Regularity Conditions to Ensure the Existence of Linearity-Generating Processes

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Abstract

"Linearity-generating" processes offer a tractable procedure to model cash-flows and pricing kernels in a way that yields exact closed form expressions for bond and stock prices. Prices are simply affine (not exponential-affine) in the factors. The linearity-generating class operates in discrete and continuous time with an arbitrary number of factors. This paper presents novel and general regularity conditions which ensure that processes are well-defined. It illustrates them with a series of economic examples. (JEL: C65, E43, G12, G13)

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1 Introduction

The last three decades of financial research have established that stocks and bonds have time-varying risk premia (Campbell 2003). As a way to represent stocks with time-varying growth rate and risk premia Gabaix (2007) defines and analyzes the "linearity-generating" class of financial processes, which yields simple closed for stocks and bonds, with an arbitrary number of factors.[1]

The LG class is a tractable and flexible class of processes for asset pricing. [2] In its generality, it is comparable to the exponential-affine class of Duffie and Kan (1996), which has proven very useful to analyze bonds. In the LG class bond and stock prices obtain in closed forms that are affine functions of the factors. This contrasts with the exponential-affine class, in which bond prices are an exponential-affine function of the factors and stocks are expressed as an infinite sum over maturities rather than a simple closed-form expression, as in the LG class.

Gabaix (2007) assumes that the processes are well-defined, and in particular that pricing kernel and dividends remain always positive. In this paper we show some simple conditions to guaranty this result. As a corollary we exhibit some new concrete LG stochastic processes.

The following two examples describe the problem and present our solution.

Two Examples of LG processes with boundaries We start with a simple "LG twisted" interest rate process (Gabaix 2007, Example 12).

Example 1 *(Simple one-factor interest rate model) Consider the interest rate process:*

$$\begin{aligned} r_t &= r_* + \widehat{r}_t \\ d\widehat{r}_t &= -\phi \widehat{r}_t dt + \widehat{r}_t^2 dt + \sigma\left(\widehat{r}_t\right) dB_t, \end{aligned} \tag{1}$$

where r_* is a constant, B_t is a standard Brownian motion. Assume $\widehat{r}_0 < \phi$, and that the process is well-defined for $t \geq 0$. Then,

$$E_t\left[e^{-\int_0^T r_{t+s} ds}\right] = e^{-r_* T}\left(1 - \frac{1 - e^{-\phi T}}{\phi}\widehat{r}_t\right), \tag{2}$$

[1] It unifies in a common framework antecedents such as Bhattacharya (1978), Menzly, Santos and Veronesi (2004), and Santos and Veronesi (2006).

[2] For instance, it has been used to think about stocks and bond puzzles (Gabaix 2008), exchange rate puzzles (Farhi and Gabaix 2008), and the econometrics of return and cash-flow predictibility (Binsbergen and Koijen 2007).

independently of the functional form for $\sigma(\widehat{r}_t)$. Also, the price of a perpetuity is:

$$E_t\left[\int_0^\infty e^{-\int_0^T r_{t+s}ds}dT\right] = \frac{1}{r_*}\left(1 - \frac{\widehat{r}_t}{r_* + \phi}\right). \tag{3}$$

The interest rate process (1) illustrates salient features of an LG process. (i) Its drift is approximately an autoregressive process (as in the term $-\phi\widehat{r}_t dt$) but with a "twist" introduced by the term $\widehat{r}_t^2 dt$. This term needs to have a coefficient of $+1$ to be in the LG class. Note that in many cases the extra twist term will be small: if the deviation of the interest rate from trend ($|\widehat{r}_t|$) is less than 5 percent then the extra drift term is less than 0.25 percent per year. Hence for many purposes, the process behaves similarly to an autoregressive process.

Then, perhaps surprisingly at first, the bond price (2) is (ii) linear in the state variable, $\widehat{r}_t$ (hence the name "linearity-generating" process) and (iii) is independent of the value of the volatility term $\sigma(\widehat{r}_t)$. As long as the process is well-defined, the value and functional form of the process volatility does not affect bond prices.

Furthermore, while many other processes have closed forms for bond prices, the distinctive feature of the LG class is that it also yields a closed form expression for the price of a perpetuity, Eq. 3. This is useful for the case of the stock, which is isomorphic to a perpetuity (see many examples in Gabaix 2007). These features are useful because they allow closed forms for perpetuities and stocks and also prices are independent of the details of the system, e.g. of some variance terms.

However, the caveat in Example 1 was "Assume that the process is well-defined". For the process to be well-defined (in particular, for it not to explode) we require that for all times, $\widehat{r}_0 < \phi$. The Feller conditions, are the well-known tools to ensure this. Qualitatively, they say that $\sigma(\widehat{r}_t)$ should to go zero smoothly enough in a right neighborhood of $\widehat{r}_t = \phi$. Volatility dies down near the boundary so the process never leaves the region $\{\widehat{r}_t < \phi\}$.

The real challenge, for which this paper proposes a solution is: How can we generalize the conditions of this example to n factors? What should the boundaries be for the process? How do we ensure that volatility dies smoothly enough? We illustrate the question with the next example (Gabaix 2007, Example 13), which generalizes Example 1.

Example 2 *(Multifactor interest rate model) Consider the interest rate process:*

$$r_t = r_* + \sum_{i=1}^n r_{it}$$

$$dr_{jt} = -\phi_j r_{jt}dt + \left(\sum_{i=1}^n r_{it}\right) r_{jt}dt + dN_{jt}, \tag{4}$$

where r_* is a constant, $N_t = (N_{1t}, ..., N_{nt})$ is a square-integrable martingale. Suppose that the process is well-defined for $t \geq 0$. Then,

$$E_t \left[e^{-\int_0^T r_{t+s} ds} \right] = e^{-r_* T} \left(1 - \sum_{i=1}^{n} \frac{1 - e^{-\phi_i T}}{\phi_i} r_{it} \right). \tag{5}$$

Also, the price of a perpetuity is:

$$E_t \left[\int_0^\infty e^{-\int_0^T r_{t+s} ds} dT \right] = \frac{1}{r_*} \left(1 - \sum_{i=1}^{n} \frac{r_{it}}{r_* + \phi_i} \right). \tag{6}$$

Eq. 4 means that each component j of the deviation of the interest rate mean-reverts with a speed ϕ_j, but with a "LG-twist", namely the term $\left(\sum_{i=1}^n r_{it} \right) r_{jt}$ in (4). Then again, bond prices are linear in the factors (Eq. 5), and the price of a perpetuity obtains in closed form (Eq. 6).

In this example work beyond Feller's conditions is needed. Indeed, it is not completely trivial to formulate "simple" conditions on $r_{1t}, ..., r_{nt}$ that ensure that bond prices (5) are always positive. In this paper, we present various sufficient conditions for the process to be well-defined which implies that bond prices are positive. In this introduction we give a flavor for the conditions. Order $\phi_1 \leq ... \leq \phi_n$, one sufficient condition is:

$$\text{Condition D at time } t\text{: } \forall k = 1...n, \ \sum_{i=k}^{m} \frac{r_{it}}{\phi_i} < 1. \tag{7}$$

Also, we show that if Condition D holds at time 0 and the noise is sufficiently small then Condition D will hold for all time. This way the condition is "self-perpetuating." We will specify the condition on volatility, and the sense in which it should go to zero "smoothly enough" near the boundary of Condition D. This will occupy us in Section 2, which is in discrete time, and Section 3, which is in continuous time.

We note that other sufficient conditions could work. Gabaix (2007) proposes a simple condition, which gives:

$$\text{Condition C at time } t\text{: } \sum_{i=1}^{m} \frac{\max(r_{it}, 0)}{\phi_i} < 1. \tag{8}$$

We will see that Condition C implies Condition D which implies that all bond prices are positive.

This paper will work this out systematically, in discrete and continuous time, with and without risk premia, with one and several factors, and with continuous and jump processes. LG processes allow a unified treatment. Appendix A reviews the basics on them.

Notations. We will use the following notation. For Z a vector in $\mathbb{R}^N$, for some $N \geq 1$:

$$Z \succ 0$$

if and only if all components of Z are strictly positive. Also, $diag(Z)$ is the diagonal matrix with diagonal elements $Z_1, ..., Z_N$, and ι is a vector with all components equal to 1.

Section 2 presents the results in discrete time. Section 3 presents the results in continuous time. Section 4 provides extensions. Section 5 concludes. Appendix A provides a concise introduction to the LG class. Appendix B shows some results on special matrices. Appendix C contains the longer proofs.

2 Discrete Time

2.1 A One-Factor Introduction

We start with a basic 1-factor example (Gabaix 2007, Example 1 and Lemma 1). It will give us the flavor for the type of regularity conditions that we will want to impose with n factors.

Example 3 *(Basic LG stock process) The dividend satisfies:*

$$\frac{D_{t+1}}{D_t} = (1 + g_t)(1 + \varepsilon_{t+1})$$

$$g_{t+1} = \frac{(1-\phi) g_t + \eta_{t+1}}{1 + g_t}, \tag{9}$$

and $\varepsilon_t > -1$ *almost surely,* $E_t[\eta_{t+1}] = E_t[\varepsilon_{t+1}] = E_t[\varepsilon_{t+1}\eta_{t+1}] = 0$. *Assume that the process is well-defined, with* $g_t > -1$*, for all non-negative times, and the price is* $P_t = E_t\left[\sum_{T=0}^{\infty} D_{t+T}/(1+r)^T\right]$. *Then, the equilibrium price is:*

$$\frac{P_t}{D_t} = \frac{1+r}{r}\left(1 + \frac{g_t}{r+\phi}\right).$$

When is the process well defined? Let us start with the case where there is no noise, $\forall t, \eta_{t+1} = 0$. The application $g \mapsto (1-\phi) g/(1+g)$ has two fixed points, an attractive one $g = 0$, and a repelling one that, $g = -\phi$. To ensure that the process is economically meaningful, we require that g_0 be on the right side of the repelling point, $g_0 > -\phi$. That will ensure (when there is no noise) that for all $t \geq 0$, $g_t > -\phi$, and in particular $g_t > -1$. If $g_0 < -\phi$, then for some t, $g_t < -1$, which is not a meaningful economic outcome. In conclusion in the deterministic growth rate case we want to impose

$$g_t > \underline{g}, \text{ for some } \underline{g} \in [-\phi, 0). \tag{10}$$

When the growth rate is stochastic we still want to ensure (10). This means that for all $\frac{(1-\phi)g_t+\eta_{t+1}}{1+g_t} > \underline{g} = -\phi$, i.e. $g_t + \eta_{t+1} > \underline{g}$. Hence the volatility of η_{t+1} has to go to zero near the boundary $\underline{g}$. For instance, suppose that $\eta_{t+1} = \sigma(g_t) v_{t+1}$, with $\sigma(g_t) \geq 0$ and that there is an $m > 0$ such that $v_{t+1} > -m$ almost surely. Then, we want: $g_t - \sigma(g_t) m \geq \underline{g}$, i.e. $\sigma(g_t) \leq \frac{g_t - \underline{g}}{m}$. To summarize:

Result 1 *(Conditions for the existence of the process in the 1-factor, discrete time case). Consider the process in Example 3:*

$$g_{t+1} = \frac{(1-\phi) g_t + \sigma(g_t) v_{t+1}}{1+g_t},$$

with $E_t[v_{t+1}] = 0$, and an $m > 0$ such that $\eta_{t+1} > -m$ with probability 1 and $0 \leq \sigma(g) \leq \frac{g-\underline{g}}{m}$ (the volatility goes to 0 fast enough close to $\underline{g}$) where $\underline{g} = -\phi$. Suppose $g_0 > \underline{g}$. Then, almost surely, for all $t \geq 0$, $g_t > \underline{g}$, and the process is well defined.

2.2 The N-Dimensional Case in Discrete Time: Initial Conditions

2.2.1 Theory

We next study the N dimensional case. The task is the following: given an LG process (with $\nu, Y_t \in \mathbb{R}^N$, $\Psi \in \mathbb{R}^{N \times N}$),

$$E_t[Y_{t+1}] = \Psi Y_t \text{ and } M_t D_t = \nu' Y_t, \tag{11}$$

we need simple conditions on Y_0, and the innovations $Y_{t+1} - E_t[Y_{t+1}]$ so that for all nonnegative t, $M_t D_t > 0$. We will say that the process is *well-defined* when it is defined for all dates $t \geq 0$, with $M_t D_t > 0$.

We will make the following assumption:

Assumption 1 *Generator matrix Ψ is diagonalizable in the space of real matrices, i.e. there is a real matrix q and a diagonal matrix Λ such that $\Psi = q \Lambda q^{-1}$.*

We order the coordinates by decreasing eigenvalues of Λ, i.e. $\Lambda_{11} \geq ... \geq \Lambda_{NN}$. We define $k_t = q^{-1} Y_t$. This way: $E_t[k_{t+1}] = q^{-1} \Psi Y_t$, and

$$E_t[k_{t+1}] = \Lambda k_t \text{ and } M_t D_t = \nu' q k_t.$$

We next make a second assumption:

Assumption 2 *The components of vector $\nu' q$ are all different from zero.*

Indeed, if that was not the case it would be enough to suppress some components of k_t. We define $Q = q \; diag(\nu' q)^{-1}$ and $K_t = Q^{-1} Y_t$ so that $K_t = diag(\nu' q)\, k_t$. Then,

$$E_t[K_{t+1}] = \Lambda K_t \text{ and } M_t D_t = \iota' K_t \tag{12}$$

with $\iota' = (1, ..., 1)$. In other terms, the state vector is now k_t, and the process is diagonal, in the sense that $E_t[k_{t+1}] = \Lambda k_t$, where Λ is a diagonal matrix.

Finally, we define the matrix $\Theta = \left(1_{\{i \geq j\}}\right)_{ij}$ as in Lemma 1, and $Z_t = \Theta K$, i.e.

$$Z_{kt} = \sum_{i=1}^{k} K_{it}, \tag{13}$$

for $k = 1...N$, and $M_t D_t = \zeta' Z_t > 0$ with $\zeta = (0, ..., 0, 1)$. Defining $F = \Theta Q^{-1}$ we have $Z_t = F Y_t$.

Lemma 1 in Appendix A shows that:

$$E_t[Z_{t+1}] = A Z_t \text{ and } M_t D_t = \xi' Z_t,$$

where A is a matrix with non-negative non-diagonal coefficients and positive diagonal coefficients, and $\xi = (0, ..., 0, 1)$. So, $Z \succ 0$ implies $AZ \succ 0$, and $A^T Z \succ 0$ for all $T \geq 0$.

In this section, we start with a partial goal: finding conditions on Y_0 such that for all t's, $M_t D_t > 0$, when there is no noise. We state the following Proposition, which is proven in Appendix B.

Proposition 1 *(Sufficient Condition D on the initial conditions, deterministic case). Suppose that the process is deterministic, $Y_{t+1} = \Psi Y_t$. Suppose Y_0's transform, Z_0, satisfies for $t = 0$, Condition D:*

$$\textit{Condition D at time } t\text{: } \forall k = 1...N, Z_{kt} > 0. \tag{14}$$

Then, for $t \geq 0$, Condition D holds, and $M_t D_t > 0$.

Proposition 2 *(Sufficient Condition D on the initial conditions for the prices to be positive). Suppose that Y_0's transform Z_0 satisfies Condition D (Eq. 14) at time 0. Then, for $t \geq 0$, $E_0[M_t D_t] > 0$.*

For completeness we state two related Propositions (proven in Gabaix 2007).

Proposition 3 *(Sufficient Condition C on the initial conditions in the deterministic case). Suppose that the process is deterministic, $Y_{t+1} = \Psi Y_t$. Suppose Y_0's transform K_0, satisfies for $t = 0$,* *Condition C:*

$$\text{Condition C at time } t\text{: } K_{1t} + \sum_{i>1} \min(0, K_{it}) > 0. \tag{15}$$

Then, for $t \geq 0$, Condition C holds and $M_t D_t > 0$.

Proposition 4 *(Sufficient Condition C on the initial conditions for the prices to be positive). Suppose Y_0's transform K_0 satisfies Condition C (Eq. 15) at time 0. Then, for $t \geq 0$, $E_0[M_t D_t] > 0$.*

Finally, Condition C is more restrictive than Condition D.

Proposition 5 *For any time t, Condition C implies Condition D.*

Proof. This follows from:

$$Z_{kt} = \sum_{i=1}^{k} K_{it} \geq K_{1t} + \sum_{i=2}^{k} \min(0, K_{it}) \geq K_{1t} + \sum_{i=2}^{N} \min(0, K_{it}).$$

■

2.2.2 Applications

Basic stock model We first study how the machinery of Proposition 2 applies to our basic stock model in Example 3. The discount rate is $M_t = (1+r)^{-t}$. So that with $Y_t = M_t D_t (1, g_t)'$, we have $E_t Y_{t+1} = \Psi Y_t$, with $\Psi = (1+r)^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1-\phi \end{pmatrix}$. Diagonalizing Ψ yields the following canonical representation:

$$K_t = M_t D_t \begin{pmatrix} 1 + \frac{g_t}{\phi} \\ -\frac{g_t}{\phi} \end{pmatrix},$$

with $E_t[K_{t+1}] = \Lambda K_t$, with $\Lambda = (1+r)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1-\phi \end{pmatrix}$, and $M_t D_t = (1,1)' K_t$. The vector Z_t is:

$$Z_t = M_t D_t \begin{pmatrix} 1 + \frac{g_t}{\phi} \\ 1 \end{pmatrix}.$$

Condition D gives: $1 + \frac{g_t}{\phi} > 0$, and $1 > 0$, i.e. $g_t > -\phi$, exactly the condition we found in the direct investigation of the 1-dimensional case. Condition C is: $1 + \frac{g_t}{\phi} + \min\left(-\frac{g_t}{\phi}, 0\right) > 0$, is the same conclusion.

Checking the conditions with 1 factor Suppose $Y_t = M_t D_t (1, x_t)$, and the generator is $\Psi = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $a > d > 0$. The condition for the process to be well-defined is:

$$a + bx_t > d. \tag{16}$$

Indeed, write $\Psi = a \begin{pmatrix} 1 & b/a \\ 0 & 1-\phi \end{pmatrix}$, $g_t = bx_t/a$. With $1-\phi = d/a$ the above condition $g_t/\phi + 1 > 0$ is (16)

$N-$**factor stock model** With N factors for the growth rate the canonical stock model is:

$$D_{t+1}/D_t = 1 + \sum_{i=1}^{n} g_{jt}$$
$$E_t [g_{it+1}] = (1 - \phi_i) g_{it} / \left(1 + \sum_{j=1}^{n} g_{jt}\right),$$

with $0 < \phi_1 \leq ... \leq \phi_n$. The discount factor is $M_t = (1+r)^{-t}$. The canonical basis is: $K_{0,t} = M_t D_t (1 + \sum g_{it}/\phi_i)$, $K_{it} = -M_t D_t g_{it}/\phi_i$, where we start the indices at $i = 0$ which is natural in this context. Indeed, the reader can verify $E_t K_{t+1} = \Lambda K_t$, with $\Lambda = (1+r)^{-1} diag (1, 1-\phi_1, ..., 1-\phi_n)$. Also, $M_t D_t = \iota' K_t$ with $\iota = (1, .., 1)$.

Condition D is: $\forall k = 0...n, \sum_{i=0}^{k} K_{it} > 0$, i.e.

$$\forall k = 1...n, 1 + \sum_{i=k}^{n} \frac{g_{it}}{\phi_i} > 0. \tag{17}$$

Condition C is: $1 + \sum g_{it}/\phi_i + \sum \min (0, -g_{it}/\phi_i)$, i.e.

$$1 + \sum_{i=1}^{n} \min \left(\frac{g_{it}}{\phi_i}, 0\right) > 0. \tag{18}$$

The idea remains that "the growth rate cannot become too negative". The weighing by x_{it} means that there is a stronger penalty for persistent processes, which makes sense, as they have a longer influence.

$N-$**factor bond model** The model is, in its simplest form:

$$\frac{M_{t+1}}{M_t} = \frac{1}{1+r_*}\left(1-\sum_{j=1}^{n} r_{jt}\right)$$
$$E\left[r_{it+1}\right] = \frac{(1-\phi_i)\, r_{it}}{1-\sum_{j=1}^{n} r_{it}},$$

and for bonds, the dividend is $D_t = 1$.

The diagonalized basis is: $K_{0,t} = M_t\left(1-\sum r_{it}/\phi_i\right)$, $K_{it} = M_t r_{it}/\phi_i$, and $M_t = \iota' K_t$ with $\iota = (1, .., 1)$. Then, $E_t K_t = (1+r_*)^{-1}\, diag\left(1, 1-\phi_1, ..., 1-\phi_n\right) K_t$.

Condition D is

$$\forall k = 1...n, 1-\sum_{i=k}^{n} \frac{r_{it}}{\phi_i} > 0. \tag{19}$$

Condition C becomes: $1-\sum_i r_{it}/\phi_i + \sum_i \min\left(0, r_{it}/\phi_i\right)$, i.e.

$$1 > \sum_{i=1}^{n} \max\left(0, \frac{r_{it}}{\phi_i}\right). \tag{20}$$

The idea remains that "the interest rate cannot become too high".

We have now a better sense of what are sufficient conditions to ensure that the process remains positive. We now turn to conditions that ensure that the noise "dies down" close enough to the boundaries.

2.3 Making Sure that the Noise "Dies Down" Close to the Boundary

2.3.1 Theory

As in the 1-factor case of section 2.1, we need to ensure that the noise is close to zero near the boundary region $\{Z_t : Z_t \succ 0\}$ for the process. We specify here sufficient conditions for that to happen.

We start from:

$$Y_{t+1} = \Psi Y_t + \sigma\left(Y_t\right) \eta_{t+1}, \tag{21}$$

where η_t is a $p-$dimensional vector with $E_t\left[\eta_{t+1}\right] = 0$, $\sigma(Y)$ is an $N \times p$ matrix.

We use a canonical basis Z_t, with $Z_t = FY_t$ and $M_t D_t = \zeta' Z_t > 0$, $\zeta = (0, ..., 0, 1)$. We want to ensure that, almost surely, for all t, $Z_t \succ 0$. That will imply $M_t D_t > 0$. To ensure that, the next Proposition states conditions that are reasonably easy to verify.

Proposition 6 *(Sufficient conditions so that $M_tD_t > 0$ for all t). Call $\lambda_1 \geq ... \geq \lambda_n$ the eigenvalues of Ψ, and $F^{(i)} = (F_{ij})_{j=1...n}$ the i-th row vector of matrix F. Suppose that $Z_0 \succ 0$ (i.e. has all its components strictly positive), and that, for all t, any one of the conditions (SC1,2,3,4) is verified almost surely:*

$$\textit{(SC 1)}\ Z_t \succ 0 \Rightarrow \forall i = 1...n,\ F^{(i)}\Psi F^{-1}Z_t + F^{(i)}\sigma\left(F^{-1}Z_t\right)\eta_{t+1} \succ 0 \tag{22}$$

$$\textit{(SC 2)}\ Z_t \succ 0 \Rightarrow \forall i = 1...n,\ F^{(i)}\Psi F^{-1}Z_t > \left\|F^{(i)}\sigma\left(F^{-1}Z_t\right)\right\|_1 \|\eta\|_\infty \tag{23}$$

$$\textit{(SC 3)}\ Z_t \succ 0 \Rightarrow \forall i = 1...n,\ \lambda_i Z_{it} + F^{(i)}\sigma\left(F^{-1}Z_t\right)\eta_{t+1} > 0\ \textit{a.s.} \tag{24}$$

$$\textit{(SC 4)}\ Z_t \succ 0 \Rightarrow \forall i = 1...n,\ \lambda_i Z_{it} > \left\|F^{(i)}\sigma\left(F^{-1}Z_t\right)\right\|_1 \|\eta\|_\infty \tag{25}$$

where $\|x\|_1 = \sum_{i=1}^N |x_i|$, $\|\eta\|_\infty = ess\sup|\eta|$. Then, with probability 1, for all $t \geq 0$, $Z_t \succ 0$, and in particular $M_tD_t > 0$.

Also, $SC4 \Rightarrow SC3 \Rightarrow SC1$ and $SC4 \Rightarrow SC2 \Rightarrow SC1$.

Proposition 6 provides sufficient conditions to express that the noise needs to be "small enough" near the boundary. We turn to concrete examples to illustrate them.

2.3.2 Applications

Simple stock model Take (9), $g_{t+1} = \frac{(1-\phi)g_t + v(g_t)u_{t+1}}{1+g_t}$, with $v(g) \geq 0$. We set $M_t = (1+r)^{-t}$. As $Y_t = M_tD_t(1, g_t)'$, $Z_t = M_tD_t\begin{pmatrix} 1 + g_t/\phi \\ 1 \end{pmatrix}$, so that:

$$Z_{t+1} = (1+r)^{-1}\begin{pmatrix} 1 & 0 \\ \phi & 1-\phi \end{pmatrix} Z_t + (1+r)^{-1} M_tD_t \begin{pmatrix} v(g_t)u_{t+1}/\phi \\ 0 \end{pmatrix}.$$

Condition SC1 simply means that $Z_{t+1} > 0$, i.e. $1 + g_t/\phi + v(g_t)u_{t+1}/\phi > 0$, which is just the condition we had seen in section 2.1.

Condition SC2 means $v(g_t) < (\phi + g_t)/\|u\|_\infty$: the volatility of the process goes to 0 near $g_t = -\phi$.

$N-$factor bond model The model is now more specified, as:

$$\frac{M_{t+1}}{M_t} = \frac{1}{1+r_*}\left(1-\sum_{j=1}^{n} r_{jt}\right)$$
$$r_{it+1} = \frac{(1-\phi_i)\, r_{it} + \sigma_{it}\cdot\eta_{t+1}}{1-\sum_{j=1}^{n} r_{it}},$$

where $\eta_{t+1} \in \mathbf{R}^p$ for some p, $E_t[\eta_{t+1}] = 0$, and for $i = 1...n$, $\sigma_{it} \in \mathbf{R}^p$. As before, the diagonalized basis is: $K_{0,t} = M_t(1-\sum r_{it})$, $K_{it} = M_t x_{it}$, and $M_t = \iota' K_t$ with $\iota = (1,..,1)$. Then, $E_t K_t = \Lambda K_t$, with $\Lambda = (1+r_*)^{-1}\, diag(1-\phi_0,..., 1-\phi_n)$, calling $\phi_0 = 0$.

The associated $Z = \Theta K_t$, with $\Theta = (1_{i\geq j})_{0\leq i,j\leq n}$. So, for $i = 0...n-1$, $Z_{it} = M_t\left(1-\sum_{j=i+1}^{n} \sigma_{jt}\right)$, and $Z_{nt} = M_t$. Also,

$$Z_{t+1} = \frac{1}{1+r_*}\left(\Theta diag(1-\phi_0,..., 1-\phi_n)\,\Theta^{-1} Z_t + M_t v_{t+1}\right),$$

with $v_{it+1} = \sigma_{it}\eta_{t+1}$ if $i > 0$ and $v_{0,t+1} = -\sum_{i=1}^{n} \sigma_{it}\eta_{t+1}$.

Then, SC4 is simply:

$$\forall i = 0,...,n-1, (1-\phi_i)\, Z_{it} > \left\|\sum_{j=i+1}^{n} \sigma_{jt}\right\|_1 \cdot \|\eta_{t+1}\|_\infty$$

and SC3 is: $\forall i = 0,...,n-1, (1-\phi_i)\, Z_{it} > \left(\sum_{j=i+1}^{n} \sigma_{jt1}\right)\cdot \eta_{t+1}$ almost surely

2.4 Killing Functions

In practice, a way to ensure the conditions of Proposition 6 is via a "killing function" at the borders.

Definition 1 *(Killing functions) A "killing" function with 1 argument $\kappa : \mathbf{R} \to \mathbf{R}_+$, is a function such that (i) $\kappa(x) = 0$ if any $x \leq 0$; (ii) here is an x^* such that $\kappa(x) = 1$ for $x \geq x^*$, (iii) κ is uniformly Lipschitz, i.e. there is a c such that $|\kappa(x)-\kappa(y)| \leq c|x-y|$, for all $x, y \in \mathbf{R}$.*

A "killing" function with N arguments $\kappa : \mathbf{R}^N \to \mathbf{R}_+^p$, (for positive integers N, p) is a function such that (i) $\kappa(x) = 0$ if $x^i \leq 0$ for at least one $i = 1...N$; (ii) here is an x^* such that $\kappa(x) = (1,...,1)$ if $\forall i = 1...N$, $x^i \geq x_0$, (iii) κ is uniformly Lipschitz, i.e. there is a c such that $\|\kappa(x)-\kappa(y)\| \leq c||x-y||$, for all $x, y \in \mathbf{R}^N$.

A "killing function" $\kappa(x)$ is equal to 0 as soon as one of the components of x non-positive, and is equal to 1 when all components of x are far enough from 0.

In practice, x_0 is small. An example of a killing function with one argument is the "ramp" function $\kappa(x) = \min(x^+/x^*, 1)$, equal to x/x when $x \in [0, x_0]$, and 1 for $x \geq x_0$. A generalization with N arguments is $\kappa(x) = \min\left(\min_{i=1...n} x_i^+/x^*, 1\right)$, for $x \in \mathbf{R}^N$.

Indeed, suppose one would like to have a process approximately equal to: $Y_{t+1} \simeq \Psi Y_t + v(Y_t)\eta_{t+1}$. To make sure the process remains well-defined, one can set up the modified process (21), with $\sigma(Y_t) = K(Y_t)v(Y_t)$, and

$$K(Y_t) = \kappa\left(\min_i \frac{F^{(i)}\Psi Y_t}{\|F^{(i)}v(Y)\|_1 \|\eta\|_\infty}\right). \tag{26}$$

Another possibility is a $K(Y_t)$ (with value in $\mathbb{R}_+$) that ensures, for all Y_t s.t. $FY_t \succ 0$,

$$\forall i = 1...n, F^{(i)}\Psi Y_t + K(Y_t) F^{(i)} v(Y_t) \eta_{t+1} > 0 \tag{27}$$

where again $F^{(i)}$ is the $i - th$ row vector of matrix F.

Proposition 7 *Consider the LG process $Y_{t+1} = \Psi Y_t + K(Y_t)v(Y_t)\eta_{t+1}$, with (26) or (27). Suppose that $FY_0 \succ 0$. Then, for all times $t \geq 0$, $FY_t \succ 0$ and $M_t D_t > 0$.*

We now end our study of the discrete time case, and move on to its analogue for continuous time.

3 Continuous Time

3.1 The N-Dimensional Case in Discrete Time: Initial Conditions

3.1.1 Standard Representations of the Processes

We start from a continuous-time LG process:

$$dY_t = -\Xi Y_t dt + dN_t \text{ and } M_t D_t = \nu' Y_t$$

where Ξ is an $N \times N$ matrix, ν, $Y_t \in \mathbb{R}^N$, and N_t a martingale with values in $\mathbb{R}^N$. We want to find conditions such that almost surely for all $t \geq 0$, $M_t D_t > 0$.

We proceed as in the discrete time case, and make the following assumption.

Assumption 3 *Generator matrix Ξ is diagonalizable in the space of real matrices, i.e. there is a real matrix q, and a diagonal matrix Λ, such that $-\Xi = q\Lambda q^{-1}$.*

We order the rows so that $\Lambda_{11} \geq ... \geq \Lambda_{NN}$. We make assume that $q'\nu$ has non-zero elements (Assumption 2). We define $Q = q\ diag\,(q'\nu)^{-1}$, and $K_t = Q^{-1}Y_t$. Then,

$$dK_t = \Lambda K_t dt + Q^{-1}dN_t \text{ and } M_t D_t = \iota' K_t,$$

with $\iota = (1, ..., 1)'$. Using matrix Θ in Lemma 1, we define: $Z_t = \Theta K_t = FY_t$, with $F = \Theta Q^{-1}$. So, Z_t satisfies:

$$dZ_t = AZ_t dt + FdN_t \text{ and } M_t D_t = \xi' Z_t \tag{28}$$

where $\xi = (0, ..., 0, 1)$ and $A = \Theta\Lambda\Theta^{-1}$ has nonnegative non-diagonal elements, in virtue of Lemma 1.

3.2 Making Sure that the Noise "Dies Down" Close to the Boundary

3.2.1 A Preliminary Theorem

We first present an abstract result which is useful to establish more concrete sufficient conditions. The reader may wish to skip this subsection in a first reading and to move directly to section 3.2.2.

Let $n, p \geq 1$ and consider the stochastic differential equation (SDE)

$$dz_t = \mu(z_{t-})dt + \sigma(z_{t-})dG_t + diag\,(z_{t-})\,dH_t, \quad z_0 = x_0 \in \mathcal{D}, \tag{29}$$

for an $n+1$-dimensional process z_t with components $z_t^0, z_t^1, \ldots, z_t^n$. z_t is understood as a column vector. $\mathcal{D}$ is the region

$$\mathcal{D} = \left\{x \in \mathbb{R}^{n+1} : x \succ 0\right\}.$$

μ is a function from $\mathcal{D}$ to $\mathbb{R}^{n+1}$ and σ maps $\mathcal{D}$ to the space of $(n+1) \times p$-matrices. G_t is an p-dimensional column vector whose components are continuous semimartingales $G_t^1, \ldots, G_t^p$. [3] H_t is a column vector consisting of $n+1$ semimartingales (possibly with jumps) $H_t^0, H_t^1, \ldots, H_t^n$ with the property (recall $\Delta H_t = H_t - H_{t^-}$)

$$\Delta H_t^m > -1 \quad \text{for all } m \text{ and } t. \tag{30}$$

This condition implies that starting from $\mathcal{D}$, z_t cannot be propelled outside $\mathcal{D}$ by a jump dH_t.

By μ^m we denote the m-th component of μ and by σ^m the row vector $(\sigma^{m1}, \ldots, \sigma^{mN})$. Moreover,

[3] For instance, every Brownian motion with drift, and, more generally, every solution of a SDE driven by a Brownian motion, is a continuous semimartingale.

we set

$$||\mu|| := \sqrt{\sum_{m=0}^{n}(\mu^m)^2} \quad \text{and} \quad ||\sigma|| := \sqrt{\sum_{m=0}^{n}\sum_{r=1}^{p}(\sigma^{mr})^2}\,.$$

Assume there exist constants $a \geq 0$, $b_1, b_2, \cdots \geq 0$, $c \in \mathbb{R}$, $d \geq 0$ and $\varepsilon > 0$ such that the following conditions hold:[4]

(i) $||\mu(x)|| \vee ||\sigma(x)|| \leq a(1 + ||x||)$ for all $x \in \mathcal{D}$

(ii) $||\mu(x) - \mu(y)|| \vee ||\sigma(x) - \sigma(y)|| \leq b_k||x - y||$ for all $x, y \in \mathcal{D}$ such that $||x||, ||y|| \leq k$

(iii) $\mu^m(x) \geq cx^m$ for all $m = 0, \ldots, n$ and $x \in \mathcal{D}$

(iv) $||\sigma^0(x)|| \leq dx^0$ for all $x \in \mathcal{D}$ with $x^0 < \varepsilon$

(v) $||\sigma^m(x)|| \leq dx^m$ for all $m \geq 1$ and $x \in \mathcal{D}$ with $x^m < \varepsilon(1 \wedge x^0)$

Then the following two properties hold.

Theorem 1 *The SDE (29) has a unique strong solution that never leaves $\mathcal{D}$.*

Proposition 8 *Suppose further than G_t and H_t are a square-integrable martingales and $\mu(z) = Bz$ for B a $(n+1) \times (n+1)$ matrix. Then for all non-negative t, T, $E_t[z_{t+T}] = e^{AT} z_t$.*

3.2.2 Theory

We assume that the noise in process Z_t (Eq. 28) can be parameterized:

$$dZ_t = AZ_t dt + Z_t^0 v\left(\frac{Z_t}{Z_t^0}\right) dB_t + diag\left(Z_{t^-}\right) dJ_t, \tag{31}$$

where A is an $N \times N$ matrix, B_t is a standard $K-$dimensional Brownian motion, $v(z)$ is a $N \times p$-dimensional matrix, and J_t has values in $\mathbb{R}^N$ and is square-integrable martingale.

For instance, we could have $J_t = j_t - \lambda t$ where j_t is a finite-activity square-integrable jump process and λt is its compensator. We assume that:

(a) A has nonnegative non-diagonal terms.

(b1) For $m = 0...N-1$, if $z^m = 0$, then $v^m(z) = 0$.

(b2) v is continuous and almost everywhere differentiable

(b3) There exists a constant C such that for all z where v is differentiable, $|\partial v^{ij}/\partial z^m| \leq C$ for all $i, 0...N-1$, $j = 1...p$ and $m = 1...N-1$

[4] We could replace assumptions (i) and (ii) by an assumption that SDE (29) has a unique strong solution.

(b4) There exists a constant C' such that for all z where v is differentiable, $\left| v^{ij} - \sum_{m=1}^{N-1} z^m \frac{\partial v^{ij}}{\partial z^m} \right| \leq C'$ for all $i = 0...N-1$, $j = 1...p'$.

(c) $\Delta J_t > -1$ (recall $\Delta J_t = J_t - J_{t^-}$)

Condition (a) ensures that the drift does not pull the process out of the positivity domain $\mathcal{D} = \{Z \text{ s.t. } Z \succ 0\}$.

Condition (b1-b3) make sure that the volatility is small enough near the boundary of the positivity domain. Condition (b1) ensures that, when $Z^i = 0$, then the volatility of the $i-th$ component is at 0, so the volatility term does not expel the process outside of the positivity domain. Condition (b2)-(b4) ensures that the volatility function $\sigma(Z) = Z^0 v(Z/Z^0)$ is Lipschitz.

Condition (c) means that the process cannot jump outside of the positivity domain $\mathcal{D}$.

Theorem 2 *(Sufficient condition to make the LG process dividend-augmented pricing kernel always positive). Suppose $Z_0 \succ 0$, and conditions (a)-(c) above hold. Then, the SDE (31) has a unique strong solution, that never leaves $\mathcal{D} = \{Z \text{ s.t. } Z \succ 0\}$. In particular, with probability 1, the process satisfies, $M_t D_t > 0$ for all $t \geq 0$. Also, for all non-negative t, T, $E_t[Z_{t+T}] = e^{AT} Z_t$.*

3.3 Applications

Simple one-factor interest rate process Consider the simple interest rate process, with $r_t = \widehat{r}_t + r_*$

$$d\widehat{r}_t = -(\phi - \widehat{r}_t)\widehat{r}_t dt + \sigma(\widehat{r}_t) dB_t, \tag{32}$$

where B_t is a standard Brownian motion. We saw that it implies (2). We form: $M_t = e^{-\int_0^t r_s ds}$ and set $D_t \equiv 1$. The LG vector is $Y_t = M_t(1, r_t)'$, and the positivity test vector is:

$$Z_t = \begin{pmatrix} M_t\left(1 - \frac{\widehat{r}_t}{\phi}\right) \\ M_t \end{pmatrix}$$

so that:

$$dZ_t^1 = -r_* Z_t^1 dt + M_t\left(-\widehat{r}_t\left(1 - \frac{\widehat{r}_t}{\phi}\right) + (\phi - \widehat{r}_t)\frac{\widehat{r}_t}{\phi} - \frac{\sigma(\widehat{r}_t)}{\phi} dB_t\right) = -r_* Z_t^1 dt + \frac{\sigma(\widehat{r}_t)}{\phi} Z_t^2 dB_t$$
$$dZ_t^2 = -M_t r_t dt = \phi Z_t^1 - (r_* + \phi) Z_t^2$$

i.e.

$$dZ_t = \begin{pmatrix} -r_* & 0 \\ \phi & -r_* - \phi \end{pmatrix} Z_t + \begin{pmatrix} \frac{\sigma(\widehat{r}_t)}{\phi} Z_t^2 dB_t \\ 0 \end{pmatrix}. \tag{33}$$

The conditions above are $Z_t^1 > 0$, $Z_t^2 > 0$. Applying Theorem 2 yields:

Result 2 *(1-factor interest rate process) In the interest rate model above, suppose that:* $\widehat{r}_0 < \phi$*;* σ *is continuous and almost everywhere differentiable;* $\sigma'(\widehat{r})$ *and* $\sigma'(\widehat{r})(\phi - \widehat{r}) + \sigma(\widehat{r})$ *are uniformly bounded for all the* $\widehat{r} \in (-\infty, \phi)$ *where* σ *is differentiable. Then the SDE (32) are (33) have a unique strong solution for all* $t \geq 0$*, and Eq. 2-3 hold for all* $t \geq 0$*. Also for all* $t \geq 0$*,* $\widehat{r}_t < \phi$*, and* $Z_t \succ 0$*.*

Other sufficient conditions can be found. In particular, we could ensure that some $\overline{r} \in (0, \phi)$ is a natural right boundary $\widehat{r}_t$, starting from $\widehat{r} < \overline{r}$. Using the general Feller conditions, it would be enough to have, in a left neighborhood of ϕ $|\sigma(\widehat{r})| \leq k(1 - \widehat{r}/\overline{r})^{\beta}$, for some $k > 0$ and $\beta > 1/2$. Then we would have, for all $t \geq 0$, $\widehat{r} \leq \phi$. Here we impose a condition with $\beta = 1$ (to ensure that v is Lipschitz). Our goal is simply to provide simple sufficient conditions.

A two-factor interest rate process Consider $M_t = \exp\left(-\int_0^t r_s ds\right)$, $r_t = r_* + r_{1t} + r_{2t}$, and

$$dr_{1t} = -(\phi_1 - r_{1t} - r_{2t}) r_{1t} dt + \sigma_1(r_{1t}, r_{2t}) dB_{1t} + w_1(r_{1t}, r_{2t}) dW_t$$
$$dr_{1t} = -(\phi_2 - r_{1t} - r_{2t}) r_{2t} dt + \sigma_2(r_{1t}, r_{2t}) dB_{2t} + w_2(r_{1t}, r_{2t}) dW_t,$$

with $0 < \phi_1 \leq \phi_2$, and B_1, B_2, W independent standard Brownian processes. The drift terms mean that r_{it} mean-reverts to 0 according to a LG-twisted AR(1), with the typical speed ϕ_i. The dB_{it} shocks are specific to component i, and the dW_t shocks are common to both components.

We set $D_t \equiv 1$. $M_t(1, r_{1t}, r_{2t})$ is a LG process, and, if the process is well-defined, then, according to Gabaix (2007, Theorem 3 and Example 13)

$$E_t\left[\exp\left(-\int_0^T r_{t+s} ds\right)\right] = e^{-r_* T}\left(1 - \frac{1 - e^{-\phi_1 T}}{\phi_1} r_{1t} - \frac{1 - e^{-\phi_2 T}}{\phi_2} r_{2t}\right). \tag{34}$$

Here, we apply Theorem 2. The diagonal representation is:

$$K_t = M_t \begin{pmatrix} 1 - \frac{r_{1t}}{\phi_1} - \frac{r_{2t}}{\phi_2} \\ \frac{r_{1t}}{\phi_1} \\ \frac{r_{2t}}{\phi_2} \end{pmatrix},$$

which satisfies $E_t\left[dK_t\right] = diag\left(0, -\phi_1, -\phi_2\right) K_t dt$, and positivity test vector is

$$Z_t = M_t \begin{pmatrix} 1 - \frac{r_{1t}}{\phi_1} - \frac{r_{2t}}{\phi_2} \\ 1 - \frac{r_{2t}}{\phi_2} \\ 1 \end{pmatrix},$$

which satisfies (by direct calculation, or application of Lemma 1)

$$dZ_t = \begin{pmatrix} -r_* & 0 & 0 \\ \phi_1 & -r_* - \phi_1 & 0 \\ \phi_1 & \phi_2 - \phi_1 & -r_* - \phi_2 \end{pmatrix} Z_t dt + M_t \begin{pmatrix} -\frac{\sigma_1(r_{1t},r_{2t})dB_{1t}+w_1(r_{1t},r_{2t})dW_t}{\phi_1} - \frac{\sigma_2(r_{1t},r_{2t})dB_{2t}+w_2(r_{1t},r_{2t})dW_t}{\phi_2} \\ -\frac{\sigma_2(r_{1t},r_{2t})dB_{2t}+w_2(r_{1t},r_{2t})dW_t}{\phi_2} \\ 0 \end{pmatrix}.$$

We note that the drift matrix of dZ_t has nonnegative non-diagonal coefficients, as expected. Applying Theorem 2 yields:

Result 3 *(2-factor interest rate model) Suppose that*

$$\frac{r_{1t}}{\phi_1} + \frac{r_{2t}}{\phi_2} < 1 \text{ and } \frac{r_{2t}}{\phi_2} < 1 \tag{35}$$

holds for $t = 0$, and

$$\begin{aligned} \sigma_1\left(r_1, r_2\right) &= \overline{\sigma}_1 \kappa_{\sigma_1}\left(1 - \frac{r_1}{\phi_1} - \frac{r_2}{\phi_2}\right) \\ w_1\left(r_1, r_2\right) &= \overline{w}_1 \kappa_{w_1}\left(1 - \frac{r_1}{\phi_1} - \frac{r_2}{\phi_2}\right) \\ \sigma_2\left(r_1, r_2\right) &= \overline{\sigma}_2 \kappa_{\sigma_2}\left(1 - \frac{r_1}{\phi_1} - \frac{r_2}{\phi_2}, 1 - \frac{r_2}{\phi_2}\right) \\ w_2\left(r_1, r_2\right) &= \overline{\sigma}_2 \kappa_{w_2}\left(1 - \frac{r_1}{\phi_1} - \frac{r_2}{\phi_2}, 1 - \frac{r_2}{\phi_2}\right), \end{aligned}$$

where $\kappa_{\sigma_1}, \kappa_{w_1}, \kappa_{\sigma_2}, \kappa_{w_2}$ are killing functions, as defined in Definition 1. Then, the process is well-defined for all $t \geq 0$, and Eq. 34holds. Also, (35) holds for all $t \geq 0$.

Stock with a stochastic trend in dividend growth and a stochastic equity premium We apply Theorem 2 to a stock model with stochastic growth rate and stochastic risk premium (Gabaix 2007, Example 9). The stochastic discount factor M_t and the dividend process D_t follow

$$dM_t/M_t = -rdt - \frac{\pi_t}{\sigma} dz_t \text{ and } dD_t/D_t = g_t dt + \sigma dz_t$$

The price of the stock is $P_t = E_t\left[\int_t^\infty M_s D_s ds\right]/M_t$. π_t is a the stochastic equity premium, and g_t is the stochastic growth rate of dividends.

We assume that π_t and g_t follow the following LG process, best expressed in terms of their deviation from trend, $\widehat{\pi}_t = \pi_t - \pi_*, \widehat{g}_t = g_t - g_*$:

$$d\widehat{g}_t = -\phi_g \widehat{g}_t dt + (\widehat{\pi}_t - \widehat{g}_t)\,\widehat{g}_t dt + \sigma_g(\widehat{g}_t, \widehat{\pi}_t) \cdot dB_t$$
$$d\widehat{\pi}_t = -\phi_\pi \widehat{\pi}_t dt + (\widehat{\pi}_t - \widehat{g}_t)\,\widehat{\pi}_t dt + \sigma_\pi(\widehat{g}_t, \widehat{\pi}_t) \cdot dB_t,$$

where the B_t is a $p-$dimensional Wiener process independent of z_t, and σ_γ and σ_π are processes with values in $\mathbb{R}^p$. In particular, the innovations of $\widehat{g}_t$ and $\widehat{\pi}_t$ can have quite general correlations. We suppose that the process is defined in $[t, \infty)$. Again the processes $d\widehat{g}_t$ and $d\widehat{\pi}_t$ are to a first order linear, but with quadratic "twist" terms added, $(\widehat{\pi}_t - \widehat{g}_t)\,\widehat{g}_t dt$ and $(\widehat{\pi}_t - \widehat{g}_t)\,\widehat{\pi}_t dt$. The stock price is

$$P_t = \frac{D_t}{R}\left(1 + \frac{g_t - g_*}{R + \phi_g} - \frac{\pi_t - \pi_*}{R + \phi_\pi}\right) \text{ with } R \equiv r + \pi_* - g_* \tag{36}$$

where R is the traditional Gordon rate. This example nests the three sources of variation in stock prices: the movements in dividends (D_t), in expected growth rate of dividends (g_t), and in the discount factor (π_t).

We next study how to ensure that the process is well-defined. For concreteness, we suppose $0 < \phi_g \leq \phi_\pi$. This may represent slow-moving innovations to the growth rate, as in Bansal and Yaron (2003). This stock model is analogous to the 2-factor interest rate model above, with $r_{1t} = -\widehat{g}_t$ and $r_{2t} = \widehat{\pi}_t$. The diagonal representation of the process is

$$K_t = M_t D_t \begin{pmatrix} 1 + \widehat{g}_t/\phi_g - \widehat{\pi}_t/\phi_\pi \\ -\widehat{g}_t/\phi_g \\ \widehat{\pi}_t/\phi_\pi \end{pmatrix},$$

and $E_t[dK_t] = \Lambda K_t dt$ with $\Lambda = (-R, -R - \phi_g, -R - \phi_\pi)$. The positivity test vector Z_t is:

$$Z_t = K_t = M_t D_t \begin{pmatrix} 1 + \widehat{g}_t/\phi_g - \widehat{\pi}_t/\phi_\pi \\ 1 - \widehat{\pi}_t/\phi_\pi \\ 1 \end{pmatrix},$$

and

$$dZ_t = \begin{pmatrix} -R & 0 & 0 \\ \phi_g & -R-\phi_g & 0 \\ \phi_g & \phi_\pi - \phi_g & -R-\phi_\pi \end{pmatrix} Z_t + Z_t dz_t + M_t D_t \begin{pmatrix} \left(\sigma_g\left(\widehat{g}_t, \widehat{\pi}_t\right)/\phi_g - \sigma_\pi\left(\widehat{g}_t, \widehat{\pi}_t\right)/\phi_\pi\right)\cdot dB_t \\ -\sigma_\pi\left(\widehat{g}_t, \widehat{\pi}_t\right)/\phi_g \cdot dB_t \\ 0 \end{pmatrix}.$$

Hence, applying 2, we obtain the following.

Result 4 *(Stock with a stochastic trend in dividend growth and a stochastic equity premium) Suppose that*

$$1 + \widehat{g}_t/\phi_g - \widehat{\pi}_t/\phi_\pi > 0 \text{ and } 1 - \widehat{\pi}_t/\phi_\pi > 0 \tag{37}$$

holds for $t = 0$*, and*

$$\sigma_g\left(\widehat{g}_t, \widehat{\pi}_t\right) = \kappa_g\left(1 + \widehat{g}_t/\phi_g - \widehat{\pi}_t/\phi_\pi\right)\overline{\sigma}_g$$
$$\sigma_\pi\left(\widehat{g}_t, \widehat{\pi}_t\right) = \kappa_\pi\left(1 + \widehat{g}_t/\phi_g - \widehat{\pi}_t/\phi_\pi, 1 - \widehat{\pi}_t/\phi_\pi\right)\overline{\sigma}_\pi,$$

where κ_g, κ_π *are killing functions with values in* $\mathbb{R}_+$*, as defined in Definition 1, and* $\overline{\sigma}_g$*,* $\overline{\sigma}_\pi$ *are* $p-$*dimensional vectors. Then, the process is well-defined for all* $t \geq 0$*, and Eq. 36 holds. Also, (37) holds for all* $t \geq 0$*.*

4 Some Other Possible Specifications

The conditions described in this paper are the existence of a vector-valued process $Z_t \in \mathbb{R}^N$, a non-zero vector $\xi \in \mathbb{R}_+^N$, a $N \times N$ matrix A with nonnegative non-diagonal elements, and positive diagonal elements, such that $E_t\left[Z_{t+1}\right] = AZ_t$ and $M_t D_t = \xi' Z_t$. This way, in the deterministic case, having $Z_t \succ 0$ implies $Z_{t+1} = AZ_t \succ 0$. In the case with noise, we had to ensure that the noise in $Z_{t+1} - E_t\left[Z_{t+1}\right]$ goes to 0 close to the boundaries.

In the above construction we have used a particular positivity vector $Z_t = \Theta K_t$, where Θ is the matrix in Lemma 1. We now show three different ways to construct other positivity vectors Z_t.

Specification with the Π matrix One such vector is $Z_t = \Pi K_t$ where Π is defined in Lemma 2, which proves that the associated A matrix, $A = \Pi \Lambda \Pi^{-1}$, satisfies the nonnegativity conditions on A, and also $\xi = (0, ..., 0, 1)$. It provides similar conditions to the ones we have expressed. In the

2-interest rate factor example of section 3.3, it gives:

$$Z_t = M_t \begin{pmatrix} \phi_1 \left(1 - \frac{r_{1t}}{\phi_1} - \frac{r_{2t}}{\phi_2}\right) \\ \phi_2 \left(1 - \frac{r_{1t}}{\phi_2} - \frac{r_{2t}}{\phi_2}\right) \\ 1 \end{pmatrix},$$

so the positivity conditions are:

$$\frac{r_{1t}}{\phi_1} + \frac{r_{2t}}{\phi_2} < 1 \text{ and } \frac{r_{1t}}{\phi_2} + \frac{r_{2t}}{\phi_2} < 1, \tag{38}$$

rather than the conditions (35), which came from using with matrix Θ. Conditions (35) imply (38), as $\frac{r_{1t}}{\phi_2} + \frac{r_{2t}}{\phi_2} = \lambda \left(\frac{r_{1t}}{\phi_1} + \frac{r_{2t}}{\phi_2} \right) + (1 - \lambda) \left(\frac{r_{2t}}{\phi_2} \right)$, with $\lambda = \phi_1/\phi_2 \in [0, 1]$. This is a general phenomenon (see Appendix B).

In this paper we chose to highlight the Θ matrix, because it is arguably simpler to use. It is easy to verify that the logic of this paper would be the same if we used the Π matrix.

Specification with an infinity of conditions Another possibility has an infinity of conditions ($N = \infty$):

$$Z_t = \left(\nu' \Psi^T Y_t\right)_{T=0,1,...}, \tag{39}$$

which basically calculates the value of all prices of finite-maturity claims $E_t \left[M_{t+T} D_{t+T}\right] = \nu' \Psi^T Y_t$ (see Appendix A).[5] Here $A_{ij} = 1_{\{j=i+1\}}$, and $\xi = (1, 0, 0, ...)$. This is the weakest sort of condition (all other conditions imply positivity bond prices, hence $\nu' \Psi^T Y_t = E_t \left[M_{t+T} D_{t+T}\right] > 0$), but it forces checking an infinity of inequalities, which is burdensome, at it requires some reasoning and truncation.

A hybrid specification We can have a mixed example. Take J a positive integer, and $N = J + n + 1$, and

$$\begin{aligned} Z_{jt} &= \iota' \Lambda^{j-1} K_t \text{ for } j = 1...J \\ &= \Theta^{(j-J)} \Lambda^J K_t \text{ for } j = J + 1...J + n + 1, \end{aligned}$$

where $\Theta^{(j-J)}$ is the $j - J$-th row vector in matrix Θ. The first part of Z_t is condition (39) for $T = 0...J - 1$ (indeed, $\nu' \Psi^T Y_t = \iota' \Lambda^T K_t$). The second part of Z_t is the condition associated with

[5] It can also be normalized $Z_t = \left(\nu' \Psi^T Y_t / \Delta_{11}^T\right)_{T=0,1,...}$, where Δ_{11} is the largest eigenvalue of Ψ.

Θ and applied to the expected value of the state vector in J periods $E_t[K_{t+J}] = \Lambda^J K_t$. So, for the first J period, the positivity of bond prices is checked while for the later periods the criterion $\Theta K_t \succ 0$ is used. [6]

5 Conclusion

This paper has provided conditions ensuring that LG processes are well-defined. We have tried to formulate them in a way that makes them easy to use in theoretical or empirical work. We have illustrated the conditions via a series of economic examples.

As we end this paper we wish to highlight one remaining question on LG processes. Is there a formulation of the volatility as function of the state variables, that allows calculation in closed form of derivative prices with LG processes (perhaps up to a Fourier transform, as Duffie, Pan and Singleton 2000 for affine processes) ? Our conditions should be useful to guide the search of such a formulation of volatility.

[6] $\xi = (1, 0, 0, ...)$ and $A = \begin{pmatrix} 1\{j = i+1\} & B \\ 0 & \Theta\Delta\Theta^{-1} \end{pmatrix}$ with $B_{ij} = 0$ for $i = 1...J$, $j = 1...n+1$, except $b_{J,n+1} = 1$.

Appendix A. Results for Linearity-Generating processes

Here we present the some results on the Linearity-Generating (LG) processes identified and analyzed in Gabaix (2007). LG processes are given by $M_t D_t$, a pricing kernel M_t times a dividend D_t, and X_t, a n-dimensional (n a non-zero integer) vector of factors (that can be thought as stationary). For instance, for bonds the dividend is $D_t = 1$.

Discrete time By definition, process $M_t D_t (1, X_t)$ is a LG process with generator $\Psi = \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$ if and only if it follows, for all t's:

$$\mathbb{E}_t \left[\frac{M_{t+1} D_{t+1}}{M_t D_t} \right] = \alpha + \delta' X_t \tag{40}$$

$$\mathbb{E}_t \left[\frac{M_{t+1} D_{t+1}}{M_t D_t} X_{t+1} \right] = \gamma + \Gamma X_t, \tag{41}$$

where $\alpha \in \mathbb{R}, \gamma \in \mathbb{R}^n, \delta \in \mathbb{R}^n$, $\Gamma \in \mathbb{R}^{n\times n}$, and almost surely $M_t D_t > 0$ for all $t \geq 0$.

Higher moments need not be specified. For instance the functional form of the noise does not matter, which makes LG processes parsimonious.

The key property of LG processes is that stocks and bonds have simple closed-form expressions. The price-dividend ratio of a "bond", $Z_t(T) = \mathbb{E}_t [M_{t+T} D_{t+T}] / (M_t D_t)$, is, with I_n the identity matrix of dimension n, and 0_n is the row vector with n zeros.

$$Z_t(T) = \begin{pmatrix} 1 & 0_n \end{pmatrix} \Psi^T \begin{pmatrix} 1 \\ X_t \end{pmatrix} \tag{42}$$

$$= \alpha^T + \delta' \frac{\alpha^T I_n - \Gamma^T}{\alpha I_n - \Gamma} X_t \text{ when } \gamma = 0. \tag{43}$$

If all eigenvalues of generator $\begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix}$ have a modulus less than 1, then the price of a stock, $P_t = \mathbb{E}_t \left[\sum_{s \geq t} M_s D_s \right] / M_t$, is finite and equal to:

$$P_t = D_t \frac{1 + \delta' (I_n - \Gamma)^{-1} X_t}{1 - \alpha - \delta' (I_n - \Gamma)^{-1} \gamma} \tag{44}$$

$$= D_t \begin{pmatrix} 1 & 0_n \end{pmatrix} (I_{n+1} - \Psi)^{-1} \begin{pmatrix} 1 \\ X_t \end{pmatrix}. \tag{45}$$

There is a more compact way to think about LG processes. Define the process with values in

$\mathbb{R}^{n+1}$

$$Y_t := \begin{pmatrix} M_t D_t \\ M_t D_t X_t \end{pmatrix} = \begin{pmatrix} M_t D_t \\ M_t D_t X_{1t} \\ \vdots \\ M_t D_t X_{nt} \end{pmatrix}, \tag{46}$$

so that with vector $\nu' = (1, 0, ..., 0)$,

$$M_t D_t = \nu' Y_t. \tag{47}$$

Y_t stacks all the information relevant to the prices of the claims derived here (other assets, e.g. options, require of course to know more moments). Conditions (40)-(41) can be written as:

$$E_t \left[Y_{t+1} \right] = \Psi Y_t. \tag{48}$$

Hence, the (dividend-augmented) stochastic discount factor of a LG process is simply the projection (47) of an autoregressive process, Y_t. The tractability of LG processes comes from the tractability of autoregressive processes. Eq. 42 comes simply from:

$$Z_0(T) = \frac{E_0 [M_T D_T]}{M_0 D_0} = \frac{E_0 [\nu' Y_T]}{M_0 D_0} = \frac{\nu' E_0 [Y_T]}{M_0 D_0} = \frac{\nu' \Psi^T Y_0}{M_0 D_0} = \nu' \Psi^T \frac{Y_0}{M_0 D_0} = \nu' \Psi^T \begin{pmatrix} 1 \\ X_t \end{pmatrix}.$$

Also, Eq. 45 comes from:

$$\frac{P_0}{D_0} = \sum_{T=0}^{\infty} \frac{E_0 [M_T D_T]}{M_0 D_0} = \sum_{T=0}^{\infty} Z_0(T) = \sum_{T=0}^{\infty} \nu' \Psi^T \begin{pmatrix} 1 \\ X_t \end{pmatrix} = \nu' \left(\sum_{T=0}^{\infty} \Psi^T \right) \begin{pmatrix} 1 \\ X_t \end{pmatrix}$$
$$= \nu' \left(I_{n+1} - \Psi \right)^{-1} \begin{pmatrix} 1 \\ X_t \end{pmatrix}.$$

If the process is well-defined, i.e. $M_t D_t > 0$ for all t, prices simply depend on (48). The central task of this paper is to provide conditions on the process, so that indeed $M_t D_t > 0$ for all t.

Continuous time The following notation is useful when using LG processes. For x_t, μ_t processes, we say $E_t [dx_t] = \mu_t dt$, or $E_t [dx_t] / dt = \mu_t$, to signify that there exists a martingale N_t such that: $x_t = x_0 + \int_0^t \mu_s ds + N_t$.

In continuous time, $M_t D_t (1, X_t)$ is a LG process with generator $\Xi = \begin{pmatrix} a & \beta \\ b & \Phi \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$

if and if only it follows:

$$E_t\left[d\left(M_tD_t\right)\right] = -\left(a+\beta' X_t\right) M_t D_t dt \tag{49}$$

$$E_t\left[d\left(M_tD_tX_t\right)\right] = -\left(b+\Phi X_t\right) M_t D_t dt, \tag{50}$$

with $a \in \mathbb{R}, \beta \in \mathbb{R}^n, b \in \mathbb{R}^n$, $\Phi \in \mathbb{R}^{n\times n}$, and almost surely $M_tD_t > 0$ for all $t \geq 0$

The price-dividend ratio of a "bond" is: $Z_t\left(T\right) = \mathbb{E}_t\left[M_{t+T}D_{t+T}\right] / \left(M_tD_t\right)$

$$\begin{aligned} Z_t\left(T\right) &= \begin{pmatrix} 1 & 0_n \end{pmatrix} \exp\left(-\Xi T\right) \begin{pmatrix} 1 \\ X_t \end{pmatrix} \\ &= \exp\left(-aT\right) + \beta' \frac{\exp\left(-\Phi T\right) - \exp\left(-aT\right) I_n}{\Phi - aI_n} X_t \text{ when } b = 0. \end{aligned} \tag{51}$$

The price of a stock, $P_t/D_t = \mathbb{E}_t\left[\int_t^\infty M_sD_sds\right] / \left(M_tD_t\right)$, is, if all eigenvalues of generator Ξ have a positive real part (finite stock price):

$$\begin{aligned} P_t/D_t &= \frac{1-\beta'\Phi^{-1}X_t}{a-\beta'\Phi^{-1}b} \\ &= \begin{pmatrix} 1 & 0_n \end{pmatrix} \Xi^{-1} \begin{pmatrix} 1 \\ X_t \end{pmatrix}. \end{aligned}$$

To ensure that the process is well-behaved (hence prevent prices from being negative), the volatility of the process has to go to zero near some boundary. The present paper sufficient conditions for this.

Similarly to the discrete time case, one defines the vector Y_t as in (46). Thus conditions (40)-(41) can be written as:

$$E_t\left[dY_t\right] = -\Xi Y_t dt, \tag{52}$$

which is the continuous time analogue of (48). The formulas for bonds and stocks are derives as in the discrete time case, observing $E_t\left[Y_{t+T}\right] = e^{-\Xi T}Y_t$. The latter equation requires some regularity conditions that are the topic of this paper.

Appendix B. Some Useful Lemmas on Positive Matrices

The following Lemma is useful in several parts of this paper. It is proven by simple verification of (54).

Lemma 1 *(Θ matrix) Given an integer $N \geq 1$, a diagonal matrix $\Lambda = diag(\lambda_1, ..., \lambda_N)$, define the $N \times N$ matrix Θ:*

$$\Theta = \left(1_{\{i \geq j\}}\right)_{ij} = \begin{pmatrix} 1 & 0 & ... & 0 \\ ... & 1 & 0 & ... \\ ... & ... & 1 & 0 \\ 1 & ... & ... & 1 \end{pmatrix}. \tag{53}$$

Then, $\left(\Theta\Lambda\Theta^{-1}\right)_{ij} = 0$ if $i < j$, $= \lambda_i$ if $i = j$, and $= \lambda_j - \lambda_{j+1}$ if $i > j$, i.e.

$$\Theta\Lambda\Theta^{-1} = \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ \lambda_1 - \lambda_2 & \lambda_2 & 0 & ... \\ ... & ... & \ddots & 0 \\ \lambda_1 - \lambda_2 & ... & \lambda_{N-1} - \lambda_N & \lambda_N \end{pmatrix}. \tag{54}$$

In particular, if $\lambda_1 \geq ... \geq \lambda_N$, then $\Theta\Lambda\Theta^{-1}$ has non-negative coefficients, and, if the λ_i are positive, positive diagonal coefficients.

The Lemma implies that, if $Z_{t+1} = \Theta\Lambda\Theta^{-1}Z_t$, then $Z_0 \succ 0$ implies that for all $t \geq 0$, $Z_t \succ 0$. The next Lemma analyzes another matrix with a similar property.

Lemma 2 *(Π matrix) Given an integer $N \geq 1$, a diagonal matrix $\Lambda = diag(\lambda_1, ..., \lambda_N)$, define the $N \times N$ matrix Π:*

$$\begin{aligned} \Pi_{ij} &= (\lambda_{i+1} - \lambda_j)\, 1_{i \geq j} \text{ for } i = 1...N-1 \\ &= 1 \text{ for } i = N, \end{aligned}$$

i.e.,

$$\Pi = \begin{pmatrix} \lambda_1 - \lambda_2 & 0 & ... & 0 \\ \lambda_1 - \lambda_3 & \lambda_2 - \lambda_3 & 0 & ... \\ ... & ... & \ddots & 0 \\ \lambda_1 - \lambda_N & ... & \lambda_{N-1} - \lambda_N & \lambda_{N-1} - \lambda_N \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{55}$$

Then, $\left(\Pi\Lambda\Pi^{-1}\right)_{ij} = 0$ if $i < j$ or $N = i > j$, $= \lambda_i$ if $i = j$, and $= \lambda_j - \lambda_{j+2}$ if $N > i > j$, and $= 1$ if $i = N$ and $j = N - 1$.

In particular, if $\lambda_1 \geq ... \geq \lambda_N$, then $\Theta\Lambda\Theta^{-1}$ has non-negative coefficients and, if the λ_i are positive, positive diagonal coefficients.

Proof. Call A the announced value for $\Pi\Lambda\Pi^{-1}$. One simply verifies that $\Pi\Lambda = A\Pi$, which is straightforward algebra. ■

For instance, when $N = 5$

$$\Pi\Lambda\Pi^{-1} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ \lambda_1 - \lambda_3 & \lambda_2 & 0 & 0 & 0 \\ \lambda_1 - \lambda_3 & \lambda_2 - \lambda_4 & \lambda_3 & 0 & 0 \\ \lambda_1 - \lambda_3 & \lambda_2 - \lambda_4 & \lambda_3 - \lambda_5 & \lambda_4 & 0 \\ 0 & 0 & 0 & 1 & \lambda_5 \end{pmatrix}.$$

The Θ matrix yields stricter conditions than Π matrix, in the sense that $\forall Z \in \mathbb{R}^N$, $\Theta Z \succ 0 \Rightarrow \Pi Z \succ 0$. This comes from the fact that $\Pi = \pi\Theta$ for a matrix π with non-negative coefficients, namely $\pi_{ij} = 0$ for $i > j$ or $i = N > j$, $\pi_{NN} = 1$, and $\pi_{ij} = \lambda_j - \lambda_{j+1}$ otherwise.

Appendix C. Additional Derivations

Proof of Proposition 1 In the deterministic case $Y_{t+1} = \Psi Y_t$, hence $Z_{t+1} = \Theta\Lambda\Theta^{-1}Z_t$.

Eq 14 is $Z_t \succ 0$. The key fact is Lemma 1 in Appendix A which shows that $\Theta\Lambda\Theta^{-1}$ has non-negative non-diagonal coefficients, and positive diagonal coefficients. So $Z_t \succ 0$ implies $Z_{t+1} \succ 0$. By induction, for all $t \geq 0$, $Z_t \succ 0$.

Finally, $M_t D_t = Z_{nt} > 0$.

Proof of Proposition 2 It is proven like Proposition 1. Define $z_t = E_0 Z_t$. Then, $z_{t+1} = \Theta\Lambda\Theta^{-1}z_t$. Observe that $z_0 \succ 0$, and $z_t \succ 0$ implies $z_{t+1} \succ 0$. Finally, $E_0 [M_t D_t] = z_{nt} > 0$.

Proof of Proposition 6 With $Z_t = FY_t$, $Z_{t+1} = F\Psi F^{-1}Z_t + F\sigma(F^{-1}Z_t)\eta_{t+1}$. So (SC1) simply expresses $Z_t \succ 0 \Rightarrow Z_{t+1} \succ 0$.

Because $F^{(i)}\sigma(F^{-1}Z)\eta_{t+1} \geq -\left\|F^{(i)}\sigma(F^{-1}Z)\right\|_1 \|\eta\|_\infty$, $SC2 \Rightarrow SC1$ and $SC4 \Rightarrow SC3$

Next, remark that $\lambda_i = (F\Psi F^{-1})_{ii}$. Because $F\Psi F^{-1}$ has nonnegative elements non-diagonal elements, when $Z_t \succ 0$,

$$F^{(i)}\Psi F^{-1}Z_t = \sum_{j=1}^{n} \left(F^{(i)}\Psi F^{-1}\right)_{ii} Z_{jt} \geq \left(F^{(i)}\Psi F^{-1}\right)_{ii} Z_{it} = \lambda_i Z_{it},$$

i.e.

$$F^{(i)}\Psi F^{-1}Z_t \geq \lambda_i Z_{it}$$

This implies $SC3 \Rightarrow SC1$ and $SC4 \Rightarrow SC2$.

Proof of Proposition 7 Immediate, given SC1 and SC2 of Proposition 6.

Proof of Theorem 1 By (ii), μ and σ have a unique continuous extension to the closure $\overline{\mathcal{D}}$ of $\mathcal{D}$. From there we extend them to $\mathbb{R}^{n+1}$ by setting $\hat{\mu}(x) := \mu(|x|)$ and $\hat{\sigma}(x) := \sigma(|x|)$, where $|x|$ denotes the vector with components $|x^m|$, $m = 0, \ldots, n$. Since $|||x||| = ||x||$ and $|||x|-|y||| \leq ||x-y||$ for all $x, y \in \mathbb{R}^n$, $\hat{\mu}$ and $\hat{\sigma}$ satisfy the conditions (i) and (ii) for all $x, y \in \mathbb{R}^{n+1}$. Therefore, the SDEs

$$dz_t = \hat{\mu}(z_{t-})dt + \hat{\sigma}(z_{t-})dG_t + diag\,(z_{t-})\,dH_t\,, \quad z_0 = x_0 \in \mathcal{D}\,, \tag{56}$$

and

$$dA_t = cA_{t-}dt + \hat{\sigma}(A_{t-})dG_t + diag\,(A_{t-})\,dH_t\,, \quad A_0 = x_0 \in \mathcal{D}\,, \tag{57}$$

both have unique solutions. If we can show that A_t does not leave $\mathcal{D}$, then z_t cannot leave $\mathcal{D}$ either. Indeed, it follows by comparison from condition (iii) that $z_t^m \geq A_t^m$ for all m and $0 \leq t \leq \tau$, where τ is the stopping time

$$\tau := \inf\{t \geq 0 : z_t \notin \mathcal{D}\}\ .$$

So if A_t does not leave $\mathcal{D}$, then $\tau = \infty$ and z_t does not leave $\mathcal{D}$.

To show that A_t does not leave $\mathcal{D}$, introduce for all $m = 0, \ldots, n$ and $k \geq 1$ the functions

$$\nu^m(x) := \begin{cases} (x^m)^{-1}\hat{\sigma}^m(x) & \text{if } x^m \neq 0 \\ (0, \ldots, 0) & \text{if } x^m = 0 \end{cases}\ .$$

and

$$\nu^{m,k}(x) := \nu^m(x)\varphi^{m,k}(x)\,,$$

where

$$\varphi^{0,k}(x) := \begin{cases} \left(\frac{|x^0|}{\varepsilon}\right)^{1/k} & \text{if } |x^0| < \varepsilon \\ 1 & \text{if } |x^0| \geq \varepsilon \end{cases}\ .$$

and for $m \geq 1$,

$$\varphi^{m,k}(x) := \begin{cases} \left(\frac{|x^m|}{\varepsilon(1\wedge|x^0|)}\right)^{1/k} & \text{if } |x^m| < \varepsilon(1 \wedge |x^0|) \\ 1 & \text{if } |x^m| \geq \varepsilon(1 \wedge |x^0|) \end{cases}\ .$$

$\nu^{0,k}$ is continuous for all $k \geq 1$,

$$\lim_{k\to\infty} \nu^{0,k}(x) = \nu^0(x) \quad \text{for all } x \in \mathbb{R}^{n+1},$$

and

$$\left|\nu^0(x) - \nu^{0,k}(x)\right| \leq d \quad \text{for all } k \geq 1 \text{ and } x \in \mathbb{R}^{n+1} .$$

Hence, $\nu^{0,k}(A_{t-})$ is a left-continuous process with right limits, and $\nu^0(A_{t-})$ is predictable and locally bounded. In particular, $\nu^0(A_{t-})$ is integrable with respect to G_t and A^0_t satisfies the SDE

$$dA^0_t = A^0_{t-} \left\{ cdt + \nu^0(A_{t-})dG_t + dH^0_t \right\} .$$

It follows that A^0_t is equal to the stochastic exponential $\mathcal{E}(z^0)_t$ of the semimartingale

$$z^0_t = ct + \int_0^t \nu^0(A_{s-})dG_s + H^0_t .$$

By (30) we have $\Lambda z^0_t > -1$ and therefore, $A^0_t = \mathcal{E}(z^0)_t > 0$ for all $t \geq 0$.

By (v), $\nu^{m,k}$ is continuous on the set $\{x \in \mathbb{R}^{n+1} : x^0 \neq 0\}$ for all $m \geq 1$ and $k \geq 1$. Moreover,

$$\lim_{k\to\infty} \nu^{m,k}(x) = \nu^m(x) \quad \text{for all } m \geq 1 \text{ and } x \in \mathbb{R}^{n+1} ,$$

and

$$\left|\nu^m(x) - \nu^{m,k}(x)\right| \leq d \quad \text{for all } m, k \geq 1 \text{ and } x \in \mathbb{R}^{n+1}.$$

It follows that $\nu^{m,k}(A_{t-})$ is left-continuous with right limits and $\nu^m(A_{t-})$ is predictable and locally bounded. In particular it is integrable with respect to G_t, and A^m_t solves the SDE

$$dA^m_t = A^m_{t-} \left(cdt + \nu^m(A_{t-})dG_t + dH^m_t \right) .$$

So it is equal to the stochastic exponential $\mathcal{E}(z^m)_t$ of the semimartingale

$$z^m_t := ct + \int_0^t \nu^m(A_{s-})dG_s + H^m_t .$$

Since $z^m_t > -1$ for all $t \geq 0$, $A^m_t = \mathcal{E}(z^m)_t > 0$ for all $t \geq 0$.

Proof of Proposition 8 Under the assumptions of this Proposition, $\xi_T \equiv \int_0^T \sigma(z_{t-})dG_t + diag\,(z_{t-})\,dH_t$ is a martingale. Define for $T \geq 0$, $f\,(T) = E_0\,[z_{t+T}]$. Taking $\varepsilon > 0$, we have:

$$\frac{f\,(T+\varepsilon) - f\,(T)}{\varepsilon} = E_0\left[\frac{z_{T+\varepsilon} - z_t}{\varepsilon}\right] = E_0\left[\frac{1}{\varepsilon}\int_{t=T}^{T+\varepsilon} \mu(z_{t-})dt + \sigma(z_{t-})dG_t + diag\,(z_{t-})\,dH_t\right]$$
$$= E_0\left[\frac{1}{\varepsilon}\int_T^{T+\varepsilon} Az_{t-}dt\right] + \frac{1}{\varepsilon}E_0\,[\xi_{T+\varepsilon} - \xi_T] = AE_0\left[\frac{\int_T^{T+\varepsilon} z_{t-}dt}{\varepsilon}\right] + 0.$$

Taking the limit $\varepsilon \to 0$, we get that $f\,(T)$ is differentiable, and

$$f'\,(T) = AE_0\,[z_t] = Af\,(T)\,,$$

which integrates to $f\,(T) = e^{AT}f\,(0) = e^{AT}z_0$.

Proof of Theorem 2 SDE (31) is of the form (29), with $\mu\,(Z) = AZ$, $\sigma\,(Z) = Z^0 v\,(Z/Z^0)$, $G_t = B_t$, and $d_t = J_t$.

Condition (a) implies condition (iii) from Theorem 1, as for $Z \succ 0$ we have:

$$\mu^m\,(Z) = A_{mm}Z^m + \sum_{j\neq m} A_{mj}Z^j \geq A_{mm}Z^m \geq cZ^m,$$

if we define $c = \min_{m=0,...,N-1} A_{mm}$.

We next verify that conditions (b2–b4) imply that $\sigma\,(Z) = Z^0 v\left(\frac{Z}{Z_0}\right)$ is Lipschitz. Indeed for $m = 1...N-1$,

$$\frac{\partial\sigma^{ij}\,(Z)}{\partial Z^m} = Z^0 \cdot \frac{1}{Z^0} \cdot \partial_m v^{ij}\left(\frac{Z}{Z^0}\right) = \partial_m v^{ij}\left(\frac{Z}{Z^0}\right),$$

where $\partial_m v^{ij}$ is the derivative of v^{ij} with respect to its m-th argument. So, $\left|\frac{\partial\sigma^{ij}(Z)}{\partial Z^m}\right| \leq C$. Also,

$$\frac{\partial\sigma^{ij}\,(Z)}{\partial Z^0} = \frac{\partial}{\partial Z^0}\left(Z^0 v^{ij}\left(\frac{Z}{Z_0}\right)\right) = v^{ij}\left(\frac{Z}{Z_0}\right) + \sum_{m=1}^{N-1} Z^0 \cdot \frac{-Z_m}{Z_0^2} \cdot \partial_m v^{ij}\left(\frac{Z}{Z_0}\right) = v^{ij} - \sum_{m=1}^{N-1} z^m \frac{\partial v^{ij}}{\partial z^m},$$

with $z^m = Z^m/Z^0$. So, $\left|\frac{\partial\sigma^{ij}(Z)}{\partial Z^0}\right| \leq C'$.

We conclude that $\sigma\,(Z)$ is Lipschitz. As $\mu\,(Z) = AZ$ is also Lipschitz this implies (i), (ii) from Theorem 1.

Conditions (iv) and (v) come from the fact that σ is Lipschitz, and condition (b1), as

$$\begin{aligned} ||\sigma^m(x)|| &= ||\sigma^m(x) - \sigma^m(x_0, ..., 0, ...x_{N-1})|| \text{ by condition (b1)} \\ &\leq \|\sigma'\|_\infty x^m. \end{aligned}$$

Proof of Result 2 We observe that $\widehat{r} = \phi(1 - Z_1/Z_2)$, and we apply Theorem 2 to $v(z) = \begin{pmatrix} \frac{\sigma(\phi(1-z))}{\phi} \\ 0 \end{pmatrix}$. Condition (b3) means $|\sigma'|$ is bounded. Calculating:

$$v^1 - z\frac{\partial v^1}{\partial z} = \frac{\sigma(\phi(1-z))}{\phi} + z\sigma'(\phi(1-z)) = \frac{\sigma(\widehat{r})}{\phi} + \sigma'(\widehat{r})\left(1 - \frac{\widehat{r}}{\phi}\right),$$

we see that condition (b4) means that $|\sigma(r) + \sigma'(r)\phi|$ is bounded.

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