# Econometrics I

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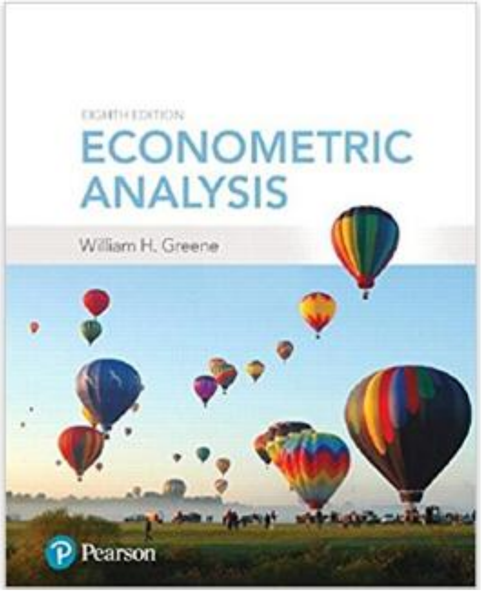

# Econometrics I

## Part 3 – Least Squares Algebra

# Vocabulary

- **Some terms** to be used in the discussion.
  - Population characteristics and entities vs. sample quantities and analogs
  - Residuals and disturbances
  - Population regression line and sample regression
- **Objective**: Learn about the conditional mean function. 'Estimate' $\boldsymbol{\beta}$ and $\sigma^2$
- **First step:** Mechanics of fitting a line (hyperplane) to a set of data

# Fitting Criteria

- The set of points in the sample
- Fitting criteria - what are they:
  - LAD: Minimize$_{\mathbf{b}}$ $\Sigma$ $|y - \mathbf{x'b}_{LAD}|$
  - Least squares: Minimize$_{\mathbf{b}}$ $\Sigma$ $(y - \mathbf{x'b}_{LS})^2$
  - and so on
- **Why least squares**?

A fundamental result:

***Sample moments are "good" estimators of their population counterparts***

We will examine this principle and apply it to least squares computation.

# An Analogy Principle for Estimating β

In the population $E[\mathbf{y} \mid \mathbf{X}] = \mathbf{X}\beta$ so

$E[\mathbf{y} - \mathbf{X}\beta \mid \mathbf{X}] = \mathbf{0}$

Continuing (assumed) $E[\mathbf{x}_i \varepsilon_i] = \mathbf{0}$ for every i

Summing, $\Sigma_i E[\mathbf{x}_i \varepsilon_i] = \Sigma_i \mathbf{0} = \mathbf{0}$

Exchange $\Sigma_i$ and E[] $E[\Sigma_i \mathbf{x}_i \varepsilon_i] = E[\mathbf{X}'\varepsilon] = \mathbf{0}$

$E[\mathbf{X}'(\mathbf{y} - \mathbf{X}\beta)] = \mathbf{0}$

So, if $\mathbf{X}\beta$ is the conditional mean, then $E[\mathbf{X}'\varepsilon] = \mathbf{0}$.

We choose **b**, the estimator of β, to mimic this population result: i.e., mimic the population mean with the sample mean

Find **b** such that $$\frac{1}{n}\mathbf{X}'\mathbf{e} = \mathbf{0} = \frac{1}{n}\mathbf{X}'(\mathbf{y} - \mathbf{X}\mathbf{b})$$

As we will see, the solution is the least squares coefficient vector.

# Population Moments

We assumed that $E[\varepsilon_i|\mathbf{x}_i] = 0$. (Slide 2:40)

It follows that $\text{Cov}[\mathbf{x}_i,\varepsilon_i] = \mathbf{0}$.

Proof: $\text{Cov}(\mathbf{x}_i,\varepsilon_i) = \text{Cov}(\mathbf{x}_i,E[\varepsilon_i|\mathbf{x}_i]) = \text{Cov}(\mathbf{x}_i,0) = \mathbf{0}$.

(Theorem B.2). If $E[y_i|\mathbf{x}_i] = \mathbf{x}_i'\boldsymbol{\beta}$, then

$$\boldsymbol{\beta} = (\text{Var}[\mathbf{x}_i])^{-1}\,\text{Cov}[\mathbf{x}_i,y_i].$$

Proof: $\text{Cov}[\mathbf{x}_i,y_i] = \text{Cov}[\mathbf{x}_i,E[y_i|\mathbf{x}_i]]=\text{Cov}[\mathbf{x}_i,\mathbf{x}_i'\boldsymbol{\beta}]$

This will provide a population analog to the statistics we compute with the data.

# U.S. Gasoline Market, 1960-1995

| G | CONST | PG | Y |
|---|---|---|---|
| 129.7 | 1 | 0.925 | 6036 |
| 131.3 | 1 | 0.914 | 6113 |
| 137.1 | 1 | 0.919 | 6271 |
| 141.6 | 1 | 0.918 | 6378 |
| 148.8 | 1 | 0.914 | 6727 |
| 155.9 | 1 | 0.949 | 7027 |
| 164.9 | 1 | 0.97 | 7280 |
| 171 | 1 | 1 | 7513 |
| 183.4 | 1 | 1.014 | 7728 |
| 195.8 | 1 | 1.047 | 7891 |
| 207.4 | 1 | 1.056 | 8134 |
| 218.3 | 1 | 1.063 | 8322 |
| 226.8 | 1 | 1.076 | 8562 |
| 237.9 | 1 | 1.181 | 9042 |
| 225.8 | 1 | 1.599 | 8867 |
| 232.4 | 1 | 1.708 | 8944 |
| 241.7 | 1 | 1.779 | 9175 |
| 249.2 | 1 | 1.882 | 9381 |
| 261.3 | 1 | 1.963 | 9735 |
| 248.9 | 1 | 2.656 | 9829 |
| 226.8 | 1 | 3.691 | 9722 |
| 225.6 | 1 | 4.109 | 9769 |
| 228.8 | 1 | 3.894 | 9725 |
| 239.6 | 1 | 3.764 | 9930 |
| 244.7 | 1 | 3.707 | 10421 |
| 245.8 | 1 | 3.738 | 10563 |
| 269.4 | 1 | 2.921 | 10780 |
| 276.8 | 1 | 3.038 | 10859 |
| 279.9 | 1 | 3.065 | 11186 |
| 284.1 | 1 | 3.353 | 11300 |
| 282 | 1 | 3.834 | 11389 |
| 271.8 | 1 | 3.766 | 11272 |
| 280.2 | 1 | 3.751 | 11466 |
| 286.7 | 1 | 3.713 | 11476 |
| 290.2 | 1 | 3.732 | 11636 |
| 297.8 | 1 | 3.789 | 11934 |

# Least Squares

- Example will be, $G_i$ regressed on

$$\mathbf{x}_i = [1, PG_i, Y_i]$$

- Fitting criterion: Fitted equation will be

$$y_i = b_1 x_{i1} + b_2 x_{i2} + \ldots + b_K x_{iK}.$$

- Criterion is based on residuals:

$$e_i = y_i - b_1 x_{i1} + b_2 x_{i2} + \ldots + b_K x_{iK}$$

Make $e_i$ as small as possible.

Form a criterion and minimize it.

# Fitting Criteria

- Sum of residuals: $\sum_{i=1}^{n} e_i$
- Sum of squares: $\sum_{i=1}^{n} e_i^2$
- Sum of absolute values of residuals: $\sum_{i=1}^{n} |e_i|$
- Absolute value of sum of residuals $\left|\sum_{i=1}^{n} e_i\right|$
- We focus on $\sum_{i=1}^{n} e_i^2$ now and $\sum_{i=1}^{n} |e_i|$ later

# Least Squares Algebra

$$\sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \mathbf{x}_i'\mathbf{b})^2 \;=\; \mathbf{e}'\mathbf{e} = (\mathbf{y} - \mathbf{X}\mathbf{b})'(\mathbf{y} - \mathbf{X}\mathbf{b})$$

Matrix and vector derivatives.

- Derivative of a scalar with respect to a vector
- Derivative of a column vector wrt a row vector
- Other derivatives

# Least Squares Normal Equations

$$\frac{\partial \sum_{i=1}^{n} e_i^2}{\partial \mathbf{b}} = \frac{\partial \sum_{i=1}^{n} (y_i - \mathbf{x}_i'\mathbf{b})^2}{\partial \mathbf{b}} = \sum_{i=1}^{n} \frac{\partial (y_i - \mathbf{x}_i'\mathbf{b})^2}{\partial \mathbf{b}}$$

$$= \sum_{i=1}^{n} 2(y_i - \mathbf{x}_i'\mathbf{b})(-\mathbf{x}_i) = -2\sum_{i=1}^{n} \mathbf{x}_i y_i + 2\sum_{i=1}^{n} \mathbf{x}_i\mathbf{x}_i'\mathbf{b}$$

$$= -2\left(\sum_{i=1}^{n} \mathbf{x}_i y_i\right) + 2\left(\sum_{i=1}^{n} \mathbf{x}_i\mathbf{x}_i'\right)\mathbf{b}$$

$$= -2\mathbf{X'y} + 2\mathbf{X'Xb}$$

# Least Squares Normal Equations

$$\frac{\partial(\mathbf{y}-\mathbf{Xb})'(\mathbf{y}-\mathbf{Xb})}{\partial\mathbf{b}} = -2\mathbf{X}'(\mathbf{y}-\mathbf{Xb}) \quad \mathbf{=} \; \mathbf{0}$$

$$\partial(1\times1)/\partial(K\times1) \qquad (-2)(n\times K)'(n\times1)$$

$$\mathbf{=} \; (-2)(K\times n)(n\times1) = K\times1$$

Note: Derivative of $(1\times1)$ wrt $K\times1$ vector is a $K\times1$ vector.

Solution: $-2\mathbf{X}'(\mathbf{y}-\mathbf{Xb}) = \mathbf{0} \Rightarrow \mathbf{X}'\mathbf{y} = \mathbf{X}'\mathbf{Xb}$

# Least Squares Solution

Assuming it exists: $\mathbf{b} = (\mathbf{X'X})^{-1}\mathbf{X'y}$

Note the analogy: $\boldsymbol{\beta} = \left(\text{Var}(\mathbf{x})\right)^{-1}\left(\text{Cov}(\mathbf{x},\text{y})\right)$

$$\mathbf{b} = \left(\frac{1}{n}\mathbf{X'X}\right)^{-1}\left(\frac{1}{n}\mathbf{X'y}\right) = \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\text{y}_i\right)$$

Suggests something desirable about least squares

# Second Order Conditions

**Necessary Condition :** First derivatives = $\mathbf{0}$

$$\frac{\partial(\mathbf{y}-\mathbf{Xb})'(\mathbf{y}-\mathbf{Xb})}{\partial\mathbf{b}} = -2\mathbf{X}'(\mathbf{y}-\mathbf{Xb})$$

**Sufficient Condition :** Second derivatives ...

$$\frac{\partial^2(\mathbf{y}-\mathbf{Xb})'(\mathbf{y}-\mathbf{Xb})}{\partial\mathbf{b}\partial\mathbf{b}'} = \frac{\partial\left(\frac{\partial(\mathbf{y}-\mathbf{Xb})'(\mathbf{y}-\mathbf{Xb})}{\partial\mathbf{b}}\right)}{\partial\mathbf{b}'}$$

$$= \frac{\partial\left\{-2\mathbf{X}'(\mathbf{y}-\mathbf{Xb})\right\}}{\partial\mathbf{b}'} = \frac{\partial\left(-2\mathbf{X}'\mathbf{y}\right)}{\partial\mathbf{b}'} + \frac{\partial\left\{-2\mathbf{X}'\left(-\mathbf{Xb}\right)\right\}}{\partial\mathbf{b}'} = \mathbf{0} + \frac{\partial 2\mathbf{X}'\mathbf{Xb}}{\partial\mathbf{b}'}$$

$$= \frac{\partial\ \text{K}\times 1 \text{ column vector}}{\partial\ 1\times\text{K row vector}} = \text{K}\times\text{K matrix}$$

$$= 2\mathbf{X}'\mathbf{X}$$

# Side Result: Sample Moments

$$\mathbf{X'X} = \begin{bmatrix} \Sigma_{i=1}^{n} x_{i1}^2 & \Sigma_{i=1}^{n} x_{i1}x_{i2} & \dots & \Sigma_{i=1}^{n} x_{i1}x_{iK} \\ \Sigma_{i=1}^{n} x_{i2}x_{i1} & \Sigma_{i=1}^{n} x_{i2}^2 & \dots & \Sigma_{i=1}^{n} x_{i2}x_{iK} \\ \dots & \dots & \dots & \dots \\ \Sigma_{i=1}^{n} x_{iK}x_{i1} & \Sigma_{i=1}^{n} x_{iK}x_{i2} & \dots & \Sigma_{i=1}^{n} x_{iK}^2 \end{bmatrix}$$

$$= \Sigma_{i=1}^{n} \begin{bmatrix} x_{i1}^2 & x_{i1}x_{i2} & \dots & x_{i1}x_{iK} \\ x_{i2}x_{i1} & x_{i2}^2 & \dots & x_{i2}x_{iK} \\ \dots & \dots & \dots & \dots \\ x_{iK}x_{i1} & x_{iK}x_{i2} & \dots & x_{iK}^2 \end{bmatrix}$$

$$= \Sigma_{i=1}^{n} \begin{bmatrix} x_{i1} \\ x_{i2} \\ \dots \\ x_{ik} \end{bmatrix} \begin{bmatrix} x_{i1} & x_{i2} & \dots & x_{iK} \end{bmatrix}$$

$$= \Sigma_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i'$$

# Does **b** Minimize **e'e**?

$$\frac{\partial^2 \mathbf{e'e}}{\partial \mathbf{b} \partial \mathbf{b'}} = 2\mathbf{X'X} = 2\begin{bmatrix} \Sigma_{i=1}^n x_{i1}^2 & \Sigma_{i=1}^n x_{i1}x_{i2} & \cdots & \Sigma_{i=1}^n x_{i1}x_{iK} \\ \Sigma_{i=1}^n x_{i2}x_{i1} & \Sigma_{i=1}^n x_{i2}^2 & \cdots & \Sigma_{i=1}^n x_{i2}x_{iK} \\ \cdots & \cdots & \cdots & \cdots \\ \Sigma_{i=1}^n x_{iK}x_{i1} & \Sigma_{i=1}^n x_{iK}x_{i2} & \cdots & \Sigma_{i=1}^n x_{iK}^2 \end{bmatrix}$$

If there were a single b, we would require this to be positive, which it would be; $2\mathbf{x'x} = 2\sum_{i=1}^n x_i^2 > 0$. OK

The matrix counterpart of a positive number is a **positive definite matrix**.

# A Positive Definite Matrix

Matrix **C** is positive definite if **a'Ca** is > 0 for every **a**.

Generally hard to check. Requires a look at characteristic roots (later in the course).

For some matrices, it is easy to verify. **X'X** is one of these.

$$\mathbf{a'X'Xa} = (\mathbf{a'X'})(\mathbf{Xa}) = (\mathbf{Xa})'(\mathbf{Xa}) = \mathbf{v'v} = \sum_{k=1}^{K} v_k^2 > 0$$

Could $\mathbf{v} = \mathbf{0}$? $\mathbf{v} = \mathbf{0}$ means $\mathbf{Xa} = \mathbf{0}$. Is this possible? No.

Conclusion: $\mathbf{b} = (\mathbf{X'X})^{-1}\mathbf{X'y}$ does indeed minimize $\mathbf{e'e}$.

# Algebraic Results - 1

In the population: $E[\mathbf{X}'\varepsilon] = \mathbf{0}$

In the sample: $\frac{1}{n}\sum_{i=1}^{n} \mathbf{x_i} e_i = \mathbf{0}$

$\mathbf{X}'\mathbf{e} = \mathbf{0}$ means for each column of $\mathbf{X}$, $\mathbf{x}_k'\mathbf{e} = 0$

(1) Each column of $\mathbf{X}$ is orthogonal to $\mathbf{e}$.

(2) One of the columns of $\mathbf{X}$ is a column of ones.

$\mathbf{i'e} = \sum_{i=1}^{n} e_i = 0$. The residuals sum to zero.

(3) It follows that $\frac{1}{n}\sum_{i=1}^{n} e_i = 0$ which mimics $E[\varepsilon_i] = 0$.

# Residuals vs. Disturbances

Disturbances (population) $y_i - \mathbf{x}_i'\beta = \varepsilon_i$

Partitioning **y**: $\mathbf{y} = E[\mathbf{y}|\mathbf{X}] + \boldsymbol{\varepsilon}$

= **conditional mean + disturbance**

Residuals (sample) $y_i - \mathbf{x}_i'\mathbf{b} = e_i$

Partitioning **y** : $\mathbf{y} = \mathbf{Xb} + \mathbf{e}$

= **projection + residual**

( **Note : Projection into the column space of X**, i.e., the set of linear combinations of the columns of **X. Xb** is one of these.)

# Algebraic Results - 2

- A "residual maker" $\mathbf{M} = (\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')$
- $\mathbf{e} = \mathbf{y} - \mathbf{Xb} = \mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \mathbf{My}$
- **My** = The residuals that result when **y** is regressed on **X**
- **MX = 0** **(This result is fundamental!)**

  How do we interpret this result in terms of residuals?

  When a column of **X** is regressed on all of **X**, we get a perfect fit and zero residuals.
- (Therefore) $\mathbf{My} = \mathbf{MXb} + \mathbf{Me} = \mathbf{Me} = \mathbf{e}$

  (You should be able to prove this.
- $\mathbf{y} = \mathbf{Py} + \mathbf{My}$, $\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = (\mathbf{I} - \mathbf{M})$.

  $\mathbf{PM} = \mathbf{MP} = \mathbf{0}$.
- **Py** is the projection of **y** into the column space of **X**.

# The **M** Matrix

- **M = I- X(X′X)$^{-1}$X′** is an nxn matrix
- **M** is **symmetric** – **M** = **M**′
- **M** is **idempotent** – **M*****M** = **M**

  (just multiply it out)
- **M** is **singular**; **M**$^{-1}$ does not exist.

  (We will prove this later as a side result in another derivation.)

# Results when **X** Contains a Constant Term

- $\mathbf{X} = [\mathbf{1},\mathbf{x}_2,\ldots,\mathbf{x}_K]$
- The first column of **X** is a column of ones
- Since $\mathbf{X'e} = \mathbf{0}$, $\mathbf{x}_1\mathbf{'e} = 0$ – the residuals sum to zero. $\mathbf{y} = \mathbf{Xb} + \mathbf{e}$

  Define $\mathbf{i} = [1,1,\ldots,1]' =$ a column of n ones

  $\mathbf{i'y} = \sum_{i=1}^{n} y_i = n\bar{y}$

  $\mathbf{i'y} = \mathbf{i'Xb} + \mathbf{i'e} = \mathbf{i'Xb}$ so $(1/n)\mathbf{i'y} = (1/n)\mathbf{i'Xb}$

  implies

  $\bar{y} = \bar{\mathbf{x}}'\mathbf{b}$ (the regression line passes through the means)

  These do not apply if the model has no constant term.

# U.S. Gasoline Market, 1960-1995

| G | CONST | PG | Y |
|---|---|---|---|
| 129.7 | 1 | 0.925 | 6036 |
| 131.3 | 1 | 0.914 | 6113 |
| 137.1 | 1 | 0.919 | 6271 |
| 141.6 | 1 | 0.918 | 6378 |
| 148.8 | 1 | 0.914 | 6727 |
| 155.9 | 1 | 0.949 | 7027 |
| 164.9 | 1 | 0.97 | 7280 |
| 171 | 1 | 1 | 7513 |
| 183.4 | 1 | 1.014 | 7728 |
| 195.8 | 1 | 1.047 | 7891 |
| 207.4 | 1 | 1.056 | 8134 |
| 218.3 | 1 | 1.063 | 8322 |
| 226.8 | 1 | 1.076 | 8562 |
| 237.9 | 1 | 1.181 | 9042 |
| 225.8 | 1 | 1.599 | 8867 |
| 232.4 | 1 | 1.708 | 8944 |
| 241.7 | 1 | 1.779 | 9175 |
| 249.2 | 1 | 1.882 | 9381 |
| 261.3 | 1 | 1.963 | 9735 |
| 248.9 | 1 | 2.656 | 9829 |
| 226.8 | 1 | 3.691 | 9722 |
| 225.6 | 1 | 4.109 | 9769 |
| 228.8 | 1 | 3.894 | 9725 |
| 239.6 | 1 | 3.764 | 9930 |
| 244.7 | 1 | 3.707 | 10421 |
| 245.8 | 1 | 3.738 | 10563 |
| 269.4 | 1 | 2.921 | 10780 |
| 276.8 | 1 | 3.038 | 10859 |
| 279.9 | 1 | 3.065 | 11186 |
| 284.1 | 1 | 3.353 | 11300 |
| 282 | 1 | 3.834 | 11389 |
| 271.8 | 1 | 3.766 | 11272 |
| 280.2 | 1 | 3.751 | 11466 |
| 286.7 | 1 | 3.713 | 11476 |
| 290.2 | 1 | 3.732 | 11636 |
| 297.8 | 1 | 3.789 | 11934 |

# Least Squares Algebra

y =

| |
|---|
| 129.7 |
| 131.3 |
| 137.1 |
| 141.6 |
| 148.8 |
| 155.9 |
| 164.9 |
| 171 |
| 183.4 |
| 195.8 |
| 207.4 |
| 218.3 |
| 226.8 |
| 237.9 |
| 225.8 |
| 232.4 |
| 241.7 |
| 249.2 |
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| 225.6 |
| 228.8 |
| 239.6 |
| 244.7 |
| 245.8 |
| 269.4 |
| 276.8 |
| 279.9 |
| 284.1 |
| 282 |
| 271.8 |
| 280.2 |
| 286.7 |
| 290.2 |
| 297.8 |

X =

| | | |
|---|---|---|
| 1 | 0.925 | 6036 |
| 1 | 0.914 | 6113 |
| 1 | 0.919 | 6271 |
| 1 | 0.918 | 6378 |
| 1 | 0.914 | 6727 |
| 1 | 0.949 | 7027 |
| 1 | 0.97 | 7280 |
| 1 | 1 | 7513 |
| 1 | 1.014 | 7728 |
| 1 | 1.047 | 7891 |
| 1 | 1.056 | 8134 |
| 1 | 1.063 | 8322 |
| 1 | 1.076 | 8562 |
| 1 | 1.181 | 9042 |
| 1 | 1.599 | 8867 |
| 1 | 1.708 | 8944 |
| 1 | 1.779 | 9175 |
| 1 | 1.882 | 9381 |
| 1 | 1.963 | 9735 |
| 1 | 2.656 | 9829 |
| 1 | 3.691 | 9722 |
| 1 | 4.109 | 9769 |
| 1 | 3.894 | 9725 |
| 1 | 3.764 | 9930 |
| 1 | 3.707 | 10421 |
| 1 | 3.738 | 10563 |
| 1 | 2.921 | 10780 |
| 1 | 3.038 | 10859 |
| 1 | 3.065 | 11186 |
| 1 | 3.353 | 11300 |
| 1 | 3.834 | 11389 |
| 1 | 3.766 | 11272 |
| 1 | 3.751 | 11466 |
| 1 | 3.713 | 11476 |
| 1 | 3.732 | 11636 |
| 1 | 3.789 | 11934 |

# Least Squares

**X′X**

| | | | |
|---|---|---|---|
| 1 | 36.0000 | 83.3980 | 332383.0 |
| 2 | 83.3980 | 248.0401 | 838669.3 |
| 3 | 332383.0 | 838669.3 | .3180537D+10 |

**X′y**

| | |
|---|---|
| 1 | 8139.4000 |
| 2 | 20561.74 |
| 3 | .7823519D+08 |

**(X′X)^-1**

| | | | |
|---|---|---|---|
| 1 | 1.6852 | .2662 | -.0002 |
| 2 | .2662 | .0792 | -.4870520D-04 |
| 3 | -.0002 | -.4870520D-04 | .3889722D-07 |

$$\mathbf{b} = \left( \square \; \square \right)^{-1} \times \square \; \square = \square = \begin{bmatrix} -79.7535 \\ -15.1224 \\ .0369 \end{bmatrix}$$

# Residuals

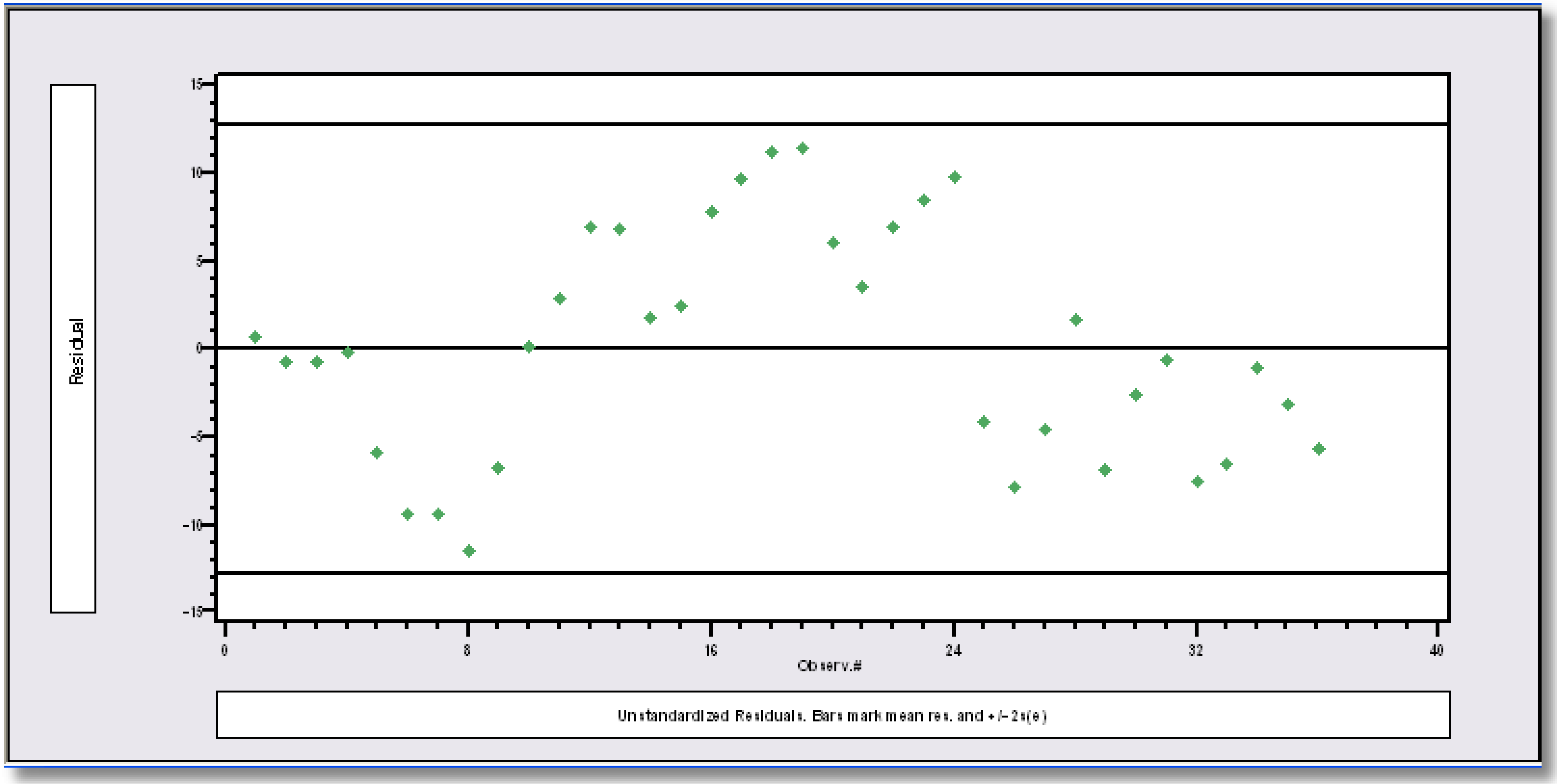
e = y − Xb =

| y | X: 1 | X: 2 | X: 3 | e |
|---|---|---|---|---|
| 129.7 | 1 | 0.925 | 6036 | 0.590391 |
| 131.3 | 1 | 0.914 | 6113 | -0.818823 |
| 137.1 | 1 | 0.919 | 6271 | -0.776628 |
| 141.6 | 1 | 0.918 | 6378 | -0.242229 |
| 148.8 | 1 | 0.914 | 6727 | -5.98792 |
| 155.9 | 1 | 0.949 | 7027 | -9.43475 |
| 164.9 | 1 | 0.97 | 7280 | -9.45803 |
| 171 | 1 | 1 | 7513 | -11.5068 |
| 183.4 | 1 | 1.014 | 7728 | -6.83297 |
| 195.8 | 1 | 1.047 | 7891 | 0.0480476 |
| 207.4 | 1 | 1.056 | 8134 | 2.8125 |
| 218.3 | 1 | 1.063 | 8322 | 6.87733 |
| 226.8 | 1 | 1.076 | 8562 | 6.71303 |
| 237.9 | 1 | 1.181 | 9042 | 1.67911 |
| 225.8 | 1 | 1.599 | 8867 | 2.36131 |
| 232.4 | 1 | 1.708 | 8944 | 7.76678 |
| 241.7 | 1 | 1.779 | 9175 | 9.61187 |
| 249.2 | 1 | 1.882 | 9381 | 11.0639 |
| 261.3 | 1 | 1.963 | 9735 | 11.319 |
| 248.9 | 1 | 2.656 | 9829 | 5.92825 |
| 226.8 | 1 | 3.691 | 9722 | 3.43036 |
| 225.6 | 1 | 4.109 | 9769 | 6.81625 |
| 228.8 | 1 | 3.894 | 9725 | 8.38944 |
| 239.6 | 1 | 3.764 | 9930 | 9.65486 |
| 244.7 | 1 | 3.707 | 10421 | -4.23501 |
| 245.8 | 1 | 3.738 | 10563 | -7.90891 |
| 269.4 | 1 | 2.921 | 10780 | -4.67559 |
| 276.8 | 1 | 3.038 | 10859 | 1.57702 |
| 279.9 | 1 | 3.065 | 11186 | -6.98764 |
| 284.1 | 1 | 3.353 | 11300 | -2.64132 |
| 282 | 1 | 3.834 | 11389 | -0.753384 |
| 271.8 | 1 | 3.766 | 11272 | -7.66202 |
| 280.2 | 1 | 3.751 | 11466 | -6.65141 |
| 286.7 | 1 | 3.713 | 11476 | -1.09526 |
| 290.2 | 1 | 3.732 | 11636 | -3.21519 |
| 297.8 | 1 | 3.789 | 11934 | -5.75549 |

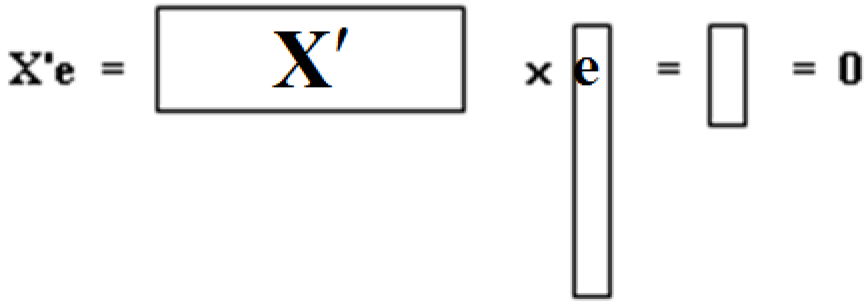
b =

| |
|---|
| -79.7535 |
| -15.1224 |
| .0369 |

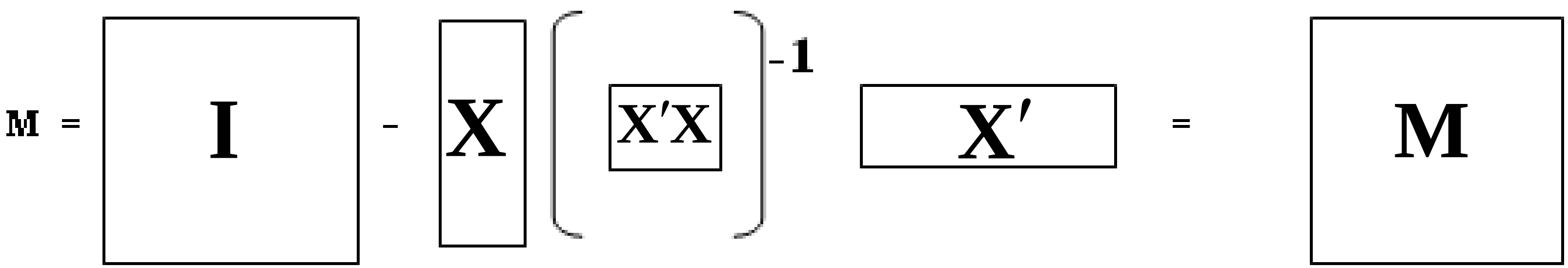
# Least Squares Residuals (autocorrelated)



# Least Squares Algebra-3




```
|-> matrix;list;x'e$
--------+--------------
 RESULT|             1
--------+--------------
       1|   -.210321E-11
       2|   -.475708E-11
       3|   -.194123E-07
```




**M is n × n potentially huge**

# Least Squares Algebra-4

**MX =**

Matrix - MX

[36, 3] Cell: 3.40151e-010

| | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 2.35922e-014 | 6.06459e-014 | 2.09639e-010 |
| 2 | 3.35287e-014 | 8.32667e-014 | 3.00815e-010 |
| 3 | 7.77156e-015 | 2.04559e-014 | 6.27551e-011 |
| 4 | 2.10942e-015 | 5.96745e-015 | 7.61702e-012 |
| 5 | 3.63043e-014 | 8.9373e-014 | 3.34808e-010 |
| 6 | 1.12688e-014 | 2.65066e-014 | 9.98739e-011 |
| 7 | 2.83662e-014 | 6.73905e-014 | 2.6381e-010 |
| 8 | 2.68535e-014 | 6.53089e-014 | 2.54417e-010 |
| 9 | 2.14967e-014 | 5.13235e-014 | 2.04778e-010 |
| 10 | 2.25098e-014 | 5.48901e-014 | 2.15817e-010 |
| 11 | 8.17402e-015 | 1.86101e-014 | 8.21814e-011 |
| 12 | 1.18239e-014 | 2.77348e-014 | 1.20366e-010 |
| 13 | 3.76088e-014 | 8.98864e-014 | 3.64821e-010 |
| 14 | 3.41394e-014 | 8.24063e-014 | 3.3674e-010 |
| 15 | 3.31124e-014 | 8.01442e-014 | 3.18664e-010 |
| 16 | 2.38976e-014 | 5.67046e-014 | 2.29704e-010 |
| 17 | 3.7248e-014 | 8.75133e-014 | 3.53964e-010 |
| 18 | 7.16094e-015 | 1.31006e-014 | 6.99174e-011 |
| 19 | 1.77636e-014 | 4.14668e-014 | 1.7792e-010 |
| 20 | 2.55143e-014 | 6.09235e-014 | 2.41414e-010 |
| 21 | 2.65898e-014 | 7.01522e-014 | 2.37378e-010 |
| 22 | 2.36478e-014 | 6.21864e-014 | 2.04267e-010 |
| 23 | 3.65263e-014 | 9.32449e-014 | 3.27248e-010 |
| 24 | 4.02456e-014 | 1.05402e-013 | 3.73177e-010 |
| 25 | 4.67126e-014 | 1.19155e-013 | 4.38263e-010 |
| 26 | 2.56739e-014 | 6.66689e-014 | 2.39311e-010 |
| 27 | 2.18159e-014 | 5.25691e-014 | 2.14072e-010 |
| 28 | 4.20775e-014 | 1.01974e-013 | 4.03702e-010 |
| 29 | 2.77556e-014 | 6.18949e-014 | 2.66596e-010 |
| 30 | 1.19904e-014 | 2.67009e-014 | 1.20508e-010 |
| 31 | 3.92186e-014 | 9.72555e-014 | 3.74598e-010 |
| 32 | 5.09176e-014 | 1.29396e-013 | 4.87944e-010 |
| 33 | 3.64014e-014 | 8.84848e-014 | 3.47654e-010 |
| 34 | 3.68316e-014 | 8.95395e-014 | 3.51747e-010 |
| 35 | 3.73868e-014 | 9.26481e-014 | 3.62661e-010 |
| 36 | 3.55271e-014 | 8.39329e-014 | 3.40151e-010 |