

# Bounded Rationality and Directed Cognition

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## Abstract

This paper proposes a psychological bounded rationality algorithm that uses partially myopic option value calculations to allocate scarce cognitive resources. The model can be operationalized even when decision problems require an arbitrarily large number of state variables. We evaluate the model using experimental data on a class of complex one-person games with full information. The model explains the experimental data better than the rational actor model with zero cognition costs.

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# 1 Introduction

Cognitive resources should be allocated just like other scarce resources. Consider a chess player who is choosing her next move. The player will cut corners when analyzing the game, ignoring certain lines of play, even if there is a chance that she will consequently overlook her best option. Similar search issues arise in almost every decision task. Decision time is scarce, implying that we have to choose how to allocate it.

Such search problems have been extensively studied since the work of Simon (1955) and Stigler (1961). The current paper proposes a new model of cognitively costly search. The model develops a middle ground between psychological models — like Simon’s satisficing framework — and purely rational models — like McCall’s (1965) search model.

Our model has three elements: a set of cognitive operations, a set of state variables that represent the current state of beliefs, and a value function. *Cognitive operations* include the different thought processes that agents use to deepen their understanding of a given problem. For example, consider the selection of an undergraduate college. Thinking more deeply about a *particular* college is a cognitive operation that will refine one’s estimate of the value of attending that college. Cognitive *state variables* encode the current state of the decision maker’s beliefs. In the college example, cognitive state variables represent the agent’s continuously updated expectations about the value of each of the colleges in her choice set. Cognitive state variables also represent the agent’s sense of how thoroughly each college has been considered (and how much remains to be learned). The *value function* is used to evaluate the expected benefit of additional cognitive operations. For example, should the decision maker invest additional time thinking about Boston University or Cornell? Thinking incrementally about Cornell may substantially deepen one’s knowledge about Cornell, but this information will not be useful if the decision maker has a strong preference for an urban lifestyle. There is little use thinking about an option that is unlikely to be ultimately chosen. The value function enables the decision maker to reason in such an option-theoretic way, directing thought in the most productive directions.

Our value function approach uses the standard dynamic programming toolbox, but our approach does not assume that the decision maker knows the optimal value function. Instead, we propose a modeling strategy that uses ‘proxy value functions’ that provide an approximation of the optimal value function. Just like optimal value functions, these proxy value functions encode knowledge states and evaluate the usefulness of potential cognitive operations. We argue that

proxy value functions have methodological advantages over optimal value functions because proxy value functions generate similar qualitative and quantitative predictions but are easier to work with. In contrast, optimal value functions cannot be calculated for realistically complex bounded rationality problems.

The paper models both the process of sequentially choosing cognitive operations as well as the ultimate decision to stop cognition (i.e., to stop thinking and make a final decision about what good to “buy”). We propose a three-step iterative rule. *First*, the algorithm evaluates the expected benefit of simple cognitive operations. The expected benefit of a specific cognitive operation is the difference between the expected value of stopping cognition immediately and the *ex ante* expected value of stopping cognition right after the cognitive operation is applied. This is a myopic calculation, since it does not incorporate the possible consequences of continuing with more cognition after the current cognitive operation is executed. *Second*, the algorithm executes the cognitive operation with the greatest expected benefit implied by this myopic rule. *Third*, the algorithm repeatedly cycles through these first two steps, stopping only when the cognitive costs of analysis outweigh the expected benefit of the best remaining cognitive operation.

We call this the directed cognition model. This paper shows how to operationalize this framework for complex decision problems with any number of state variables.

To illustrate our approach, we analyze a one-player decision problem. We then compare the results of this analysis to the decisions of experimental subjects. In this application, we use hundreds of continuous (cognitive) state variables. Nevertheless, such an application is easy to execute with our proxy value functions, though it can not be implemented with optimal value functions (unless decision costs are zero, in which case the optimal value function is trivial to solve). In our experiment, the data reject the rational model with zero cognition cost in favor of our bounded rationality alternative.

Our approach combines numerous strands in the existing literature on bounded rationality. Our conceptual approach extends the satisficing literature which was pioneered by Simon (1955) and subsequently formalized with models of deliberation costs by Conlisk (1996), Payne *et al.*(1993) and others. Our approach also extends the heuristics literature started by Kahneman and Tversky (1974) and Thaler (1991). Our framework is motivated by experimental and theoretical research (Camerer *et al.* 1993, Jehiel 1995) which argues that decision makers solve problems by looking forward, rather than using backward induction. Our emphasis on experimental evaluation is motivated by the work of Erev and Roth (1998), Camerer and Ho (1999), and Çelen and Kariv

(2004) who develop and then econometrically test learning models.

Our framework is also motivated by computer science research that uses techniques to break the “curse of dimensionality” in dynamic programming problems – for instance, by limiting the horizon of the analysis. Useful references to this literature are contained in Bertsekas and Tsitsiklis (1996), Do Val and Basar (1999) and Jehiel (2004). In our model agents optimize over the cognitive operators they choose.

Our work is also related to a new literature on attention allocation (Payne, Bettman, and Johnson 1993, Camerer *et al.* 1993, Costa-Gomes, Crawford, and Broseta 2001, Johnson *et al.* 2002). In recent years many authors have argued that the scarcity of attention has large effects on economic choices. For example, Gabaix and Laibson (2002) and Lynch (1996) study the effects of limited attention on consumption dynamics and the equity premium puzzle. Mankiw and Reis (2002), Sims (2003), and Woodford (2002) study the effects on monetary transmission mechanisms. D’Avolio and Nierenberg (2002), Della Vigna and Pollet (2003), and Peng and Xiong (2002) study the effects on asset pricing. Finally, Daniel, Hirshleifer, and Teoh (2002) and Hirshleifer, Lim, and Teoh (2003) study the effects on corporate finance, especially corporate disclosure. Some of these papers analyze limited attention from a theoretical perspective. Others analyze limited attention indirectly, by studying its associated market consequences. Our paper contributes to this literature by developing and experimentally testing a tractable model of attention allocation.

In section 2 we describe our theoretical approach and motivate our use of proxy value functions. In section 3 we introduce a class of one-player decision trees and describe the way in which our model applies to this class of problems. We also describe other algorithms that could be used to approximate human analysis of these decision trees. We then describe an experiment in which we asked undergraduates to solve these decision trees. In section 4 we compare the predictions of the models to the experimental results. In section 5 we conclude.

## **2 Models of decision making**

### **2.1 A simple setting**

Consumers routinely choose between multiple goods. We begin with the simplest version of such a decision problem. The consumer must choose either good  $B$  or good  $C$ . The value of the two goods is not immediately known, though careful thought will better reveal their respective values. For example, two detailed job offers may be hard to immediately compare. It will be necessary

to thoughtfully consider all of the features of the offers (e.g., health plan, pension benefits, vesting rules, maternity leave, sick days, etc.). After such consideration, the potential employee will be in a better position to compare the two options. Similar issues arise whenever a consumer chooses from a set of goods.

To analyze this problem, it is helpful to note that a decision between uncertain good  $B$  and uncertain good  $C$  is equivalent mathematically to a decision between a sure thing with payoff zero and a single uncertain good  $A$ , where  $A$ 's payoff is the difference between the payoffs of goods  $B$  and  $C$ . In this section, we work with the 0 vs.  $A$  problem because it simplifies notation.<sup>1</sup>

Let  $a_t$  represent the agent's expectations about the payoff of good  $A$  at time  $t$ . We assume that thinking about  $A$  leads to a change in the expectations,

$$da_t = \sigma(t)dz_t. \tag{1}$$

Here  $dz_t$  represents an independent<sup>2</sup> Brownian increment, which reflects knowledge gained from introspection, and  $\sigma(t)$  is the (time dependent) standard deviation of the noise. Finally, we assume that thinking generates instantaneous shadow cost  $q$ , which is either known to the decision maker (in a familiar setting) or is learned through experience. For example, if a cognitive operation takes  $\kappa$  units of time and the opportunity cost of time is  $\$w$ , then  $q$  would be  $\$\kappa w$ .

We can now formally state the agent's stopping problem. At every moment in time, the agent decides whether to continue analyzing her problem (paying shadow flow cost  $q$ ), or to stop and realize expected termination payoff  $\max\{a_t, 0\}$ : upon termination the agent chooses good  $A$  if its expected value exceeds 0, the value of the outside option. We consider applications in which decisions are made in the course of minutes or days, so that time discounting can be reasonably omitted from the model. If  $\tau$  represents the total time spent on the problem, so that  $\tau$  is the endogenous stopping time, then the agent's objective function is given by

$$E_0 [\max\{a_\tau, 0\} - q\tau]. \tag{2}$$

Note that the stopping time,  $\tau$ , will *not* be fixed in advance. Instead the stopping decision is continuously revised as the consumer learns more about the likely value of  $A$ . Finally, it is important to emphasize that the state variable in this problem is  $a$ , the agent's expectation about

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<sup>1</sup>The two problems are mathematically equivalent, but not necessarily psychologically equivalent.

<sup>2</sup>Rational expectations must be martingales, so the  $dz_t$  increments must be independent.

the state of the world. Throughout the analysis that follows, state variables will represent the state of knowledge of the decision maker.

It is also helpful to represent the value function that captures the expected continuation value of *any* policy function, including suboptimal policy functions. For any stopping rule, we can represent the expected continuation value of that stopping rule as  $V(t, a)$ , where  $t$  is the current time period and  $a$  is the agent's expectation about the state of the world.

For completeness, we can unify our two representations of the problem by noting that for any stopping rule,

$$V(t, a_t) = E_t [\max\{a_\tau, 0\} - q(\tau - t)].$$

Since the value function is constructed recursively, the value function at date  $t$  reflects the expected value of all future expected payoffs (i.e.,  $E[\max\{a_\tau, 0\}]$ ) and expected costs ( $E[q(\tau - t)]$ ).

## 2.2 Rational cognition

Suppose the decision maker executes the stopping rule that maximizes the value function characterized in Eqs. (1)-(2).<sup>3</sup> We solve this 'rational cognition' policy rule in Appendix A assuming either that  $\sigma^2(t)$  is constant or that  $\sigma^2(t)$  is affine and decreasing. Such dynamic programming analysis uses continuous-time Bellman equations and boundary conditions to solve for value functions and optimal policies.

The rational cognition approach (classical dynamic programming) has two clear advantages. It makes sharp predictions and it is highly parsimonious. However, as economists well know, it can be applied to only a limited set of practical situations. Generic variations of our decision problem are not analytically tractable.<sup>4</sup> For example, we would like to let the number of goods go from two to  $N$ . We would like to let  $\sigma$  vary with the particular information that was recovered during previous searches. We would like to allow for the possibility that information about good  $i$  contains some information about good  $j$  (i.e., covariances in information revelation).

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<sup>3</sup>This assumption is potentially internally *inconsistent*. We assume that the solution of the policy rule is costless, but the evaluation of information inside the decision problem is not (having shadow cost  $q$ ). One can motivate this cost asymmetry by assuming that learning has revealed the optimal policy rule.

<sup>4</sup>A strand of the operations research literature is devoted to finding particular classes of problems whose solutions have a simple characterization. One of them uses Gittins indices (Gittins 1979), basically a means to decouple  $n$  choice problems. The Gittins solution applies only to setups with stationary environments (such as multi-arm bandit problems), and in particular does not apply (as stressed by Gittins) to finite horizon problems. See Weitzman (1979) and Roberts and Weitzman (1980) for related techniques that solve special cases of the divine cognition problem. We consider problems that do not fall within this class of special cases.

Many real-world problems have too many state variables to be amenable to standard dynamic programming methods. Thinking about  $N$  goods with  $M$  attributes with history dependent variances and covariances generates  $\frac{1}{2}MN(MN + 3)$  state variables. To make this more concrete, assume that a researcher is trying to model the decision process of a college student making a residential choice. The student picks among 13 dormitories and 4 off-campus apartments each of which has 8 attributes. The rational cognition model requires that the researcher solve a dynamic programming problem with 9452 continuous state variables. To solve such problems, economists could try to develop new methods to solve (very) high-dimension dynamic programming problems. We suggest that a different — computationally less taxing — approach should also be considered.

We motivate this alternative approach with two observations. First, as we have noted, the rational cognition framework is frequently not feasible. Second, even if it were feasible, the complexity of the analysis makes us worry that it might not represent real-world decision making.

### 2.3 Directed cognition

We suggest an alternative framework that can handle arbitrarily complex problems, but is nevertheless computationally tractable. Moreover, this alternative is parsimonious and makes quantitative predictions. We call this approach directed cognition. Specifically, we propose the use of bounded rationality models in which decision makers use proxy value functions that are not fully forward looking.<sup>5</sup> We make this approach precise below and characterize the proxy value functions. This approach is particularly apt for bounded rationality applications, since modeling cognitive processes necessarily requires a large number of state variables. Memories and beliefs are all state variables in decisions about cognition.

Reconsider the boundedly rational decision problem described above. The consumer picks a stopping time,  $\tau$ , to maximize

$$E [\max\{a_\tau, 0\} - q\tau] \tag{3}$$

where  $a_t = a_0 + \int_0^t \sigma(t) dz_t$ . This is typically a difficult problem to solve, as the stopping time is not deterministic. Instead, the stopping rule depends on  $a_t$ , a stochastic variable.

If the consumer had access to a proxy value function, say  $\hat{V}(a, t)$ , the consumer would continue thinking as long as she expected a net improvement in expected value. Formally, the agent will

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<sup>5</sup>Gabaix and Laibson (2005) proposes a microfoundation for these value functions.

continue thinking if

$$\max_{h \in \mathbf{R}^+} E_t \left[ \hat{V}(a_{t+h}, t+h) - \hat{V}(a_t, t) \right] - qh > 0. \quad (4)$$

We could assume that the agent continues thinking for (initially optimal) duration  $h^*$  before re-calculating a (newly optimal duration)  $h^*$ . Or we could assume that she continuously updates her evaluation of Eq. (4) and only stops thinking when  $h^* = 0$ . In this section, we make the latter assumption for tractability.

We now turn to the selection of the proxy value function. In this paper we consider the simplest possible candidate (for the choice between  $a$  and 0)

$$\hat{V}(a_t, t) = \max\{a_t, 0\}. \quad (5)$$

Eq. (5) implies that the decision maker holds partially myopic beliefs, since she fails to recognize option values *beyond* the current thinking horizon of  $h^*$ .

Note that this framework can be easily generalized to handle problems of arbitrary complexity. For example, if the agent were thinking about  $N$  goods, then her proxy value function would be,

$$\hat{V}(a_t^1, a_t^2, \dots, a_t^N, t) = \max\{a_t^1, a_t^2, \dots, a_t^N\}.$$

We now consider this general  $N$ -good problem and analyze the behavior that it predicts.

Suppose that thinking for interval  $h$  about good  $A^i$  transforms one's expectation from  $a_t^i$  to  $a_t^i + \eta(i, h, t)$ , where  $\eta(i, h, t)$  is a mean zero random variable. Let  $a_t^{-i} = \max_{j \neq i} a_t^j$ . Then the expression

$$E_t \left[ \hat{V}(a_t^1, \dots, a_t^i + \eta(i, h, t), \dots, a_t^N, t+h) - \hat{V}(a_t^1, a_t^2, \dots, a_t^N, t) \right] \quad (6)$$

can be rewritten as,

$$E \left[ \max(a_t^i + \eta(i, h, t), a_t^{-i}) \right] - \max(a_t^i, a_t^{-i}) = E \left[ (a_t^i + \eta(i, h, t) - a_t^{-i})^+ \right],$$

where  $x^+ = \max\{0, x\}$ . Let  $\sigma(i, h, t)$  represent the standard deviation of  $\eta(i, h, t)$ , and define  $u$  such that  $\eta(i, h, t) = \sigma(i, h, t)u$ . Then the cognition decision reduces to

$$(i^*, h^*) \equiv \arg \max_{i, h} w(a_t^i - a_t^{-i}, \sigma(i, h, t)) - qh, \quad (7)$$



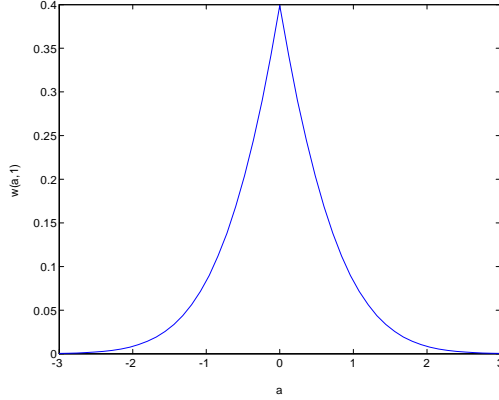


Figure 1: This Figure plots the expected benefit from continued search —  $w$ , defined in Eq. 11 — when the difference between the value of the searched alternative and its best alternative is  $a$ , and the standard deviation of the information gained is  $\sigma = 1$ . This  $w$  function is homogeneous of degree one.

where

$$v(a, \sigma) = E[(a + \sigma u)^+] \quad (8)$$

$$w(a, \sigma) = v(a, \sigma) - a^+. \quad (9)$$

When thinking generates Gaussian increments to knowledge  $u$ , we can write,

$$v(a, \sigma) = a\Phi\left(\frac{a}{\sigma}\right) + \sigma\phi\left(\frac{a}{\sigma}\right). \quad (10)$$

$$w(a, \sigma) = -|a|\Phi\left(-\frac{|a|}{\sigma}\right) + \sigma\phi\left(\frac{a}{\sigma}\right). \quad (11)$$

where  $\Phi$  and  $\phi = \Phi'$  are respectively the cumulative distribution and the density of a standard Gaussian. Figure 1 shows  $w$ .

Our boundedly rational agents won't necessarily use the “true” distribution of the information shocks  $u$ . Our agent may use simplified versions of it. Hence, we also consider the symmetric distribution with atoms at  $\pm 1$ . For this special case, which we denote with the superscript 0, we

get,

$$v^0(a, \sigma) = \begin{cases} \frac{1}{2}(a + \sigma) & \text{if } |a| \leq \sigma \\ a^+ & \text{if } |a| > \sigma \end{cases} \quad (12)$$

$$w^0(a, \sigma) = \frac{1}{2}(\sigma - |a|)^+. \quad (13)$$

The simple, linear form above makes  $w^0$  convenient to use.

Let us apply this to our Brownian problem with  $a_t = a_0 + \int_0^t \sigma(s) dz_s$ . Thinking for  $h$  more minutes will reveal an amount of normally distributed information with standard deviation

$$\sigma(t, h) = \left( \int_t^{t+h} \sigma^2(s) ds \right)^{1/2}.$$

The strategy in the directed cognition model is to continue thinking iff there is a  $h > 0$  such that

$$w(a_t, \sigma(t, h)) - qh > 0.$$

This leads to the policy “continue iff  $|a_t| < \bar{a}_t$ ”, where  $\bar{a}_t$  is implicitly defined by<sup>6</sup>

$$\sup_{h \geq 0} w(\bar{a}_t, \sigma(t, h)) - qh = 0. \quad (14)$$

For the standard case of the directed cognition model (i.e., normally distributed information innovations, so that  $w$  is defined in Eq. 11),  $\bar{a}_t$  is defined by

$$\sup_{h \geq 0} -\bar{a}_t \Phi \left( -\frac{\bar{a}_t}{\sigma(t, h)} \right) + \sigma \phi \left( \frac{\bar{a}_t}{\sigma(t, h)} \right) - qh = 0.$$

For the case with atoms of information at  $\pm 1$  ( $w^0$  is defined in Eq. 13)  $\bar{a}_t$  is defined by  $\sup_{h \geq 0} w^0(\bar{a}_t, \sigma(t, h)) - qh = 0$ , so that

$$\bar{a}_t = \sup_{h \geq 0} \sigma(t, h) - 2qh.$$

This contrasts with the policy of rational cognition, which maximizes (3). Finding the threshold policy is typically much more complex and requires backward induction with a free boundary (i.e., solving a differential equation with boundary conditions). The complexity of rational cognition

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<sup>6</sup>Limiting oneself to very small  $h$  would lead an agent to always stop if  $a_t \neq 0$ , as  $\lim_{h \rightarrow 0} w(a_t, \sigma(t, h)) / h - q = -q$  if  $a_t \neq 0$ . Hence, the decision maker is not so myopic that he can only look at the immediate future.

explodes when the number of state variables increases. This is the curse of dimensionality.

## 2.4 Comparing directed cognition to rational cognition

We can now compare directed cognition to rational cognition. We start with the special case in which  $\sigma^2(t) = \sigma^2$ , so  $\sigma^2(t, h) = h\sigma^2$ . We call this the constant variance case. Appendix A also provides a closed form solution for the case in which the variance falls linearly with time until the variance hits zero — the affine variance case.

**Proposition 1** *For the constant variance case the value function is independent of  $t$ , and the stopping rule is “stop iff  $|a_t| \geq \bar{a} = \alpha \frac{\sigma^2}{q}$ ”. For rational cognition  $\alpha = 1/4$ . For directed cognition with normally distributed innovations,  $\alpha$  is the solution of  $\max_{h \geq 0} w(\alpha, \sqrt{h}) - h = 0$  (so  $\alpha \simeq 0.1012$ ).<sup>7</sup> For the directed cognition model with atoms at  $\pm 1$ ,  $\alpha = 1/8$ .*

The proof is in Appendix A. This Proposition implies that rational cognition and directed cognition have the same comparative statics with respect to the variables of interest (the quantity of information learned by unit of time,  $\sigma^2$ , and the flow cost of thinking,  $q$ ). But directed cognition will stop thinking earlier, as its threshold is proportional to 0.1012, while rational cognition’s threshold is  $1/4$ .

The next Proposition characterizes the consequences of these policies. This Proposition presents the true (objective) payoff expectations generated by stopping rules indexed by  $\alpha$ . Recall that the stopping threshold is given by  $\bar{a} = \alpha \frac{\sigma^2}{q}$ . Also, note that time-independence of the decision process allows us to drop the  $t$ -subscripts from each of the expressions.

**Proposition 2** *Using the notation from Proposition 1, for both rational cognition and directed cognition, the expected consumption utility is:*

$$C(a) = E[a_\tau^+ | a_t = a] = \begin{cases} 0 & \text{if } a \leq -\bar{a} \\ \frac{1}{2}(\alpha\sigma^2 + aq) & \text{if } -\bar{a} < a < \bar{a} \\ a & \text{if } \bar{a} \leq a \end{cases} \quad (15)$$

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<sup>7</sup>There is a semi-explicit formula for  $\alpha$ . If  $\xi \simeq 0.6210$  is the solution of  $2\xi\Phi(-\xi) = \phi(\xi)$ , then an exact expression for  $\alpha \simeq 0.1012$  is  $\alpha = \xi\phi(\xi)/2$ .

the decision maker will, on average, keep thinking about the problem for an amount of time:

$$T(a) = E[\tau - t | a_t = a] = \begin{cases} 0 & \text{if } |a| \geq \bar{a} \\ -a^2/\sigma^2 + \alpha^2\sigma^2/q^2 & \text{if } |a| < \bar{a} \end{cases} \quad (16)$$

and the value function  $V(a) = C(a) - qT(a)$  is:

$$V(a) = E[a_\tau^+ - q(\tau - t) | a_t = a] = \begin{cases} 0 & \text{if } a \leq -\bar{a} \\ \alpha(\frac{1}{2} - \alpha)\frac{\sigma^2}{q} + \frac{a}{2} + \frac{q}{\sigma^2}a^2 & \text{if } -\bar{a} < a < \bar{a} \\ a & \text{if } \bar{a} \leq a \end{cases} \quad (17)$$

The proof is in Appendix A.

Eq. (17) — the value function — allows us to compare the economic performance of the algorithms. This value function depends on  $\alpha$ , which is sufficient to characterize the stopping rules of the algorithms (cf. Proposition 1).

Proposition 2 also decomposes payoffs into their two subcomponents: payoffs arising from consumption utility<sup>8</sup> —  $C(a) = E[a_\tau^+ | a]$  — and costs arising from cognition costs —  $qE[\tau - t | a]$ . The integrated measure of economic performance is the value function  $V = E[a_\tau^+ - q(\tau - t) | a]$ , which is the expected consumption utility net of cognition costs.

Proposition 2 enables us to plot (Figure 2) the value function under the three different cases: rational cognition (solid line,  $\alpha = 1/4$ ), directed cognition with atomic information ( $\alpha = 1/8$ , dashed line), and directed cognition with Gaussian information ( $\alpha = 0.1012$ , dot-dashed line). By definition, the value function generated by rational cognition lies above the other two (suboptimal) value functions. The three curves are compared in Figure 2.

This simple case allows us to study the relative performance of the algorithms in a case where we can calculate the values in simple closed forms. Appendix A derives closed form solutions for the affine variance case:  $\sigma^2(t) = A - Bt$ . We can also derive more general results. We turn to them now.

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<sup>8</sup>When the consumer stops thinking, at time  $\tau$ , he will pick the good iff  $a_\tau \geq 0$ , so his consumption utility will be  $a_\tau^+$ .

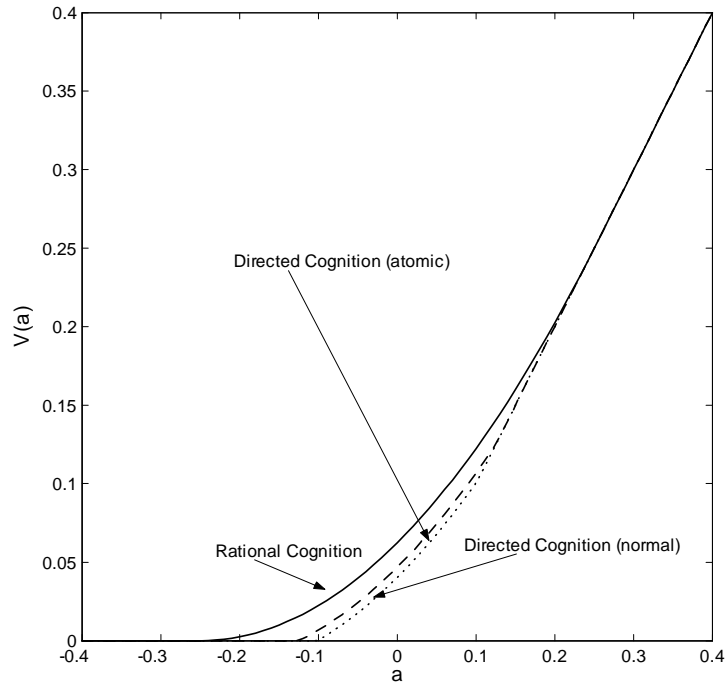


Figure 2: This Figure plots three objective value functions. We plot  $a$  on the horizontal axis and value functions on the vertical axis. We do this for three different policies: divine cognition (solid line,  $\alpha = 1/4$ ), directed cognition with atomic information ( $\alpha = 1/8$ , dashed line), and directed cognition with Gaussian information ( $\alpha = 0.1012$ , dotted line). The three curves are close to each other (in the sense that the maximum difference between the values is 0.022), which means that directed cognition has a performance close to that of divine cognition. The values come from Propositions 1 and 2.

## 2.5 Quantitative relationship between the value functions of divine and directed cognition

We can formalize the normative gap between divine and directed cognition by bounding the distance between their respective value functions. We analyze the leading case of monotonically declining variance functions.

The following Proposition bounds the gap between the true (objective) payoffs generated by the optimal stopping rule and the true (objective) payoffs generated by the directed cognition stopping rule. The Proposition is expressed in terms of “subjective value functions,” which can be shown to bound the objective value functions. Such subjective value functions represent the expected utility outcomes under the mistaken assumption that the current cognitive operation (with incremental stopping time  $\tau$ ) is the final cognitive operation. The subjective value functions are therefore based on static beliefs, which counterfactually assume that there will be no future cognitive operations. Such subjective value functions are relatively easy to calculate because they assume deterministic stopping times. By contrast, optimal value functions incorporate optimally chosen stopping times that depend on future realizations of random variables.

**Proposition 3** *Let  $V(x)$  represent the objective value function under the optimal policy (divine cognition). Let  $V^1$  represent the objective value function under the directed cognition policy. So  $V$  and  $V^1$  are the true expected payoffs associated with the optimal policy and the directed cognition policy. Suppose the variance function,  $\sigma(\cdot) = (\sigma(t))_{t \geq 0}$ , is deterministic and weakly decreasing.<sup>9</sup> Then,*

$$\begin{aligned} S^0\left(x, \sqrt{2/\pi}\sigma(\cdot)\right) &\leq S^1(x, \sigma(\cdot)) \leq V^1(x, \sigma(\cdot)) \leq V(x, \sigma(\cdot)) \leq S^2(x, \sigma(\cdot)) \leq \dots \\ &\dots \leq S^0(x, \sigma(\cdot)) + \min(|x|, S^0(0, \sigma(\cdot)))/2 \end{aligned} \quad (18)$$

where  $S^i$  is the subjective value function,

$$S^i(x, \sigma(\cdot)) = \sup_{h \in \mathbb{R}^+} v^i\left(x, \left(\int_0^h \sigma^2(s) ds\right)^{1/2}\right) - qh,$$

$v^0$  is defined in (12),  $v^1$  is defined in (10), and  $v^2(x, k) = \left(x + \sqrt{x^2 + k^2}\right)/2$ .

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<sup>9</sup>The nonincreasing  $\sigma$  implies that “high payoff” cognitive operations are done first. Hence, thinking displays decreasing returns over time.

Proposition 3 is proved in Appendix A.

Proposition 3 implies that the objective value functions for divine cognition ( $V$ ) and directed cognition ( $V^1$ ) are bounded below by  $S^1$  and above by  $S^2$ .  $S^1$  represents the subjective value function associated with Gaussian innovations.  $S^2$  represents the subjective value function associated with the density  $f^2(u) = [2(1 + u^2)^{3/2}]^{-1}$ .

It is useful to evaluate the bounds at the point  $x = 0$ , as this is where the option value  $V(x) - x^+$  is greatest.  $V(0, \sigma(\cdot))$  gives the order of magnitude of the option value.

**Proposition 4**

$$V\left(0, \sqrt{2/\pi}\sigma(\cdot)\right) \leq V^1(0, \sigma(\cdot)) \leq V(0, \sigma(\cdot)). \tag{19}$$

This Proposition gives us a way of comparing the directed cognition model (with objective value function  $V^1$ ) to the optimal policy function (with objective value function  $V$ ). The difference between  $V^1$  and  $V$  is bounded by the difference between the value function  $V$  generated by standard deviation  $\sqrt{2/\pi}\sigma \approx 0.8\sigma$  and the value function  $V$  generated by standard deviation  $\sigma$ . Thus, the directed cognition model performs as well as an optimal policy rule in a problem with payoffs scaled down by factor  $\sqrt{2/\pi} = 0.8$ .<sup>10</sup> This holds for one dimensional problems – results for cases with more dimensions are harder to derive.

Comparative statics analysis also implies that our directed cognition model is “close” to the divine cognition model. The subjective and objective value functions are all increasing in  $\sigma(\cdot)$  and decreasing in  $q$ . They also share other properties. All of the value functions tend to  $x^+$  as  $|x| \rightarrow \infty$ ; rise with  $x$ ; have slope 1/2 at 0; and are symmetric around 0 (e.g.,  $V(x) - x^+$  is even).

This section has described our directed cognition model. We were able to compare it to the divine cognition model for a special case in which the decision maker chooses between  $A$  and 0.<sup>11</sup> For this extremely simple example, the divine cognition approach is well-defined and solvable in practice. In most problems of interest, the divine cognition model will not be of any use, since economists can’t solve dynamic programming problems with a sufficiently large number of state variables. Below, we turn to an example of such a problem. Specifically, this problem involves the analysis of  $10 \times 10$  (or  $10 \times 5$ ) decision trees. Compared to real-world decision problems, such trees are not particularly complex. However, even simple decision trees like these generate hundreds of

<sup>10</sup>Note that this scaling argument only applies at  $a = 0$ . For comparisons that apply at all values of  $a$ , refer back to Proposition 3.

<sup>11</sup>All of the bounding results in this subsection assume that the decision maker selects among two options ( $n = 2$ ). We can show that the lower bounds that we derive generalize to the case  $n \geq 3$ . Obtaining analogous upper bounds for the general case remains an open question.

cognitive state variables. Divine cognition is useless here. But directed cognition, with its proxy value function, can be applied without difficulty. We now turn to this application.

### 3 An experimental implementation of the directed cognition model

#### 3.1 Decision trees

We use the directed cognition model to predict how experimental subjects will make choices in complex decision trees. We analyze decision trees, since trees can be used to represent a wide class of problems. We describe our decision trees at the beginning of this section, and then apply the model to the analysis of these trees.

Consider the decision tree in Figure 3, which is one of twelve randomly generated trees that we asked experimental subjects to analyze. Each starting box in the left-hand column leads probabilistically to boxes in the second column. Branches connect boxes and each branch is associated with a given probability.

For example, the first box in the last row contains a 5. From this box there are two branches with respective probabilities 0.65 and 0.35. The numbers inside the boxes represent flow payoffs. Starting from the box in the first column of the last row, there exist 7,776 possible outcome paths. For example, the outcome path that follows the highest probability branch at each node is  $\langle 5, 4, 4, 3, 1, -2, 5, 5, -3, 4 \rangle$ . Integrating with appropriate probability weights over all 7,776 paths, the expected payoff of starting from the first box in the last row is 4.12.

Formally, we define an *outcome path* as a feasible sequence of boxes that begins in column 1 and ends in the last (right-most) column. We define the *expected value of a starting row* as the probability-weighted value of all outcome paths that begin at a particular box in the left-most column of the tree.

We asked undergraduate subjects to choose one of the boxes in the first column of Figure 3. We told the subjects that they would be paid the expected value associated with whatever box they chose. What box would you choose?

To clarify exposition, we will refer to the final choice of a starting row as the “consumption choice.” This observable choice represents a subject’s response in the experiment. The subject



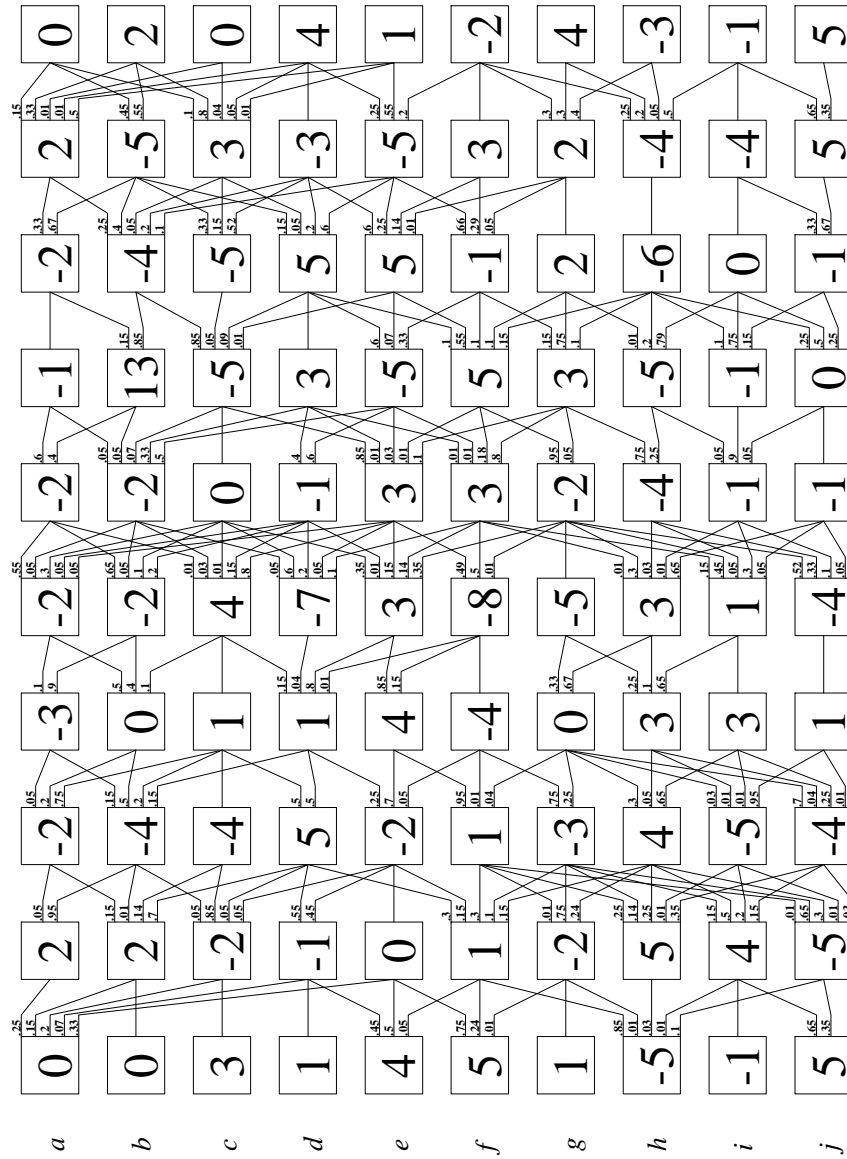


Figure 3: This Figure reproduces one of our 12 randomly-generated games. Each starting box in the left-hand column leads probabilistically to boxes in the second column. Branches connect boxes and each branch is associated with a given probability. For example, the first box in the last row contains a 5. From this box there are two branches with respective probabilities 0.65 and 0.35. The numbers inside the boxes represent flow payoffs. Starting from the last row, there exist 7,776 outcome paths. For example, the outcome path that follows the highest probability branch at each node is  $\langle 5, 4, 4, 3, 1, -2, 5, 5, -3, 4 \rangle$ . We asked undergraduate subjects to guess which of the boxes  $a, \dots, j$  in the first column of this Figure had the highest expected payoff. Which one would you pick?

also makes many “cognition choices” that guide her internal thought processes. We jointly model consumption and cognition choices, and then test the model with the observed consumption choices.

### 3.2 Application of the directed cognition model

The directed cognition model can be used to analyze trees like those in Figure 3. We describe such an application of the model in this subsection.<sup>12</sup>

Our application uses a basic cognitive operation: extending a partially examined path one column deeper into the tree. Such extensions enable the decision maker to improve her forecast of the expected value of a particular starting row.

For example, consider again the first box in the last row — row  $j$  — of Figure 3. Imagine that a decision maker is trying to estimate the expected value of starting in row  $j$ . Assume that she has already calculated the expected value of the two *truncated* paths leading from row  $j$  — column 1 — into column 2. Conditional on only that (limited) information, the expected value of row  $j$  is

$$a^j = 5 + (0.65)(4) + (0.35)(-5).$$

This conditional expectation equals the probability weighted value of the revealed nodes in columns 1 and 2.<sup>13</sup>

Following the highest probability path (i.e., the upper path with probability 0.65), the decision maker could look ahead one additional column and calculate a revised expected value:

$$a^{j'} = a^j + .65[.15(-3) + .5(4) + .2(-5) + .15(-4)].$$

Such path extensions are the basic cognitive operations that we consider.

Path extensions represent the reduced form cognitive operators in the model discussed in section 2. In that previous analysis we did not specify the nature of the cognitive operators, assuming only that the operators had shadow cost  $q$  and revealed information about the available choices. In the current application, the cognitive operators reveal structural information about the decision problem. Path extensions refine the decision maker’s expectations about the value of a starting row.

We will also allow *concatenated path extensions*. Specifically, if the decision maker executes a two-step path extension from a particular node, he follows the branch with the highest probability

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<sup>12</sup>Appendix B provides a formal description.

<sup>13</sup>For our application, the unconditional expected value of the nodes in columns 3-10 is zero.

for two steps deeper into the tree. Hence, starting from the last row of Figure 3, his updated estimate would be,

$$a^{j''} = a^{j'} + (.65)(.5)[.3(0) + .05(3) + .65(3)].$$

Such concatenated path extensions generalize naturally. An  $h$ -step path extension looks  $h$  columns more deeply into the tree. At each node along this path the extension follows the branch with highest probability.

A multi-column explored path originating from a node in column 1 can be extended in many ways, since an explored path may contain numerous nodes that have unexplored branches. Let  $f$  represent a feasible path extension;  $f$  embeds information about the starting row of the path that will be extended, the specific previously explored node from which the new extension will originate, and the number of steps of the new extension (e.g., starting at a previously explored node, proceed  $h$  columns deeper into the tree by starting with a particular branch originating at that node; for the remaining  $h - 1$  steps, always follow the branch with highest probability).

Let  $\sigma_f$  represent the standard deviation of the updated estimate resulting from application of  $f$ . Hence,

$$\sigma_f^2 = E(f(a^{i(f)}) - a^{i(f)})^2,$$

where  $i(f)$  represents the starting row which begins the path that  $f$  extends,  $a^{i(f)}$  represents the initial estimate of the value of row  $i(f)$ , and  $f(a^{i(f)})$  represents the updated value resulting from the path extension.

We now apply the model of section 2. The decision maker selects the path extension with the greatest expected benefit. Hence, the decision maker picks the cognitive operation,  $f$ , which is maximally useful:<sup>14</sup>

$$f^* \equiv \arg \max_f w(a^{i(f)} - a^{-i(f)}, \sigma_f) - q_f. \quad (20)$$

Note that  $q_f$  is the cognitive cost associated with cognitive operator  $f$ . Hence, if  $f$  is an  $h$ -step concatenated operation, then  $q_f = hq$ .<sup>15</sup> The decision maker keeps identifying and executing such

---

<sup>14</sup>Here  $w$  is the function in Eq. 11, which is associated with Gaussian innovations. Three arguments motivate this choice of  $w$ . First, it is simple to implement. Second, since an innovation is a sum of several shocks, the distribution of the innovation may be approximately Gaussian. Third, when decision makers do not know the distribution of innovations, they may use the general-purpose function  $w$  given by Eq. 11.

<sup>15</sup>This linear cost function is chosen for its simplicity. It is not clear whether the 'true' cost function is concave or convex in  $h$ .

cognitive operations until he concludes his search. Specifically, we implement the directed cognition algorithm with the following three-step iterative procedure.

**Step 1** Evaluate the expected gain of the available cognitive operations. The expected gain of a cognitive operation is the option value of changing one’s consumption choice as a result of knowledge generated by the cognitive operation. The expected gain of the best cognitive operation is given by:

$$G := \max_f w(a^{i(f)} - a^{-i(f)}, \sigma_f) - q_f. \quad (21)$$

For the decision tree application, the cognitive operations are path extensions.

**Step 2** Choose and execute the best<sup>16</sup> cognitive operation, i.e. select cognitive operation  $f^*$  such that

$$f^* = \arg \max_f w(a^{i(f)} - a^{-i(f)}, \sigma_f) - q_f. \quad (22)$$

Execute  $f^*$ , yielding updated information about the problem. For the decision tree application, the path extension with the highest expected gain is executed.

**Step 3** Keep identifying and executing cognitive operations — i.e., cycling back to step 1 — as long as the ex-ante marginal benefit of the previous cycle exceeded the fixed cost of that cycle, i.e. as long as  $G \geq F$ , where  $F$  is the fixed (cognitive) cost of step 1. When the fixed cost *exceeds* cognition benefits, stop thinking and probabilistically<sup>17</sup> choose to consume the starting row that is currently best given what is already known about the problem.

Before turning to the results, it is helpful to discuss several observations about the model.

First, Step 1 is computationally complex. We view Step 1 as an approximation of the less technical intuitive insights that decision makers use to identify cognitive operations that are likely to be useful. Even if Step 1 is intuition-based, these intuitions generate some cost. We assume that each pass through Step 1 has shadow cost  $F$ .

Second, if decision makers use their intuition to identify cognitive operations with high expected gains, they may sometimes make mistakes and fail to pick the operation with the highest expected gain. Hence, it may be appropriate to use a probabilistic model to predict which cognitive operation

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<sup>16</sup>In the case of ties, which are non-generic, apply a tie-breaking rule. In our application, we pick the row that is closest to the top of the matrix.

<sup>17</sup>Specifically, we use a logistic choice function in our implementation.

the decision maker will execute in Step 2. However, we overlook this issue in the current paper and assume that the decision maker always selects the cognitive operation with the highest expected gain.

Third, in the decision tree application we use rational expectations to calculate  $\sigma$  and  $a^j$ ,<sup>18</sup> but many other approaches are sensible. For example,  $\sigma^2$  could be estimated using past observations of  $(f(a^{i(f)}) - a^{i(f)})^2$ .

Fourth, we assume that Step 3 is backward looking. Specifically, the decision maker uses past information about the expected gains of cognitive operators to determine whether it is worth paying cost  $F$  to generate a new set of intuitions. No doubt this treatment could be improved in several ways. But, consistent with our modeling strategy, we choose the simplest approach that is broadly in accord with the psychology and the “cognitive economics” of the task.

Fifth, if Figure 3 contained a box with an extremely large payoff (e.g. 100), that box would no doubt draw the attention of subjects. To capture this effect, we could add another mental operator: global scanning. For example, “Scan the tree. Look for the box with the largest unexplored positive or negative payoff. Determine how much this box affects the estimated value of each row.” A rigorous definition for global scanning is easy to formulate though such a definition is notationally cumbersome. We did not include global scanning in our set of mental operators to keep the model as simple as possible, and also because the stochastic structure we use to generate decision trees does not generate outlier boxes.

To sum up, the directed cognition algorithm executes the path extension with the highest expected value net of cognition cost,  $q$ . The algorithm continues in this way — evaluating the expected value of path extensions and executing the most promising path extension — until no remaining path extension has a positive expected value net of cognition cost.

This procedure radically simplifies analysis of our decision trees. Each unsimplified tree has approximately 100,000 paths leading from the first column to the last column. Our directed cognition algorithm restricts attention to only a handful of these paths. For example, Figure 4 plots the path extensions generated by the directed cognition model for the game in Figure 3. Using option-theoretic considerations, the directed cognition algorithm restricts analysis to  $\frac{7}{100,000}$  of the possible paths in this tree. After these seven path extensions are examined, the algorithm

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<sup>18</sup>To justify the use of rational expectations in estimating  $\sigma$  and  $a^j$ , our experimental design allows the decision maker to scan the entire tree and form an estimate of the branching structure and the distribution of node values.

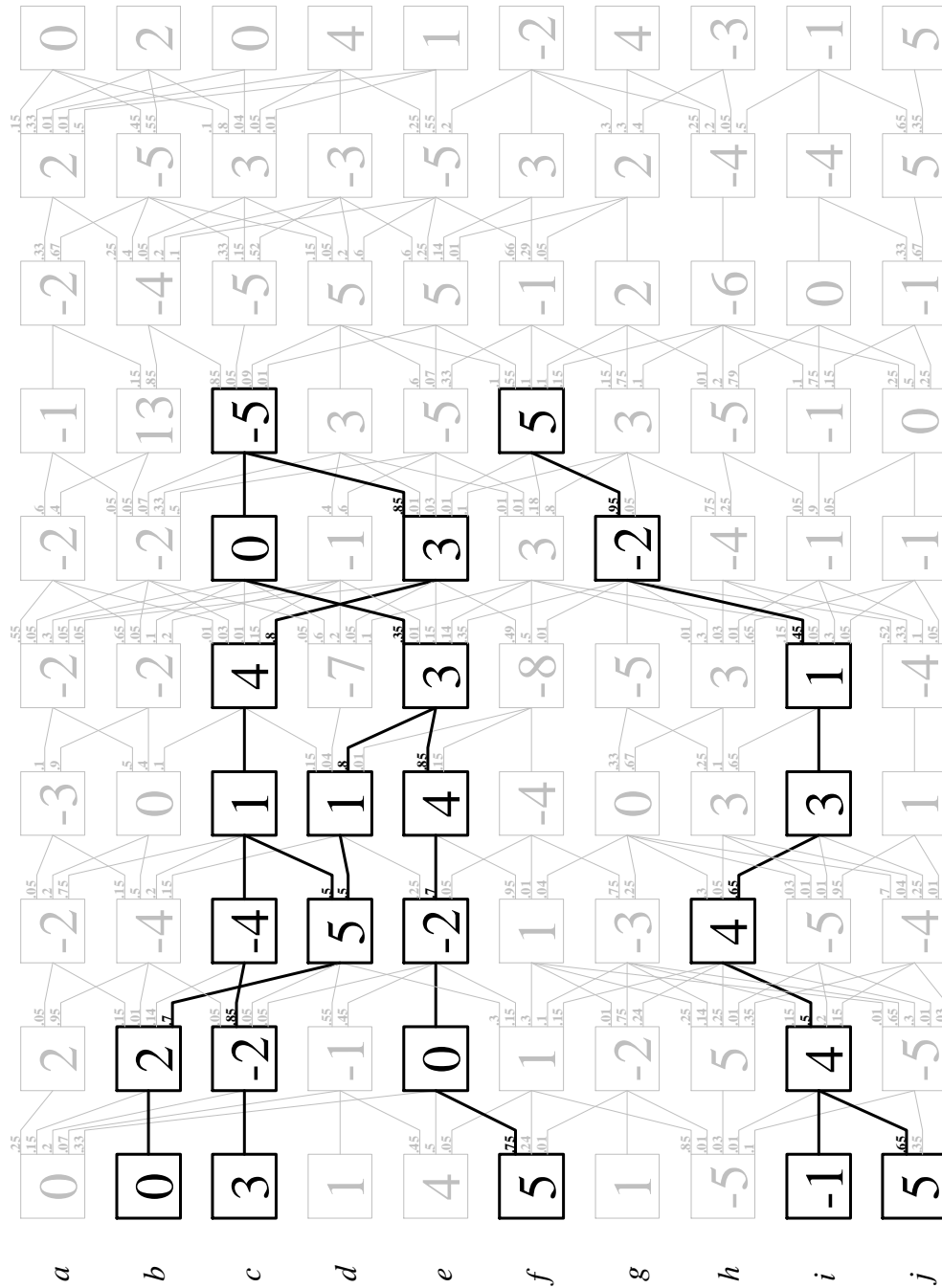


Figure 4: In the game presented in Figure 3 the directed cognition model looked at only 7 paths out of approximately 100,000 paths. This Figure highlights those 7 paths (including 1 path extension). The algorithm was very frugal. Table 1 compares the row choices implied by directed cognition to the row choices actually made by our subjects.

stops. Option value calculations enable the decision maker to simplify his problem by restricting attention to a tiny fraction of the available information.

### 3.3 Other decision algorithms

In the next section, we empirically compare the directed cognition model to the rational cognition model with zero cognition costs and three other choice models, which we call the column cutoff model, the discounting model, and the FTL model.

The rational cognition model with zero cognition cost is the standard assumption in economic models when information is freely available as it is in this experiment. For presentational simplicity, we refer to this benchmark as the *costless cognition model*, but we emphasize that this costless cognition model assumes both rationality and zero cognition costs. We do not study the rational cognition model with strictly positive costs, since this model is not computationally tractable for our (complex) problem. However, Gabaix et al. (2004) do experimentally analyze a simple (tractable) problem and study the predictions of the rational cognition model with strictly positive costs. Gabaix et al. (2004) find that the directed cognition model has far greater predictive power in this tractable setting.

The column cutoff model assumes that decision makers calculate perfectly (with zero cognition costs) but pay attention to only the first  $Q$  columns of the  $C$ -column tree, completely ignoring the remaining  $C - Q$  columns. The discounting model assumes that decision makers follow all paths, but exponentially discount payoffs according to the column in which those payoffs arise. Both of these algorithms are designed to capture the idea that decision makers ignore information that is relatively less likely to be useful. For example, the discounting model can be interpreted as a model in which the decision maker has a reduced probability of seeing payoffs in “later” columns.

We also consider a bounded rationality model that we called “follow the leaders” (FTL) in previous work (Gabaix and Laibson, 2000). This algorithm features a cutoff probability  $p$ , and can be described as:

>From any starting box  $b$  follow all branches that have a probability greater than or equal to  $p$ . Continue in this way, moving from left to right across the columns in the decision tree. If a branch has a probability less than  $p$ , consider the box to which the branch leads but do not advance beyond that box. Weight all boxes that you consider with their associated cumulative probabilities, and calculate the weighted sum of the

boxes.

This heuristic describes the FTL value of a starting box  $b$ . In our econometric analysis, we estimate  $p = 0.30$ ; costless cognition corresponds to  $p = 0$ .

More precisely, we take the average payoffs over trajectories starting from box  $b$ , with the transition probabilities indicated on the branches, but with the additional rule that the trajectory stops if the last transition probability has been  $< p$ .<sup>19</sup> For example, consider the following FTL ( $p = 0.60$ ) calculation for row  $b$  in Figure 3. FTL follows all paths that originate in row  $b$ , stopping only when the last branch of a path has a probability less than  $p$ .

$$\mathcal{E}^{FTL(p)}(b) = 0 + 2 + .15 \cdot (-2) + .01 \cdot (-4) + .14 \cdot (-4) + .7(5 + .5 \cdot 1 + .5 \cdot 1) = 5.3.$$

This FTL value can be compared to the costless cognition value of 5.78.

We consider FTL because it corresponds to some of our intuitions about how decision makers “see” decision trees that do not contain large outlier payoffs. Subjects follow paths that have high probability branches at every node. In the process of determining which branches to follow, subjects see low probability branches that branch off of high probability paths. Subjects encode such low probability branches, but subjects do not follow them deeper into the tree.

All three of the quasi-rational algorithms described in this section, (column cutoff, column discounting, and FTL) are likely to do a poor job when they are applied to a wider class of decision problems. None of these algorithms are based on a structural microfoundation (like cognition costs). The parameters in column cutoff, column discounting, and FTL are unlikely to generalize to other games. In addition, we do not believe that column cutoff, column discounting, and FTL are as psychologically plausible as the directed cognition model. Only the directed cognition model features the property of “thinking about when to stop thinking,” an important component of any decision task. For all of these reasons we do not propose column cutoff, column discounting, or FTL as general models of bounded rationality. However, they are interesting benchmarks with which to compare the directed cognition model.

Finally, we need to close *all* of our models by providing a theory that translates payoff evaluations into choices. Since subjects rarely agree on the “best choice,” it would be foolish to assume that subjects all choose the starting row with the highest evaluation (whether that evaluation is

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<sup>19</sup>Formally,  $\mathcal{E}^{FTL(p)}(b) = \sum_{i=1}^C \sum_{n_i \text{ in column } i} u_{n_i} \sum_{\substack{(n_j)_{j=2,\dots,i-1} \\ \text{in column } j}} p(n_i | n_{i-1}) \dots p(n_2 | b) \mathbf{1}_{p(n_{i-1} | n_{i-2}) \geq p} \dots \mathbf{1}_{p(n_2 | b) \geq p}$ , where  $C$  is the number of columns in the decision tree.



rational or quasi-rational). Instead, we assume that actions with the highest evaluations will be chosen with the greatest frequency. Formally, we choose a logit specification, which is the most common modeling device for these purposes. Specifically, assume that there are  $B$  possible choices with evaluations  $\{\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_B\}$  given by one of the candidate models. Then the probability of picking box  $b$  is given by:

$$P_b = \frac{\exp(\beta \mathcal{E}_b)}{\sum_{b'=1}^B \exp(\beta \mathcal{E}_{b'})}. \quad (23)$$

We estimate the nuisance parameter  $\beta$  in our econometric analysis.

### 3.4 The details of the experimental design

We tested the model using experimental data from 259 subjects recruited from the general Harvard undergraduate population.<sup>20</sup> Subjects were guaranteed a minimum payment of \$7 and could receive more if they performed well. They received an average payment of \$20.08.

#### 3.4.1 Decision Trees

The experiment was based around twelve randomly generated trees, one of which is reproduced in Figure 3. We chose large trees because we wanted undergraduates to use heuristics to “solve” these problems. Half of the trees have 10 rows and 5 columns of payoff boxes; the other half of the trees are  $10 \times 10$ . The branching structure which exits from each box was chosen randomly and is i.i.d. across boxes within each tree. The distributions that we used and the twelve resulting trees themselves are presented in an accompanying working paper. On average the twelve trees have approximately three exit branches per box (excluding the boxes in the last column of each tree which have no exit branches).

The payoffs in the trees were also chosen randomly and are i.i.d. across boxes.<sup>21</sup> The distribution is given below:

$$\text{individual box payoff} = \begin{cases} \text{uniform}[-5,5] & \text{probability .9} \\ \text{uniform}[-5,5] + \text{uniform}[-10,10] & \text{probability .1} \end{cases}$$

Payoffs were rounded to whole numbers and these rounded numbers were used in the payoff boxes.

<sup>20</sup>This experiment was briefly described in Gabaix and Laibson (2003).

<sup>21</sup>One caveat applies. The payoffs in the boxes in the first three columns were drawn from the uniform distribution with support  $[-5, 5]$ .

Experimental instructions (available on the authors' web pages) described the concept of expected value to the subjects. Subjects were told to choose one starting row from each of the twelve trees. Subjects were told that one of the trees would be randomly selected and that they would be paid the true expected value for the starting box that they chose in that tree. Subjects were given a maximal time of 40 minutes to read the experimental instructions and make all twelve of their choices. We analyzed data only from the 251 subjects who made a choice in all of the twelve decision trees.

### 3.4.2 Debriefing form and expected value calculations

After passing in their completed tree forms, subjects filled out a debriefing form which asked them to “describe how [they] approached the problem of finding the best starting boxes in the 12 games.” Subjects were also asked for background information (e.g., major and math/statistics/economics background). Subjects were then asked to make 14 expected value calculations in simple trees (three  $2 \times 2$  trees, three  $2 \times 3$  trees, and one  $2 \times 4$  tree). Subjects were told to estimate the expected value of each starting row in these trees. Figure 5 provides an example of one such tree. Subjects were told that they would be given a fixed reward ( $\approx \$1.25$ ) for every expected value that they calculated correctly (i.e., within 10% of the exact answer). We gave subjects these expected value questions because we wanted to identify any subjects who did not understand the concept of expected value. We exclude any subject from our econometric analysis who did not answer correctly at least half of the fourteen expected value questions. Out of the 251 subjects who provided an answer for all twelve decision trees, 230, or 92%, correctly answered at least half of the fourteen expected value questions. Out of this subpopulation, the median score was 12 out of 14 correct; 190 out of 230 answered at least one of the two expected values in Figure 5 correctly.

## 4 Experimental results

We restrict analysis to the 230 subjects who met our inclusion criteria<sup>22</sup>, but the results are almost identical when we include all 251 subjects who completed the decision trees.

Had subjects chosen randomly, they would have chosen starting boxes with an average payoff

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<sup>22</sup>They chose a row in all twelve decision trees and answered correctly at least half of the expected value questions on the post-experiment debriefing form.

Expected payoff for starting in box "k": \_\_\_\_\_  
 Reward of \$1.75 for answers within 10% of correct answer.

Expected payoff for starting in box "l": \_\_\_\_\_  
 Reward of \$1.75 for answers within 10% of correct answer.

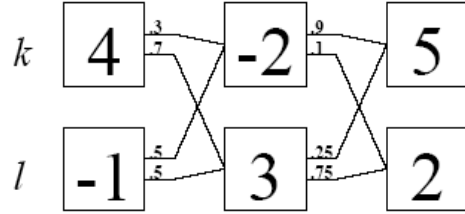


Figure 5: This Figure presents a representative game from the debriefing quiz. The answer for box  $k$  is:  $4 + .3(-2 + .9 \cdot 5 + .1 \cdot 2) + .7(3 + .25 \cdot 5 + .75 \cdot 2) = 8.835$ . These games allow us to test that subjects understand well the concept of expected value. Out of the 251 subjects who provided an answer for all twelve decision trees, 230, or 92%, correctly answered at least half of the fourteen expected value questions. Out of this subpopulation, the median score was 12 out of 14 correct.

of \$1.30.<sup>23</sup> Had the subjects chosen the starting boxes with the highest payoff, the average chosen payoff would have been \$9.74. In fact, subjects chose starting boxes with average payoff \$6.72.

We are interested in comparing five different classes of models: costless cognition, directed cognition, column cutoff, discounting, and FTL. All of these models have a nuisance parameter,  $\beta$ , which captures the tendency to pick starting boxes with the highest evaluations (see Eq. 23). For each model, we estimate a different  $\beta$  parameter. Directed cognition, column cutoff, discounting, and FTL also require an additional parameter which we estimate in our econometric analysis.

#### 4.1 The Euclidean distance $L^m$ between a model $m$ and the data

We call  $\hat{P}_{bg}$  the fraction of subjects playing game  $g$  who chose box  $b$ . Say that a model  $m$  (costless cognition, directed cognition, FTL etc.) assigns the value  $\mathcal{E}_{bg}^m$  to box  $b$  in game  $g$ . Following standard practice in the discrete choice literature, we parameterize the link between this value and the probability that a player will choose  $b$  in game  $g$  by

$$P_{bg}^m(\beta) = \exp(\beta \mathcal{E}_{bg}^m) / \sum_{b'=1}^B \exp(\beta \mathcal{E}_{b'g}^m) \quad (24)$$

for a given  $\beta$ . To determine  $\beta$ , we minimize the Euclidean distance  $\|P^m(\beta) - \hat{P}\|$  between the empirical distribution  $\hat{P}$  and the distribution predicted by the model,  $P^m(\beta)$ . If we optimize over

<sup>23</sup>In expectation this would have been zero since our games were randomly drawn from a distribution with zero mean. Our realized value is well within the two-standard error bands.

$\beta$ , we get the distance:

$$L^m(\hat{P}) = \min_{\beta \in \mathbb{R}^+} \|P^m(\beta) - \hat{P}\| \quad (25)$$

Those are the distances we report in Table 2.

## 4.2 Empirical results

For the directed cognition model we estimate  $q = 0.012$ , implying that the shadow price of extending a path one-column deeper into the tree is a little above 1 cent. We can translate this estimate into real time processing speed. Given a cost  $q$  and the directed cognition algorithm, people will make  $N_g(q)$  elementary operations in game  $g$  ( $g \in \{1, \dots, 12\}$ ). Say that people need  $\kappa$  seconds per elementary mental operation: for instance, extending a path 3 columns ahead and updating the point estimate of a box accordingly takes  $3\kappa$  seconds. Then the total time spent on the experiment is:  $T(q) = \kappa \sum_{g=1}^{12} N_g(q)$ . Given that our subjects had roughly 35 minutes (40 minutes total minus roughly 5 minutes to read the instructions), we can infer from our calibrated value of  $q = 0.012$  the value of  $\kappa = (35 \text{ minutes}) / \left( \sum_{g=1}^{12} N_g(q) \right) = 2.2$  seconds per box. We view both of these numbers as reasonable. We had chosen 1 cent per box and 1 second per box as our preferred values before we estimated these values.

For the column-cutoff model, we estimate  $Q = 8$ , implying that the decision maker evaluates only the first 8 columns of each tree (and hence all of the columns in  $10 \times 5$  trees). For the column-discounting model we estimate  $\delta = 0.91$ , implying that the probability that the decision maker sees a payoff in column  $c$  is  $(.91)^c$ . Finally, for FTL we estimate  $p = 0.30$ , implying that the decision maker follows branches until she encounters a branch probability which is less than .30. We had exogenously imposed a  $p$  value of 0.25 in our previous work (Gabaix and Laibson, 2000).

< **Insert Table 1 here** >

The first panel (Values) in Table 1 presents the predictions of our models for the sample game in Figure 3. “Values” plots the row values estimated by each of the models. Using these estimated values and a logistic transformation (24) generates choice probabilities, which are reported in the second panel of the Table.

Directed cognition estimated that row  $j$  has the highest value (the estimate is 7.19). Our subjects apparently reached the same conclusion since more of them chose row  $j$  (36.5%, see the empirical probabilities) than any other row. By contrast, row  $f$  actually has the highest value (6.88,

see the costless cognition values). Hence directed cognition failed to pick the optimal box. This is a desirable result, since directed cognition makes the same “mistake” as our subjects. As Figure 4 shows, directed cognition overlooks the fact that row  $j$  has many overlooked paths that lead to negative payoffs.

The sample game analyzed in Table 1 is just one of our 12 games. To gain a complete picture of our results we now turn to Table 2, which reports test statistics derived from the data from all 12 games. The rows of the Table represent the different models,  $m \in \{\text{costless cognition, directed cognition, column-cutoff, column-discounting, FTL}\}$ . Column 1 reports distance metric  $\widehat{L}^m$ , which measures the distance between the estimated model and the empirical data. A small distance  $\widehat{L}^m$  indicates a good fit between model and data. Columns 2-5 report the differences  $\widehat{L}^m - \widehat{L}^{m'}$ , and the associated asymptotic standard errors. When  $\widehat{L}^m - \widehat{L}^{m'} > 0$ , model  $m$  performs relatively poorly, since the distance of model  $m$  from the empirical data ( $\widehat{L}^m$ ) is greater than the distance of the model  $m'$  from the empirical data ( $\widehat{L}^{m'}$ ).

**<Insert Table 2 here.>**

As column 2 of Table 2 shows, all of the decision cost models beat costless cognition. The differences between the distance measures are:  $\widehat{L}^{\text{costless cognition}} - \widehat{L}^{\text{directed cognition}} = 0.399$ ,  $\widehat{L}^{\text{costless cognition}} - \widehat{L}^{\text{column cutoff}} = 0.221$ ,  $\widehat{L}^{\text{costless cognition}} - \widehat{L}^{\text{column discounting}} = 0.190$ , and  $\widehat{L}^{\text{costless cognition}} - \widehat{L}^{\text{FTL}} = 0.488$ . All of these differences have associated t-statistics of at least 5.

Among the decision cost models the two winners are directed cognition and FTL. Both of these models dominate column cutoff and column discounting. The difference between directed cognition and FTL is not statistically significant. The general success of the directed cognition model is somewhat surprising, since column cutoff, column discounting, and FTL were designed explicitly for this game against nature, whereas directed cognition is a more generally applicable algorithm.

## 5 Conclusion

We have proposed an algorithm for boundedly rational choice. The model provides a theory of cognitive resource allocation and is tractable even in highly complex environments. We empirically evaluate the model in such a setting. Developing models of decision-making that can be practically applied to complex problems is an important research frontier.

The directed cognition model is modular and there are many improvements that could be made to each of its parts: expanding the set of mental operators, improving the sophistication of the myopic proxy value function, adding trembles in the choice of the best cognitive operation, etc. In general, we have adopted the simplest possible version of each of the model's features.

We wish to highlight several shortcomings of the model. First, the implementation of the model requires the researcher to specify the set of cognitive operators. Ideally, a more general model would internally generate these cognitive operators. Second, the model only captures a few of the simplifying short-cuts that decision makers adopt. Third, the model does not exploit the fact that people often "solve" hard problems by developing specialized heuristics for those problems. To paraphrase Tolstoy, rational models are all rational in the same way, but boundedly rational models are all different. A unifying framework for bounded rationality represents a formidable research challenge.

## 6 Appendix A: Longer proofs

The following Proposition will be useful throughout the proofs.

**Proposition 5** *Let  $C(a, t) = E[a_\tau 1_{a_\tau > 0}] = E[a_\tau^+]$  represent the expected value of consumption, not including cognition costs. Let  $T(a, t) = E[\tau - t | a_t = a]$  represent the expectation of the remaining time spent on the problem. Let  $V = C - qT$  be the value function. They each satisfy a partial differential equation in the continuation region  $a \in (-\bar{a}_t, \bar{a}_t)$ :*

$$C_t + \frac{1}{2}\sigma^2(t)C_{aa} = 0, \quad (26)$$

$$1 + T_t + \frac{1}{2}\sigma^2(t)T_{aa} = 0, \quad (27)$$

$$-q + V_t + \frac{1}{2}\sigma^2(t)V_{aa} = 0. \quad (28)$$

The boundary conditions are:  $C(-\bar{a}_t, t) = 0$ ,  $C(\bar{a}_t, t) = \bar{a}_t$ ;  $T(-\bar{a}_t, t) = T(\bar{a}_t, t) = 0$ ; and  $V(-\bar{a}_t, t) = 0$ ,  $V(\bar{a}_t, t) = \bar{a}_t$ .

**Proof.** This is standard dynamic programming. See, e.g., Duffie (2001). Also, because of the symmetry of the problem, the decision maker will stop when  $a_t$  crosses either of the time-contingent thresholds  $-\bar{a}_t$  and  $\bar{a}_t$ . At those thresholds standard value matching conditions apply:  $V(-\bar{a}_t, t) = 0$ ,  $V(\bar{a}_t, t) = \bar{a}_t$ . ■

For the next two proofs, we observe that smooth pasting conditions also apply when the thresholds are chosen optimally,  $V_a(-\bar{a}_t, t) = 0$ ,  $V_a(\bar{a}_t, t) = 1$ .

### 6.1 Derivation of Propositions 1 and 2

In the constant case, the problem is stationary and  $t$  drops out of all expressions. The solution to the Bellman equation is

$$V(a) = C_0 + C_1 a + \frac{q}{\sigma^2} a^2$$

$$C(a) = C'_0 + C'_1 a$$

$$T(a) = C''_0 + C''_1 a - a^2/\sigma^2$$

Given  $\bar{a}$ , the 6 constants  $(C_i, C'_i, C''_i)_{i=0,1}$  are determined by the 6 boundary conditions of Proposition 5. There just remains to determine  $\bar{a}$ , which we do as follows.

For rational cognition, the smooth pasting condition  $V_a(-\bar{a}) = 0$  readily gives  $\alpha = \sigma^2 / (4q)$ .

For directed cognition with Gaussian innovations,  $\bar{a}$  is given by (14). Here variance is constant, so we get:  $\sup_{h \geq 0} w(\bar{a}, \sigma\sqrt{h}) - qh = 0$ . By homogeneity, this gives  $\bar{a} = \alpha\sigma^2/q$ , with  $\alpha$  defined by  $\sup_{h \geq 0} w(\bar{a}, \sigma\sqrt{h}) - qh = 0$ . By solving the optimization over  $h$ , we get the semi-closed form of the footnote of Proposition 1.

For directed cognition with atoms at  $\pm 1$ , we remark that  $\sup_{h \geq 0} w^0(\bar{a}, \sigma\sqrt{h}) - qh = 0$ , has a solution  $\bar{a} = \sigma^2 / (8q)$ .

This concludes the proof of Propositions 1 and 2.

## 6.2 Two Propositions for the case of linearly declining variance

When the variance declines linearly (the affine case)  $\sigma^2(t) = A - Bt$ . It is convenient to shift time indices by a factor  $A/B$ , so that the variance reaches 0 at time  $t = 0$ . Hence the process runs from  $t = t_0 < 0$  to  $t = 0$ . Defining  $s = \sqrt{B/2}$ , we get the normalization:

$$\begin{aligned} \sigma^2(t) &= -2s^2t, \text{ for } t \in [t_0, 0], t_0 < 0 \\ &= 0 \text{ for } t > 0. \end{aligned} \tag{29}$$

**Proposition 6** *In the case of linearly declining variance (29), the stopping rule is “at  $t$ , stop iff  $|a_t| \geq \bar{a}_t = -\alpha st$ ”. For the rational model (rational cognition),  $\alpha$  solves:*

$$\frac{\phi(\alpha)}{2\Phi(\alpha) - 1} = \frac{q}{s}. \tag{30}$$

*For boundedly rational agents reasoning with atoms at  $\pm 1$ , we have  $\alpha = 1 / (2q/s + \sqrt{1 + (2q/s)^2})$ . For the directed cognition model with Gaussian increments,  $\alpha$  is given implicitly by (14).*

**Proposition 7** *With the notation of Proposition 6, and  $v$  defined in (10), the expected consumption utility is*

$$C(a, t) = E[a_\tau^+ | a_t = a] = \frac{\alpha v(a, -st) + (v(\alpha, 1) - \alpha) a}{2v(\alpha, 1) - \alpha}, \tag{31}$$



the decision maker will, on average, keep thinking about the problem for an amount of time

$$T(a, t) = E[\tau - t | a_t = a] = -\frac{2v(a, -st) - a}{(2v(\alpha, 1) - \alpha)s} - t, \quad (32)$$

and the value function  $V = C - qT$  is

$$V(a, t) = \frac{\alpha + 2q/s}{2v(\alpha, 1) - \alpha} \left( v(a, -st) - \frac{a}{2} \right) + \frac{a}{2} + qt. \quad (33)$$

**Proof.** The solution of the Bellman equation is a relatively simple expression,

$$V(a, t) = C_1 v(a, -st) + C_2 a + qt. \quad (34)$$

The solution proposed in (34) can be verified by noting that  $v_a(a, \sigma) = \Phi(a/\sigma)$  and  $v_\sigma(a, \sigma) = \phi(a/\sigma)$ . Solving for constants  $C_1$  and  $C_2$  implies that the value function is given by (33) for the stopping value  $\bar{a}_t = -\alpha st$ . The derivation of the expressions of  $T$  and  $C$  is by verification of the Bellman conditions.

For rational cognition one uses the smooth pasting condition  $V_a(\bar{a}_t, t) = 1$  to solve for  $\alpha$ . For the directed cognition model one uses (14) to solve for  $\alpha$ . ■

### 6.3 Proof of Proposition 3

We prove each of the five inequalities in turn.

1.  $S^0(x, \sqrt{2/\pi}\sigma) \leq S^1(x, \sigma)$ : because  $v^1$  is convex in  $x$  and  $\frac{\partial}{\partial x} v^1(x=0, \sigma) = 1/2$ , we have

$$v^1(x, \sigma(0, t)) - qt \geq v^1(0, \sigma(0, t)) + x/2 - qt = \frac{1}{\sqrt{2\pi}}\sigma(0, t) + x/2 - qt.$$

Taking the  $\sup_t$  of both sides we get

$$S^1(x, \sigma) \geq x/2 + \sup_t \frac{1}{\sqrt{2\pi}}\sigma(0, t) - qt = x/2 + S^0(0, \sqrt{2/\pi}\sigma).$$

Applying  $\max(\cdot, x^+)$  to both sides, we get:  $S^1(x, \sigma) \geq S^0(x, \sqrt{2/\pi}\sigma)$ .

2.  $S^1(x, \sigma) \leq V^1(x, \sigma)$ : Call  $S^1(x_t, t, k) = E[x_{t+k}^+ - qk]$ , so that  $S^1(x_t, t) = \sup_{k \in \mathbb{R}^+} S^1(x_t, t, k)$ . For  $0 \leq s \leq k$ ,  $S^1(x_t, t, k) = E[S^1(x_{t+s}, t+s, k-s)] - qs \leq E[S^1(x_{t+s}, t+s)] - qs$ . Taking

the  $\sup_{k \in \mathbb{R}^+}$  on the left-hand side, we get  $S^1(x_t, t) \leq E[S^1(x_{t+s}, t+s)] - qs$  for  $s$  small enough. As  $V^1(x_t, t) = E[V^1(x_{t+s}, t+s)] - qs$ , we get, for  $D^1(x_t, t) = V^1(x_t, t) - S^1(x_t, t)$ , that  $D^1(x_t, t) \geq E[D^1(x_{t+s}, t+s)]$ , i.e.  $D^1$  is a submartingale. The search process stops at a random time  $T$ , and then  $S^1(x_T, T) = V^1(x_T, T) = x_T^+$ , so  $D^1(x_T, T) = 0$ . As  $D^1(x_t, t) \geq E[D^1(x_T, T)] = 0$ , we can conclude  $V^1(x_t, t) \geq S^1(x_t, t)$ .

3.  $V^1(x, \sigma(\cdot)) \leq V(x, \sigma(\cdot))$  : this is immediate because this holds for any policy  $v^i$ , as  $V$  is the best possible policy.
4. First, observe (with, again,  $x_0 = x$ )

$$\begin{aligned}
E[\max(x_\tau, 0)] &= E[x_\tau/2 + |x_\tau|/2] \text{ by the identity } \max(b, 0) = (b + |b|)/2 \\
&= x/2 + E[|x_\tau|]/2 \text{ as } x_t \text{ is a martingale which stops almost surely} \\
&\leq x/2 + E[x_\tau^2]^{1/2}/2 \\
&= x/2 + (x^2 + E[\sigma(0, t)^2])^{1/2}/2 \text{ as } x_t^2 - \sigma(0, t)^2 \text{ is a martingale} \tag{35} \\
&\leq x/2 + (x^2 + \sigma(0, m)^2)^{1/2}/2 \text{ with } m = E[\tau|x_0 = x] \text{ as } \sigma^2(0, t) \text{ is concave}
\end{aligned}$$

so we get:  $V(x) \leq x/2 + (x^2 + m)^{1/2}/2 - qm$ . Given that

$$S^2(x, \sigma) = x/2 + \sup_{t \in \mathbb{R}^+} (x^2 + \sigma(0, t)^2)^{1/2}/2 - qt$$

we get  $V(x) \leq S^2(x)$ .

5. We use the fact that for non-negative  $a, b$ ,  $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$  :

$$x/2 + \sqrt{x^2 + \sigma^2(0, t)}/2 - qt \leq (x + \sqrt{x^2})/2 + \sigma(0, t)/2 - qt = x^+ + \sigma(0, t)/2 - qt$$

so by taking the  $\sup_t$  of both sides we get:  $S^2(x, \sigma) \leq x^+ + S^0(0, \sigma)$ , whose right-hand side is less than or equal to  $S^0(x, \sigma) + \min(|x|, S^0(0, \sigma))/2$  by direct verification using the definition of  $v^0$ .

## 6.4 Proof of Proposition 4

We apply  $x = 0$  to (18) and get

$$S^0\left(0, \sqrt{2/\pi}\sigma(\cdot)\right) \leq V^1(0, \sigma(\cdot)) \leq V(0, \sigma(\cdot)) \leq S^0(x, \sigma(\cdot)).$$

Applying that inequality to  $\sqrt{2/\pi}\sigma(\cdot)$  rather than  $\sigma(\cdot)$ , we get

$$S^0\left(0, \left(\sqrt{2/\pi}\right)^2 \sigma(\cdot)\right) \leq V^1\left(0, \sqrt{2/\pi}\sigma(\cdot)\right) \leq V\left(0, \sqrt{2/\pi}\sigma(\cdot)\right) \leq S^0\left(x, \sqrt{2/\pi}\sigma(\cdot)\right).$$

Putting those two inequalities together, we extract

$$V\left(0, \sqrt{2/\pi}\sigma(\cdot)\right) \leq S^0\left(x, \sqrt{2/\pi}\sigma(\cdot)\right) \leq V^1(0, \sigma(\cdot)) \leq V(0, \sigma(\cdot)),$$

which allows us to conclude.

## 6.5 Derivation of the standard errors of distance measures

There are  $G = 12$  games,  $B = 10$  starting boxes per game, and  $I = 230$  subjects who satisfy our inclusion criterion. Call  $X^i \in \mathbb{R}^{BG}$  the choice vector such that  $X_{bg}^i = 1$  if subject  $i$  played box  $b$  at game  $g$ , and 0 otherwise. Call  $P = E[X^i]$  the true distribution of choices,  $\hat{P} := \sum_{i=1}^I X^i / I$  the empirical distribution of choices. We call  $\Sigma = E[X^i X^{i'}] - PP'$  the  $(BG \times BG)$  variance-covariance matrix of the choices  $X^i$ , and its estimator  $\hat{\Sigma} := (\sum_{i=1}^I X^i X^{i'} / I - \hat{P}\hat{P}') / (1 - 1/I)$ . We take the Euclidean norm  $\|P\| = \sum_g \left(\sum_b P_{bg}^2\right)^{1/2}$  for  $P \in \mathbb{R}^{BG}$ . We call  $L^m(\hat{P}, \beta) = \|P^m(\beta) - \hat{P}\|$  so that  $\hat{L}^m = \min_{\beta} L^m(\hat{P}, \beta)$  is the empirical distance between the model and the data. Call  $L^m$  and  $\beta^m$  the analogues for the true distribution of choices, i.e.  $L^m = \min_{\beta} \|P^m(\beta) - P\|$ . We also set  $\hat{\beta}^m := \arg \min_{\beta \in \mathbb{R}^+} L^m(\hat{P}, \beta)$ .

We now derive statistics<sup>24</sup> on  $\hat{L}^m$ .  $\varepsilon := \hat{P} - P$  has a distribution such that  $\sqrt{I}\varepsilon \sim N(0, \Sigma)$  for large  $I$ 's. Because  $\varepsilon = O(I^{-1/2})$  is small,  $L^m(\hat{P}) = L^m(P) + \partial_P L^m(P) \cdot \varepsilon + O(1/I)$ . By the envelope theorem,  $\partial_P L^m(P) = \Delta^m$ , where:

$$\Delta_{bg}^m = \left(\hat{P}_{bg} - P_{bg}^m(\beta^m)\right) \left(\sum_{b'} \left(\hat{P}_{b'g} - P_{b'g}^m(\beta^m)\right)^2\right)^{-1/2}. \quad (36)$$

<sup>24</sup>We use the fact, exploited by Vuong (1989) and Hansen-Heaton-Luttmer (1995), that we do not have to explicitly derive the distribution of  $\hat{\beta}^m$  to get the distribution of  $\hat{L}^m$ .

So we get the representation formula

$$\widehat{L}^m = L^m + \Delta^m \cdot \varepsilon + O(1/I). \quad (37)$$

Suppose that  $\Delta^m \neq 0$ , i.e. the model  $m$  is misspecified (this will be the case for all the models considered <sup>25</sup>). Eq. (37) implies that  $\sqrt{I}(\widehat{L}^m - L^m) \sim N(0, \Delta^{m'} \Sigma \Delta^m)$ , and gives us a consistent estimator of the standard deviation on  $\widehat{L}^m$ ,

$$\widehat{\sigma}_{L^m} = \left( \widehat{\Delta}^{m'} \widehat{\Sigma} \widehat{\Delta}^m / I \right)^{1/2}. \quad (38)$$

We will be interested in the difference  $L^m - L^n$  of the performance of two theories  $m$  and  $n$ . Eq. (37) implies that  $\widehat{L}^m - \widehat{L}^n = L^m - L^n + (\Delta^m - \Delta^n) \cdot \varepsilon + O(1/I)$ . Hence we get a consistent estimator of the standard deviation of  $\widehat{L}^m - \widehat{L}^{m'}$ ,

$$\widehat{\sigma}_{L^m - L^n} = \left( (\widehat{\Delta}^m - \widehat{\Delta}^n)' \widehat{\Sigma} (\widehat{\Delta}^m - \widehat{\Delta}^n) / I \right)^{1/2}. \quad (39)$$

Those formulae carry over to models that have been optimized. Our models have a free parameter: in directed cognition, the parameter is  $q$ , the cost (per box) of cognition; in column-discounting, it is  $\delta$ , the discount rate; in column cutoff, it is  $Q$ , the final column analyzed; in FTL it is  $p$ , the cutoff probability. In general, we parametrize a class of models  $M$  with a real (or vector) parameter  $x$ :  $M = \{M(x), x \in X_M\}$ . For instance, if  $M = FTL$ , and  $x = 0.25$ , then  $M(x)$  represents FTL with probability cutoff  $p = 0.25$ , and the space of parameter  $x$  is  $X_{FTL} = [0, 1]$ . The space can be discrete: for  $M = \text{Column Cutoff}$ ,  $X_M = \{1, 2, \dots, 10\}$ . For each  $x$ ,  $M(x)$  gives a vector of values  $\mathcal{E}^M(x)$  for the boxes, and hence a distance to the empirical distribution, for a given  $\beta$ :  $\|P(\mathcal{E}^M(x), \beta) - \widehat{P}\|$ .

For model  $M$ , we can optimize over parameter  $x$  to find the best possible fit by calculating  $\widehat{L}^M := \min_{x \in X_M} \min_{\beta \in \mathbb{R}^+} \|P(\mathcal{E}^M(x), \beta) - \widehat{P}\|$ . As this is again an application of the delta method, the standard errors calculated in (38) and (39) are still valid even though we optimize on parameter  $x$ .

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<sup>25</sup> A test that  $L^m \neq 0$  can be readily performed using the estimates  $\widehat{L}^m$  and  $\widehat{\sigma}_{L^m}$  given in the tables.

## 7 Appendix B: A formal statement of the application of the directed cognition model

(Should the paper be accepted, we would like to post it as Supplementary Material on the web page of the *Journal of European Economic Association*.)

This Appendix formally describes our applied model. The following notation is used to formally define the algorithm. A node  $n$  is a box in the game, and  $n'$  represents a node in the next column. Let  $p(n'|n)$  represent the probability of the branch leading from  $n$  to  $n'$ .<sup>26</sup> The flow payoff of node  $n$  is  $u_n$ . For the games that we study the  $u_n$  have mean 0 and standard deviation  $\sigma_u$ .

A path  $\theta$  is a set of connected nodes  $(n_1, \dots, n_m)$ , with node  $n_i$  in column  $i$ . The starting node is  $b(\theta) = n_1$ . Let  $\pi(\theta)$  represent the probability (along the path) of the last node of the path, i.e.  $\pi(\theta) = \prod_{i=1}^{m-1} p(n_{i+1}|n_i)$ .<sup>27</sup> Let  $l(\theta) = n_m$  represent the last node in path  $\theta$ .  $C$  is the total number of columns in a given decision tree and  $col(\theta) = m$  is the column of the last box in  $\theta$ . Call  $\theta_{[1,j]} = (n_1, \dots, n_j)$  the truncation of  $\theta$  to its first  $j$  nodes, and  $\theta_{-1} = (n_1, \dots, n_{m-1})$  its truncation by the last node. If  $\theta = (n_1, \dots, n_m)$  is a path,  $e(\theta)$  is the path constructed by adding, to the last node of  $\theta$ , the node with the highest probability branch:  $e(\theta) = (n_1, \dots, n_{m+1})$ , where<sup>28</sup>

$$p(n_{m+1}|n_m) = \sup\{p(n|n_m)|n \text{ branching from } n_m\}.$$

If  $\Theta$  is a set of examined paths,  $e(\Theta)$  is the set of its extensions: i.e. a path  $\theta = (n_1, \dots, n_{m+1}) \in e(\Theta)$  iff  $(n_1, \dots, n_m) = \theta'_{[1,m]}$  for some  $\theta' \in \Theta$  (i.e.  $\theta$  is the extension of a segment of a path already in  $\Theta$ ), and  $(n_m, n_{m+1})$  has the highest possible probability among branches starting from  $n_m$  that are not already traced in  $\Theta$ :

$$p(n_{m+1}|n_m) = \sup\{p(n|n_m)|\text{there is no } \theta' \in \Theta, \theta'_{[1,m+1]} = (n_1, \dots, n_m, n)\}.$$

Note that extensions can branch from the “middle” of a pre-existing path.

A time index  $t$  represents the accumulated number of incremental extensions which have been implemented by the algorithm. Each tick in  $t$  represents an additional one node extension of a

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<sup>26</sup>If no such branch exists  $p(n'|n) = 0$ .

<sup>27</sup>If  $m = 1$ ,  $\pi(\theta) = 1$ .

<sup>28</sup>In case of ties, take the top branch. Note that  $e(\theta)$  is not defined if  $\theta$  already has reached the last column and cannot be extended.

path.  $\theta(t)$  is the path which was extended/examined during period  $t$ .  $\Theta(t)$  is the current set of all paths that have been extended/examined (including  $\theta(t)$ ).

At time  $t$ ,  $\mathcal{E}(b, t)$ , is the estimate of the value of starting node  $b$ . At time  $t$ , the “best alternative” to starting node  $b$  is starting node  $a(b, t)$ . If  $b$  is the best node, then  $a(b, t)$  is the second-best node. If  $b$  is not the best node, then  $a(b, t)$  is the best node.

## 7.1 The directed cognition algorithm

The algorithm uses two cost variables. Recall that Step 1 of the algorithm generates intuitions about the relative merits of different cognitive operations. Building these intuitions costs  $F = qF^*$ , where  $F^*$  is the cost in number of elementary cognitive operations, and  $F$  is the cost in dollar units. Step 2 executes those cognitive operations. The cost of implementing a single cognitive operation is  $q$ , and our algorithm allows the decision maker to implement strings of such operations. Hence, Step 2 costs will always be multiples of  $q$ :  $hq$  where  $h \in \{0, 1, \dots, C - 2\}$ .

To start the algorithm, set  $t = 0$ . Let  $\Theta(0)$  contain the set of paths made of starting boxes.

$$\mathcal{E}(b, 0) = u_n + \pi(\theta(t + 1)) \sum_{n' \text{ branching from } n} p(n'|n)u_{n'}$$

where  $n$  represents node  $(b, 1)$ . Hence,  $\mathcal{E}(b, 0)$  represents the flow payoff of starting box  $b$  plus the probability weighted payoffs of the boxes that branch from starting box  $b$ .

Now iterate through three steps, beginning with Step 1.

### 7.1.1 Step 1

For  $\theta \in e(\Theta(t))$ , let

$$(\theta^*, h^*) = \arg \max_{\theta \in e(\Theta(t)), 0 \leq h \leq C - \text{col}(\theta)} w(\mathcal{E}(b(\theta), t) - \mathcal{E}(a(b(\theta), t), t), \pi(\theta)\sigma(h)) - qh. \quad (40)$$

The interpretation of the maximum is how much gain we can expect to get by exploring path  $\theta^*$ ,  $h^*$  steps ahead. Recall,  $b(\theta)$  is the starting box of  $\theta$ , and  $a(b, t)$  is the best alternative to  $b$ .<sup>29</sup> In the informal description of the algorithm, we assumed that the new information has standard deviation  $\sigma$ . In the equation above, the standard deviation is  $\pi(\theta)\sigma(h)$ . The function  $\sigma(h)$  is the rational expectations standard deviation generated by a concatenated  $h$ -column extension of any path, given

<sup>29</sup>We adopt the convention that  $\max S = -\infty$  if  $S$  is empty.

that the probability of reaching the beginning of the extended portion of the path is unity. The factor  $\pi(\theta)$  adjusts for the probability of actually reaching the extended portion of the path. Recall that  $\pi(\theta)$  is the probability of reaching the last node of path  $\theta$ , i.e.  $\pi(\theta) = \prod_{i=1}^{m-1} p(n_{i+1}|n_i)$ . The  $\sigma(h)$  function is derived below, after Step 3.

Call  $G$  the value of the maximum in (40). Go to step 2.

### 7.1.2 Step 2

In this step we execute the (concatenated) cognitive operation with the highest expected gain. We “explore path  $\theta^*$  at depth  $h^*$ ”. In detail:

- i. If  $h^* = 0$ , go directly to Step 3.
- ii. Let  $\theta(t+1) = \theta^*$ . In the set of examined paths, add  $\theta(t+1)$ , and, if it is the extension of a path  $\theta'$  at its terminal node, remove that path:

$$\begin{aligned}\Theta(t+1) &:= \Theta(t) \cup \{\theta(t+1)\} \setminus \{\theta_{-1}(t+1)\} \text{ if } \theta_{-1}(t+1) \in \Theta(t) \\ &:= \Theta(t) \cup \{\theta(t+1)\} \text{ otherwise.}\end{aligned}$$

- iii. Update the value of the starting box  $b(\theta(t+1))$ . If  $b \neq b(\theta(t+1))$ , then set  $\mathcal{E}(b, t+1) = \mathcal{E}(b, t)$ . If  $b = b(\theta(t+1))$ , then set

$$\mathcal{E}(b, t+1) = \mathcal{E}(b, t) + \pi(\theta(t+1)) \sum_{n' \text{ branching from } n} p(n'|n) u_{n'} \quad (41)$$

where  $n = l(\theta(t+1))$ .

- iv. Set  $h^* = h^* - 1$ . Set  $t = t + 1$ .
- v. If  $h^* \geq 1$ , set  $\theta(t+1) = e(\theta(t))$  and set  $\theta^* = e(\theta(t))$ . Then, go to substep ii above.
- vi. If  $h^* = 0$ , go to Step 3.

### 7.1.3 Step 3

- i. If  $G \geq qF$ , go to Step 1.
- ii. If  $G < qF$ , stop the search and select as choice node  $\arg \max_{b \in B} \mathcal{E}(b, t)$ .

This ends the algorithm, but we have one remaining loose end. We still need to derive the standard deviation function  $\sigma(h)$ .

## 7.2 The value of $\sigma(h)$

For a generic draw of probabilities at a node  $n'$ , we have  $r(n')$  branches and probabilities that we can order:  $p_1(n') \geq \dots \geq p_{r(n')}(n')$ . They are random variables. Recall that  $\sigma_u$  represents the standard deviation of the payoff in a box. Exploring one column ahead will give an expected square value:  $\sigma^2(1) = \sigma_u^2 E[p_1^2 + \dots + p_r^2]$ . Exploring two steps ahead, following the highest probability path between columns, gives:  $\sigma^2(2) = E[(p_1^2 + \dots + p_r^2)\sigma_u^2 + p_1^2 \cdot \sigma^2(1)] = (1 + E[p_1^2])\sigma^2(1)$ . In general, for  $h \geq 0$ :  $\sigma^2(h) = (1 + E[p_1^2] + \dots + E[p_1^2]^{h-1})\sigma^2(1) = \frac{1 - E[p_1^2]^h}{1 - E[p_1^2]}\sigma^2(1)$ .

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**Table 1: Predictions of the algorithms for the game in Figure 3**

| Box      | Values             |                    |               |                    |       | Probabilities (%) |                    |                    |               |                    |      |
|----------|--------------------|--------------------|---------------|--------------------|-------|-------------------|--------------------|--------------------|---------------|--------------------|------|
|          | Costless Cognition | Directed Cognition | Column Cutoff | Column Discounting | FTL   | Empirical         | Costless Cognition | Directed Cognition | Column Cutoff | Column Discounting | FTL  |
| <i>a</i> | -0.57              | 0.33               | 0.39          | -0.07              | 0.77  | 6.1               | 3.1                | 3.6                | 2.8           | 3.2                | 2.7  |
| <i>b</i> | 5.78               | 5.75               | 6.76          | 5.37               | 5.88  | 4.3               | 17.6               | 17.9               | 17.8          | 17.0               | 18.2 |
| <i>c</i> | -1.31              | 0.91               | -0.27         | -0.20              | -1.90 | 2.6               | 2.5                | 4.3                | 2.3           | 3.0                | 1.0  |
| <i>d</i> | 4.11               | 0.00               | 4.96          | 3.65               | 4.35  | 3.0               | 11.1               | 3.3                | 10.6          | 10.0               | 10.3 |
| <i>e</i> | 6.19               | 3.60               | 6.99          | 5.83               | 6.22  | 11.3              | 19.8               | 9.5                | 19.0          | 19.5               | 20.6 |
| <i>f</i> | 6.88               | 5.36               | 7.68          | 6.56               | 6.21  | 23.9              | 23.9               | 16.0               | 23.1          | 24.5               | 20.6 |
| <i>g</i> | -7.29              | -1.00              | -6.27         | -5.40              | -8.29 | 1.3               | 0.5                | 2.5                | 0.4           | 0.6                | 0.1  |
| <i>h</i> | -1.89              | -4.48              | -0.88         | -2.25              | -1.36 | 1.7               | 2.1                | 0.9                | 2.0           | 1.6                | 1.2  |
| <i>i</i> | 3.00               | 5.06               | 4.59          | 2.70               | 4.64  | 9.1               | 8.2                | 14.6               | 9.5           | 7.4                | 11.5 |
| <i>j</i> | 4.12               | 7.19               | 5.54          | 4.56               | 5.13  | 36.5              | 11.2               | 27.4               | 12.5          | 13.2               | 13.8 |

This Table presents the algorithms' estimate of the value of each of the boxes in the first column of Figure 1. The five models are Costless Cognition, Directed Cognition, Column Cutoff, Column Discounting, and Follow The Leader (FTL). After a logistic transformation, these estimates give predictions for the probability of choosing a box. Directed Cognition concluded that row *j* is the most promising row. Our subjects reached the same conclusion. A fully rational subject who could compute all paths would discover that box *f* has the highest payoff.

**Table 2**

| <b>m=Model</b>               | <b>L(m)</b>      | <b>L(m) - L(Costless Cognition)</b> | <b>L(m) - L(DC)</b> | <b>L(m) - L(CC)</b> | <b>L(m) - L(CD)</b> |
|------------------------------|------------------|-------------------------------------|---------------------|---------------------|---------------------|
| Costless Cognition           | 2.969<br>(0.120) |                                     |                     |                     |                     |
| Directed cognition (q=0.012) | 2.570<br>(0.107) | -0.399<br>(0.080)                   |                     |                     |                     |
| Column Cutoff (Q=8)          | 2.748<br>(0.119) | -0.221<br>(0.026)                   | 0.177<br>(0.071)    |                     |                     |
| Column Discounting (d=.91)   | 2.780<br>(0.120) | -0.190<br>(0.037)                   | 0.209<br>(0.070)    | 0.032<br>(0.017)    |                     |
| FTL (p=0.30)                 | 2.481<br>(0.095) | -0.488<br>(0.046)                   | -0.089<br>(0.057)   | -0.267<br>(0.033)   | -0.298<br>(0.034)   |

L(m) is the distance between the predictions of model m and the empirical data.

Parameters q, Q, d and p have been set to maximize the fit of the models they correspond to.

Standard errors are in parentheses.