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Theory and Applications of Stochastic Systems
Lecture 11

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1 Heavy-tailed distribution

1.1 Heavy-tailed distribution

Let X be a non-negative random variable. $F(x) = \mathbb{P}[X \le x]$ being its distribution and $\overline{F}(x) = 1 - F(x)$ be its tail distribution.

Definition 1.1 (Heavy-tailed random variables) *F (or X) is said to be heavy-tailed if $\overline{F}(x) > 0$ for $x \ge 0$ and*

$$\lim_{x\to\infty} \mathbb{P}[X > x + y | X > x] = \lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1 \qquad \textit{for } y \ge 0.$$

This can be also written as $\overline{F}(x+y) \sim \overline{F}(x)$, for all $y \ge 0$.

Intuitively this means that if X ever exceeds a large value, then it is likely to exceed any larger as well; its tail is *heavy* or *fat*. Let $\mathcal{L}$ be the class of heavy-tailed distributions. We will say either that $X \in \mathcal{L}$ or $F \in \mathcal{L}$.

Example 1.1 Let $X \sim \exp(\lambda)$ then

$$\frac{\overline{F}(x+y)}{\overline{F}(x)} = \frac{e^{-\lambda(x+y)}}{e^{-\lambda x}} = e^{-\lambda y}.$$

An exponential random variable is not heavy-tailed.

Now let X have a Pareto distribution, $F(x) = 1 - cx^{-\alpha}$ for $x \geq 0$, with c and $\alpha > 0$.

$$\frac{\overline{F}(x+y)}{\overline{F}(x)} = \frac{c(x+y)^{-\alpha}}{cx^{-\alpha}} \quad = \quad \left(\frac{x+y}{x}\right)^{-\alpha} \to 1 \ \ as\ x \to \infty$$

therefore a Pareto random variable is heavy-tailed.

Lemma 1.1 *If $X \in \mathcal{L}$ then $\mathbb{E}[e^{\epsilon X}] = \infty$ for all $\epsilon > 0$. In other words, heavy-tailed distributions have infinite moment generating functions.*

Definition 1.2 (Light-tailed random variables) *A random variable X is said to be light-tailed if $\mathbb{E}[e^{\epsilon X}] < \infty$ for some $\epsilon > 0$.*

Remark 1.1 *Note that in theory a random variable can have an infinite moment generating function without being necessarily heavy-tailed.*

Proposition 1.2 *Let $X \in \mathcal{L}$, $Y \geq 0$ be a random variable independent of X then*

$$\mathbb{P}[X - Y > x] \sim \mathbb{P}[X > x]$$

Proof.

$$\begin{aligned}
\mathbb{P}[X - Y > x] &= \mathbb{P}[X - Y > x | X > x]\overline{F}(x) \\
&= \overline{F}(x) \int_0^\infty \mathbb{P}[X > x + y | X > x] f_Y(y) dy \\
&\to \overline{F}(x) \qquad \text{since } \mathbb{P}[X > x + y | X > x] \to 1 \text{ as } x \to \infty
\end{aligned}$$

$\square$

Remark 1.2 If $\exists n$ such that $\mathbb{E}[X^n] = \infty \not\Rightarrow x \in \mathcal{L}$. In other words, the fact that some of its moments are infinite does not imply that the random variable is heavy-tailed even though it is often the case.

Lemma 1.3 *Alternative definition of heavy-tailed random variable*

$$x \in \mathcal{L} \Leftrightarrow \lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1 \qquad \textit{for } y \geq 0$$

Note that for any random variable X, and for $y \geq 0$, we have

$$\frac{\overline{F}(x-y)}{\overline{F}(x)} \geq 1$$

and hence,

$$\liminf_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} \geq 1$$

So the previous lemma can be stated as:

$$x \in \mathcal{L} \Leftrightarrow \limsup_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} \leq 1 \qquad \text{for } y \geq 0$$

1.2 Sub-exponential random variables

Definition 1.3 (sub-exponential random variable) *F (or X) is said to be sub-exponential if $\overline{F}(x) > 0$ for $x \geq 0$ and for $n \geq 2$:*

$$\lim_{x\to\infty} \frac{\overline{F^{*n}}(x)}{\overline{F}} = n. \tag{1}$$

*where * denotes the convolution symbol.*

Let $\mathcal{S}$ be the class of sub-exponential distributions. We will say either that $X \in \mathcal{S}$ or $F \in \mathcal{S}$.

Remark 1.3

- It can be shown that if the relation given by (1) holds for a particular n (e.g. $n = 2$) then it holds for all n.

- Let $X_1, ..., X_n$ be an i.i.d. sequence of random variables distributed as X. In general for all X (not necessarily sub-exponential), we have:

$$\begin{aligned}
\mathbb{P}[\max\{X_1, ..., X_n\} < x] &= \mathbb{P}[X_1 < x, ..., X_n < x] \\
&= (\mathbb{P}[X_1 < x])^n \\
&= [F(x)]^n \\
\mathbb{P}[\max\{X_1, ..., X_n\} > x] &= 1 - [F(x)]^n \\
&= [1 - F(x)]\left(1 + F(x) + [F(x)]^2 + ... + [F(x)]^{n-1}\right) \\
&\sim n\overline{F}(x)
\end{aligned}$$

as $x \to \infty$. Since $\left(1 + F(x) + [F(x)]^2 + ... + [F(x)]^{n-1}\right) \to n$ *as* $x \to \infty$.

On the other hand, by definition, if X is sub-exponential, we have:

$$\mathbb{P}[S_n > x] = \overline{F^{*n}}(x) \sim n\overline{F}(x)$$

In other words we see that for a sub-exponential distribution, the sum of i.i.d. random variables behaves asymptotically as their maximum. Or in words, this means that the sum is likely to get large because one of the r.v.'s gets large. It is this interpretation that justifies using sub-exponential distribution in stochastic modeling, for it could represent a "disaster" in an insurance risk business, or an "unusually long" processing time in a telecommunications network.

- Because for every sequence of i.i.d. random variable, $\mathbb{P}[X_1 + ... + X_n > x] \geq \mathbb{P}[\max\{X_1, ..., X_n\} > x]$, taking limits as $x \to \infty$ we obtain that

$$\liminf_{x \to \infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \geq n.$$

Therefore to show that $x \in \mathcal{S}$ we only need to prove that for $n \geq 2$:

$$\limsup_{x \to \infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \leq n.$$

Lemma 1.4 *If $x \in \mathcal{S}$ then $\mathbb{E}[e^{\epsilon X}] = \infty$, for all $\epsilon > 0$. In other words, sub-exponential distributions have infinite moment generating functions.*

Example 1.2 The following random variables are sub-exponential:

(1) Pareto: $F(x) = 1 - cx^{-\alpha}$ or $F(x) = 1 - \left(\frac{c}{c+x}\right)^{\alpha}$ with $c \geq 0, \alpha > 0$ for $x \geq 0$.

(2) Lognormal: Let $Y \sim \mathcal{N}(\mu, \sigma^2)$ then $X = \log Y \sim \text{Lognormal}(\mu, \sigma^2)$.

(3) Heavy-tailed Weibull: Let $Y \sim \exp(\lambda)$ with $\lambda > 0$ then $X = Y^{1/\alpha}$ with $0 < \alpha < 1$ has a heavy-tail Weibull distribution.

In this case it is interesting to notice that for all $\epsilon > 0$, we have $\mathbb{E}[e^{\epsilon Y}] < \infty$ which implies that $\mathbb{E}[Y^m] < \infty$ for all m. But also implies that $\mathbb{E}[Y^{m/\alpha}] = \mathbb{E}[X^m] < \infty$ but $\mathbb{E}[e^{\epsilon X}] = \infty$ by lemma 1.4. In other words, a heavy-tailed Weibull is sub-exponential and therefore its moment generating function is infinite but all its moments are well-defined.

(4) Regularly-varying: for $\alpha \geq 0$, F is said to be regularly varying with index $-\alpha$ with index $-\alpha$, denoted $R_{-\alpha}$, if

$$\lim_{x \to \infty} \frac{\overline{F}(tx)}{\overline{F}(x)} = t^{-\alpha}$$

or similarly if F can be written as $\overline{F}(x) = L(x)x^{-\alpha}$ where $L(x)$ is "slowly varying" that is

$$\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1$$

Proposition 1.5 *$\mathcal{S} \subset \mathcal{L}$. In other words, all sub-exponential distributions are heavy-tailed.*

Proof. Let X be sup-exponential. By lemma 1.3, we need to prove that:

$$\limsup_{x \to \infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} \leq 1 \qquad \text{for } y \geq 0$$

For $y \leq x$, consider

$$\begin{aligned}
\mathbb{P}[X_1 + X_2 > x] &= \mathbb{P}[X_1 + X_2 > x I_{\{X_1 > x\}}] + \mathbb{P}[X_1 + X_2 > x I_{\{X_1 \leq y\}}] + \mathbb{P}[X_1 + X_2 > x I_{\{y < X_1 \leq x\}}] \\
&= \mathbb{P}[X_1 > x] + \int_0^y \overline{F}(x-u) dF(u) + \int_y^x \overline{F}(x-u) dF(u) \\
\frac{\mathbb{P}[X_1 + X_2 > x]}{\mathbb{P}[X_1 > x]} &= 1 + \int_0^y \frac{\overline{F}(x-u)}{\overline{F}(x)} dF(u) + \int_y^x \frac{\overline{F}(x-u)}{\overline{F}(x)} dF(u) \\
&\geq 1 + F(y) + \frac{\overline{F}(x-u)}{\overline{F}(x)} \left[F(x) - F(y) \right]
\end{aligned}$$

This implies that

$$\frac{\overline{F}(x-y)}{\overline{F}(x)} \leq \left[\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} - 1 - F(y) \right] \frac{1}{F(x) - F(y)}$$

Taking limits: $\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} \to 2$, and

$$\limsup_{x \to \infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} \leq 1.$$

$\square$

In general if $X, Y \in \mathcal{S}$ then it is not necessarily true that $X + Y \in \mathcal{S}$. The following propositions shed some lights on this subject.

Proposition 1.6 *If $F \in \mathcal{S}$ and G is "lighter" than F, meaning that:*

$$\lim_{x \to \infty} \frac{\overline{G}(x)}{\overline{F}(x)} = 0$$

then $F * G \in \mathcal{S}$ and $\overline{F * G}(x) \sim \overline{F}(x)$.

Proposition 1.7 (Equivalent tails) *If $F \in \mathcal{S}$ and G is any distribution such that*

$$\lim_{x \to \infty} \frac{\overline{F}(x)}{\overline{G}(x)} = c \qquad \text{or equivalently,} \quad \overline{F}(x) \sim c\overline{G}(x)$$

then $G \in \mathcal{S}$, $F * G \in \mathcal{S}$ and $\overline{F * G}(x) \sim (1 + c)\overline{F}(x)$.

We say that F and G have "equivalent tails".

For a sequence of i.i.d. sub-exponential random variables, we know that

$$\begin{aligned} \mathbb{P}[X_1 + ... + X_n > x] &\sim n\overline{F}(x) \\ \mathbb{P}[\max\{X_1, ..., X_n\} > x] &\sim n\overline{F}(x) \end{aligned}$$

From proposition 1.7, it follows that the sum and maximum of of i.i.d. sub-exponential random variables is sub-exponential.

Proposition 1.8 *Let $Y = \sum_{i=1}^{N} X_i$ where $X = (X_i, i \geq 1)$ is an i.i.d. sequence of random variants. N is a non-negative integer-valued random variable that is independent of X and such that $0 < \mathbb{E}[N] < \infty$.*

- *if $F \in \mathcal{S}$ and $\mathbb{E}[e^{\epsilon N}] < \infty$ for some $\epsilon > 0$ then,*

  $$\mathbb{P}[Y > x] \sim \mathbb{E}[N]\overline{F}(x).$$

  Also, by proposition 1.7, it also follows that $Y \in \mathcal{S}$.

- *Conversely, if $\mathbb{P}[Y > x] \sim \mathbb{E}[N]\overline{F}(x)$ and there exists $n \geq 2$ such that $\mathbb{P}[N = n] > 0$ then $F \in \mathcal{S}$ and $Y \in \mathcal{S}$.*

Remark 1.4 Note that $F \in \mathcal{S}$ does not imply that the equilibrium distribution F_e defined by

$$F_e(x) = \mu \int_0^x \overline{F}(u)du = \mu\mathbb{E}[X - x]^+ \qquad \text{for } \mathbb{E}[X] = \frac{1}{\mu} < \infty$$

is also sub-exponential. Even though F is light-tailed if and only if F_e is light-tailed.

1.2.1 Application: $G/G/1$ queue

Consider a $G/G/1$ queuing system with sub-exponential service time. Let $(U_n : n \geq 1)$ be an i.i.d. sequence of inter-arrival times independent of $(V_n : n \geq 1)$, the i.i.d. sequence of service times. By letting $X_{n+1} = V_n - U_{n+1}$, we obtain the waiting time of the n^{th} customer through the Lindley relation:

$$W_{n+1} = [W_n + X_{n+1}]^+.$$

Now, consider a queueing system for which the distribution of its service time, $V_1 \in \mathcal{S}$. We still have

$$\begin{aligned} W_n &\stackrel{D}{=} \max_{1 \leq k \leq n} S_k \\ &\Rightarrow \max_{n \geq 0} S_n \end{aligned}$$

as $n \to \infty$. In this context the Cramér-Lundberg approximation cannot be applied to estimate $\mathbb{P}[W_\infty > x]$. As we argued in a previous lecture such approximation holds when the moment generating function is finite and more specifically, when there exists $\theta^* > 0$ such that $\mathbb{E}[e^{\theta^* X}] = 1$. However we have the following results when service times are sub-exponential[1].

Theorem 1.9

(i) *If V_1 is sub-exponential, then*

$$\mathbb{P}[W_\infty > x] \sim \frac{\rho}{1-\rho}\mathbb{P}[V_1 > x] \tag{2}$$

where $\rho = \mathbb{E}V_1/\mathbb{E}U_1 < 1$. (In particular W is sub-exponential since it is tail equivalent to V_1.)

[1]Obtaining exact asymptotics when the moment generating function exists around the origin but when such θ^* does not exist is still an open question.

(ii) *If the arrival process is a homogeneous Poisson process at rate* λ *($M/G/1$ case), then* W *is sub-exponential if and only if* V_1 *is sub-exponential if and only if (2) holds.*

This heavy-tailed asymptotic for the steady-state waiting time distribution contrasts sharply with the classical light-tailed (Cramér-Lundberg approximation.)