{k_2+r} \bar{F}(j) \le (k_2 - k_1)\bar{F}(k_1 + r + 1) \to 0$ as $r \to \infty$, evidently

$$\sum_{j=0}^{\infty} (\bar{F}(k_1 + j) - \bar{F}(k_2 + j)) = \sum_{j=k_1}^{k_2-1} \bar{F}(j).$$

Consequently,

$$\begin{aligned}
&\log \mathbb{E} \exp(\theta_1 X_n(t_1) + \theta_2 X_n(t_2)) \\
&= \big((e^{-\tilde{\theta}_1-\tilde{\theta}_2} - e^{-\tilde{\theta}_1}) \sum_{j=n_2-n_1}^{n_2-1} \bar{F}(j) + (e^{-\tilde{\theta}_1} - 1) \sum_{j=0}^{n_1-1} \bar{F}(j) \\
&\quad + (e^{-\tilde{\theta}_2} - 1) \sum_{j=0}^{n_2-n_1-1} \bar{F}(j) + (e^{\tilde{\theta}_1+\tilde{\theta}_2} - 1) \sum_{j=0}^{n_1-1} \bar{F}(j) \\
&\quad + (e^{\tilde{\theta}_2} - 1) \sum_{j=n_1}^{n_2-1} \bar{F}(j)\big)(1 + o(1)) \\
&= \frac{1}{2}\big([(\theta_1 + \theta_2)^2 - \theta_1^2 + O(n^{-H})]n^{-2H} \sum_{j=n_2-n_1}^{n_2-1} \bar{F}(j) \\
&\quad + [\theta_1^2 + O(n^{-H})]n^{-2H} \sum_{j=0}^{n_1-1} \bar{F}(j) \\
&\quad + [\theta_2^2 + O(n^{-H})]n^{-2H} \sum_{j=0}^{n_2-n_1-1} \bar{F}(j) \\
&\quad + [(\theta_1 + \theta_2)^2 + O(n^{-H})]n^{-2H} \sum_{j=0}^{n_1-1} \bar{F}(j) \\
&\quad + [\theta_2^2 + O(n^{-H})]n^{-2H} \sum_{j=n_1}^{n_2-1} \bar{F}(j)\big)(1 + o(1))
\end{aligned}$$

as $n \to \infty$.

Choose $\delta \in (0, 1-\alpha)$, and observe that for $t > 0$,

$$\begin{aligned}
&n^{-2H} \sum_{j=0}^{\lfloor nt \rfloor} \bar{F}(j) \\
&= n^{\alpha-1} \sum_{j=0}^{\lfloor n^\delta \rfloor} \bar{F}(j) + n^{\alpha-1} \sum_{j=\lfloor n^\delta \rfloor+1}^{\lfloor nt \rfloor} cj^{-\alpha}(1+o(1)) \\
&= o(1) + c \sum_{j=\lfloor n^\delta \rfloor+1}^{\lfloor nt \rfloor} (\frac{1}{n})(\frac{j}{n})^{-\alpha}\,(1+o(1)) \\
&= o(1) + c \int_{n^{\delta-1}}^{t} x^{-\alpha} dx\,(1+o(1)) \\
&= \frac{ct^{1-\alpha}}{1-\alpha} + o(1)
\end{aligned}$$

as $n \to \infty$.

Thus, we find that

$$\begin{aligned}
&\log \mathbb{E} \exp(\theta_1 X_n(t_1) + \theta_2 X_n(t_2)) \\
\to\ & \frac{c}{2(1-\alpha)} \big( [\theta_2^2 + 2\theta_1\theta_2](t_2^{2H} - (t_2-t_1)^{2H}) + \theta_1^2 t_1^{2H} \\
&\qquad + [\theta_2^2 (t_2 - t_1)^{2H} + [\theta_1^2 + \theta_2^2 + 2\theta_1\theta_2] t_1^{2H} + \theta_2^2 (t_2^{2H} - t_1^{2H}) \big) \\
=\ & \frac{c}{1-\alpha} \big( \theta_1^2 t_1^{2H} + \theta_2^2 t_2^{2H} + \theta_1\theta_2 (t_2^{2H} + t_1^{2H} - |t_2 - t_1|^{2H}) \big)
\end{aligned}$$

as $n \to \infty$, which is precisely the joint log-moment generating function of the Gaussian finite-dimensional distribution of the limit process.

**Proposition 2.2.** *The sequence $(X_n : n \geq 0)$ is tight in $D[0, \infty)$.*

*Proof.* Note that because we established convergence of the moment generating functions in Proposition 2.1, it follows that $(|X_n(t)|^p : n \geq 0)$ is uniformly integrable for all $t \geq 0$ and $p > 0$. Hence, in view of Proposition 2.1 above, all the requisite conditions of Theorem 2.1 of Taqqu (1979) are satisfied, so that $((X_n(u) : 0 \leq u \leq t) : n \geq 0)$ is tight in $D[0, t]$ for each $t \geq 0$.

Propositions 2.1 and 2.2 together prove Theorem 1.

**Remark 1** A very similar proof holds for a time-stationary scheduled arrival process formulated in continuous time. In particular, let $N(t)$ be the number of scheduled arrivals in $(0, t]$, so that

$$N(t) = \sum_{j=-\infty}^{\infty} I(jh + Uh + \xi_j \in (0, t]),$$

where customers are scheduled to arrive at times in $h\mathbb{Z}$, and $U$ is a uniform [0,1] rv independent of $(\xi_j : j \in \mathbb{Z})$; the uniform rv $U$ is introduced in order to induce time-stationarity. If the distribution of $\xi_0$ satisfies (1), then

$$\frac{N(nt) - nt/h}{n^H} \Rightarrow \sqrt{\frac{2c}{1-\alpha}} B_H(t/h)$$

as $n \to \infty$ in $D[0, \infty)$.

## 3. Implications for Queues

We now briefly describe the implications of our limit theorem for a queue that is fed by a scheduled arrival process with iid heavy-tailed perturbations $(\xi_n : n \in \mathbb{Z})$ satisfying (1). In particular, we consider such a queue in "heavy traffic", in an environment in which the service times are deterministic. (We view this deterministic assumption as being realistic in this setting, given that a service provider would likely only attempt to schedule arrivals when the service times were of highly predictable duration.)

Specifically, we consider a family of queues, indexed by $\rho \in (0, 1)$, in which the number of arrivals in $(0, t]$ to the $\rho$'th system is given by $N(\rho t)$, where $N$ satisfies the conditions of Theorem 1. If the $\rho$'th system starts off idle and the service times have unit duration, then the workload process $(W_\rho(t) : t \geq 0)$ for the $\rho$'th system is given by

$$W_\rho(t) = N(\rho t) - t - \min_{0 \leq s \leq t} [N(\rho s) - s].$$

Clearly, the utilization (or traffic intensity) of system $\rho$ is $\rho$. Heavy traffic is therefore obtained by letting $\rho \uparrow 1$.

**Theorem 2.** *Under the same conditions as for Theorem 1,*

$$(1-\rho)^{\frac{1-\alpha}{1+\alpha}} W_\rho(\cdot/(1-\rho)^{\frac{2}{1+\alpha}})$$
$$\Rightarrow \sigma B_H(\cdot) - e(\cdot) - \min_{0 \leq s \leq e(\cdot)} [\sigma B_H(s) - s]$$

*as $\rho \uparrow 1$ in $D[0, \infty)$, where $H = (1-\alpha)/2$, $\sigma^2 = 2c/(1-\alpha)$, and $e(t) = t$ for $t \geq 0$.*

*Proof.* Note that

$$
\begin{aligned}
& (1-\rho)^{\frac{1-\alpha}{1+\alpha}} W_\rho(t/(1-\rho)^{\frac{2}{1+\alpha}}) \\
= \; & (1-\rho)^{\frac{H}{1-H}} W_\rho(t/(1-\rho)^{\frac{1}{1-H}}) \\
= \; & (1-\rho)^{\frac{H}{1-H}} \big[(N(\rho t/(1-\rho)^{\frac{1}{1-H}}) - t/(1-\rho)^{\frac{1}{1-H}}) \\
& \qquad - \min_{0\le s\le t} N(\rho s/(1-\rho)^{\frac{1}{1-H}}) - s/(1-\rho)^{\frac{1}{1-H}}\big] \\
= \; & \rho^H \big(\frac{(1-\rho)^{\frac{1}{1-H}}}{\rho}\big)^H \big[\big(N(\frac{\rho t}{(1-\rho)^{\frac{1}{1-H}}}) - \frac{\rho t}{(1-\rho)^{\frac{1}{1-H}}} - (1-\rho)^{-\frac{H}{1-H}} t\big) \\
& \qquad - \min_{0\le s\le t} \big(N(\frac{\rho s}{(1-\rho)^{\frac{1}{1-H}}}) - \frac{\rho s}{(1-\rho)^{\frac{1}{1-H}}} - (1-\rho)^{-\frac{H}{1-H}} s\big)\big] \\
= \; & \rho^H \big[(\frac{(1-\rho)^{\frac{1}{1-H}}}{\rho})^H (N(\frac{\rho t}{(1-\rho)^{\frac{1}{1-H}}}) - \frac{\rho t}{(1-\rho)^{\frac{1}{1-H}}}) - t - (\rho^{-H}-1)t \\
& \qquad - \min_{0\le s\le t} (\frac{(1-\rho)^{\frac{1}{1-H}}}{\rho})^H (N(\frac{\rho s}{(1-\rho)^{\frac{1}{1-H}}}) - \frac{\rho s}{(1-\rho)^{\frac{1}{1-H}}}) - s - (\rho^{-H}-1)s\big].
\end{aligned}
$$

But Theorem 1 implies that

$$
(\frac{(1-\rho)^{\frac{1}{1-H}}}{\rho})^H (N(\frac{\rho\,\cdot}{(1-\rho)^{\frac{1}{1-H}}}) - \frac{\rho e(\cdot)}{(1-\rho)^{\frac{1}{1-H}}}) \Rightarrow \sigma B_H(\cdot)
$$

in $D[0,\infty)$ as $\rho \uparrow 1$. Since $(\rho^{-H}-1)e(\cdot) \to 0$ uniformly on compact time intervals, as $\rho \uparrow 1$, the continuous mapping principle (see for example, Billingsley (1999), p. 20) implies the theorem.

Theorem 2 suggests the approximation

$$
W(t) \stackrel{\mathcal{D}}{\approx} (1-\rho)^{\frac{\alpha-1}{\alpha+1}} Z((1-\rho)^{\frac{2}{1+\alpha}} t)
$$

when $\rho \stackrel{\Delta}{=} \mathbb{E}(N_1 - N_0)$ is close to one, where $\stackrel{\mathcal{D}}{=}$ denotes "has approximately the same distribution as" (and has no rigorous meaning, other than that associated with Theorem 2 itself) and $Z = (Z(t) : t \geq 0)$ is the *regulated* fBm given by $Z(t) = \sigma B_H(t) - t - \min_{0\le s\le t}[\sigma B_H(s) - s]$. One implication is that when $\rho$ is close to 1, the rough magnitude of $W(\cdot)$ is of order $(1-\rho)^{\frac{\alpha-1}{\alpha+1}}$ (where $\frac{\alpha-1}{\alpha+1} \in (-2,0)$) and the time scale over which $W(\cdot)$ fluctuates (in a relative sense) is of order $(1-\rho)^{\frac{-2}{1+\alpha}}$. In particular, when $\alpha$ is close to 1 (so that $N$ is almost deterministic), the magnitude of $W$ is small and the fluctuations of $W$ occur over time scales of order $(1-\rho)^{-1}$.

On the other hand if the service times $(V_n : n \in \mathbb{Z})$ associated with the scheduled arrival sequence are iid with unit mean and positive finite variance, then the corresponding heavy traffic limit theorem for the workload process $W_\rho(t) = \sum_{i=1}^{N(\rho t)} V_i - t - \min(\sum_{i=1}^{N(\rho s)} V_i - s)$ is easily shown to be

$$(1-\rho)W_\rho(\cdot/(1-\rho)^2) \Rightarrow \eta B(\cdot) - e(\cdot) - \min_{0 \le s \le e(\cdot)} (\eta B(s) - s) \tag{2}$$

as $\rho \uparrow 1$ in $D[0,\infty)$, where $B = (B(t) : t \ge 0)$ is standard Brownian motion with $B(0) = 0$, and $\eta^2 = \mathbb{V}ar V_1$. This limit theorem is identical to that obtained for a D/G/1 queue in heavy traffic, so that in this asymptotic regime with positive variance service times, scheduled traffic behaves similarly to a deterministic arrival sequence. Furthermore, in this positive variance setting, the fluctuations of a scheduled queue in heavy traffic are larger (of order $(1-\rho)^{-1}$) and occur over longer time scales (of order $(1-\rho)^{-2}$ than in the context of deterministic service times (which are, as noted earlier, of order $(1-\rho)^{-\frac{1-\alpha}{1+\alpha}}$ and $(1-\rho)^{\frac{-2}{1+\alpha}}$ respectively).

A great deal is known about the behavior of the limiting regulated fBm process $Z$, and how its behavior contrasts with that of the regulated Brownian motion appearing in (2):

1. As for the reflected Brwonian motion in (2), $Z(t) \Rightarrow Z(\infty)$ as $t \to \infty$. However, in contrast to the Brownian case, $Z(\infty)$ has super-exponential tails (so that the tails are lighter than the exponential tails that arise in the conventional heavy traffic setting of (2)). In particular,
$$\mathbb{P}(Z(\infty) > x) \sim \frac{\mathcal{H}_{2H}\sqrt{\pi} D^{1/2H} A^{(2-H)/(2H)}}{B^{1/2} 2^{(1-H)/(2H)} \sigma^{(1-H)/H}} \, x^{(1-H)^2/H} \, \bar{\Phi}(A\,\sigma^{-1}\,x^{1-H}) \tag{3}$$
as $x \to \infty$, where $\bar{\Phi}$ is the tail of the standard normal distribution, $A = \left(\frac{H}{1-H}\right)^{-H} \frac{1}{1-H}$, $B = \left(\frac{H}{1-H}\right)^{-(H+2)} H$ and $D = \left(\frac{H}{1-H}\right)^{-2H}$ and $\mathcal{H}_{2H}$ is the so called Pickands constant; see Hüsler and Piterbarg (1999) for details.

2. The convergence to equilibrium of $Z(t)$ to $Z(\infty)$ occurs roughly at rate $\exp(-\theta^* t^{2-2H})$ (in "logarithmic scale"), where $\theta^*$ involves solving a variational problem. This is faster than the roughly exponential rate to equilibrium associated with (2); see Mandjes, Norros, and Glynn (2009) for details.

3. The dynamics of the process $Z$, conditioned on an unusually long busy period of duration $t$, forces the process $Z$ to make a large positive excursion (reaching a level of order $t$) during the busy period, whereas regulated Brownian motion (under the same conditioning) tends to exhibit much smaller positive fluctuations; see Mandjes et al. (2006) for details.

All these results point to the intuition that a scheduled arrival process (with deterministic service times) behaves much more predictibly than does a queue fed by (for example) renewal input. This is in strong contrast to queues which can be approximated by regulated fBm with $H > 1/2$, which generally have much worse behavior than conventional queues (i.e. equilibrium distributions with fatter than exponential tails, subexponential rates of convergence to equilibrium, etc).

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